1. The motivic complex after Beilinson

1.1. **Motivic complex as a cone complex.** In the paper [1] the authors construct an exact sequence

$$CH^r_{\mathrm{cont}}(X_\bullet) \to CH^r(X_1) \overset{ob}{\to} H^2r(X_\bullet, p(r)\Omega^{<r}_{X_\bullet}). \tag{1}$$

Here *ob* is a morphism induced by the the morphism

$$\varphi : \mathbb{Z}_{X_1}(r) \to p(r)\Omega^{<r}_{X_\bullet} \tag{2}$$

in the derived category of pro-sheaves. Here φ is represented by

$$\mathbb{Z}_{X_1}(r) \overset{\varphi'}{\to} q(r)W_\bullet\Omega^\bullet_{X_1} \overset{\text{qis.}}{\leftarrow} I(r)\Omega^\bullet_{D_\bullet} \overset{\text{qis.}}{\to} p(r)\Omega^\bullet_{X_\bullet} \to p(r)\Omega^{<r}_{X_\bullet} \tag{3}$$

and φ' is the composition of

$$\mathbb{Z}_{X_1}(r) \to \mathscr{K}^M_{X_1,r}[-r] \overset{\mathrm{dlog}[-]}{\to} W_\bullet\Omega^r_{X_1,\log}[-r] \to q(r)W_\bullet\Omega^\bullet_{X_1}. \tag{4}$$

A naive attempt is to define $CH^r_{\mathrm{cont}}(X_\bullet)$ as the hypercohomology of cone(φ). However, since the cone construction in triangulated categories is only determined up to non-unique isomorphisms, cone(φ) is **NOT** functorial! A solution, suggested by Beilinson, is to lift φ to a morphism between complexes.

1.2. **Beilinson's construction of motivic sheaves.** Motivated by (3) one define

$$\widetilde{\mathbb{Z}}_{X_\bullet}(r) := \mathrm{Cone}\left(\mathbb{Z}_{X_1}(r) \oplus I(r)\Omega^\bullet_{D_\bullet} \to p(r)\Omega^{<r}_{X_\bullet} \oplus q(r)W_\bullet\Omega^\bullet_{X_1}\right). \tag{5}$$

Here the morphisms

$$I(r)\Omega^\bullet_{D_\bullet} \to p(r)\Omega^{<r}_{X_\bullet}$$

and

$$I(r)\Omega^\bullet_{D_\bullet} \to q(r)W_\bullet\Omega^\bullet_{X_1}$$

are the natural ones.

$$\mathbb{Z}_{X_1}(r) \to p(r)\Omega^{<r}_{X_\bullet}$$

is the zero morphism and

$$\varphi' : \mathbb{Z}_{X_1}(r) \to q(r)W_\bullet\Omega^\bullet_{X_1}$$

is defined as (4).

The aim of this lecture is to show that there is a natural quasi-isomorphism

$$\mathbb{Z}_{X_\bullet}(r) \simeq_{\text{qis.}} \widetilde{\mathbb{Z}}_{X_\bullet}(r). \tag{6}$$

By [1, Definition 7.1], it suffices to construct a quasi-isomorphism

$$\mathfrak{S}_{X_\bullet}(r) \simeq_{\text{qis.}} \widetilde{\mathfrak{S}}_{X_\bullet}(r), \tag{7}$$

where

$$\widetilde{\mathfrak{S}}_{X_\bullet}(r) := \mathrm{Cone}\left(I(r)\Omega^\bullet_{D_\bullet} \oplus \Omega^{\geq r}_{X_\bullet} \oplus W_\bullet\Omega^r_{X_1,\log}[-r] \to p(r)\Omega^\bullet_{X_\bullet} \oplus q(r)W_\bullet\Omega^\bullet_{X_1}\right)[-1]. \tag{8}$$

The morphisms in (8) are either the natural ones or 0.

The proof is divided into 3 steps.

Step 1: $\tau_{\leq r}\widetilde{\mathfrak{S}}_{X_\bullet}(r) \simeq \widetilde{\mathfrak{S}}_{X_\bullet}(r)$. This is apparent by (8).

Step 2: There is a canonical quasi-isomorphism

$$\epsilon^*\widetilde{\mathfrak{S}}_{X_\bullet}(r) \simeq_{\text{qis.}} \mathfrak{S}_{X_\bullet}(r)_{\text{ét}}. \tag{9}$$

Here $\epsilon : X_{1,\text{ét}} \to X_{1,Nis}$ is the natural morphism between sites.

Consider the diagram

$$\begin{array}{ccccc}
W_\bullet\Omega^r_{X_1,\log}[-r] & \longrightarrow & W_\bullet\Omega^r_{X_1,\log}[-r] & \longrightarrow & 0 \\
\downarrow{\scriptstyle (0,0,Id)} & & \downarrow & & \downarrow \\
I(r)\Omega^\bullet_{D_\bullet} \oplus \Omega^{\geq r}_{\mathfrak{X}_\bullet} \oplus W_\bullet\Omega^r_{X_1,\log}[-r] & \longrightarrow & p(r)\Omega^\bullet_{\mathfrak{X}_\bullet} \oplus q(r)W_\bullet\Omega^\bullet_{X_1} & \longrightarrow & \epsilon^*\widetilde{\mathfrak{S}}_{X_\bullet}(r)[1] \\
\downarrow & & \downarrow{\scriptstyle (Id,1-F_r)} & & \downarrow \\
I(r)\Omega^\bullet_{D_\bullet} \oplus \Omega^{\geq r}_{\mathfrak{X}_\bullet} & \longrightarrow & p(r)\Omega^\bullet_{\mathfrak{X}_\bullet} \oplus q(r)W_\bullet\Omega^\bullet_{X_1} & \longrightarrow & \mathscr{C}
\end{array} . \tag{10}$$

The horizontal lines and the left vertical line are distinguished by constructions. The vertical line in the middle is distinguished by [1, Corollary 4.6]. Therefore we have a quasi-isomorphism

$$\epsilon^*\widetilde{\mathfrak{S}}_{X_\bullet}(r)[1] \simeq \mathscr{C}. \tag{11}$$

Step 2 is finished by the following diagram

$$\begin{array}{ccccc}
J(r)\Omega^\bullet_{D_\bullet} & \xrightarrow{1-f_r} & \Omega^\bullet_{D_\bullet} & \xrightarrow{(10)+(11)} & \mathfrak{S}_{X_\bullet}(r)_{\text{ét}}[1] \\
\downarrow & & \downarrow & & \downarrow \\
I(r)\Omega^\bullet_{D_\bullet} \oplus \Omega^{\geq r}_{\mathfrak{X}_\bullet} & \longrightarrow & p(r)\Omega^\bullet_{\mathfrak{X}_\bullet} \oplus q(r)W_\bullet\Omega^\bullet_{X_1} & \longrightarrow & \epsilon^*\widetilde{\mathfrak{S}}_{X_\bullet}(r)[1] \\
\downarrow & & \downarrow & & \downarrow \\
I(r)\Omega^\bullet_{D_\bullet} & \longrightarrow & I(r)\Omega^\bullet_{D_\bullet} & \longrightarrow & 0
\end{array} . \tag{12}$$

Step 3: Define

$$\psi : \widetilde{\mathfrak{S}}_{X_\bullet}(r) \overset{\text{Step1}}{\simeq} \tau_{\leq r}\widetilde{\mathfrak{S}}_{X_\bullet}(r) \overset{\text{Step2}}{\to} \tau_{\leq r}R\epsilon_*\mathfrak{S}_{X_\bullet}(r)_{\text{ét}} =: \mathfrak{S}_{X_\bullet}(r). \tag{13}$$

We would like to show that ψ is a quasi-isomorphism. Consider the diagram

$$\begin{array}{ccccc}
I(r)\Omega^\bullet_{D_\bullet} \oplus \Omega^{\geq r}_{\mathfrak{X}_\bullet} \oplus W_\bullet\Omega^r_{X_1,\log}[-r] & \longrightarrow & p(r)\Omega^\bullet_{\mathfrak{X}_\bullet} \oplus q(r)W_\bullet\Omega^\bullet_{X_1} & \longrightarrow & \widetilde{\mathfrak{S}}_{X_\bullet}(r)[1] \\
\downarrow & & \downarrow{\scriptstyle (Id,1-F_r)} & & \downarrow{\scriptstyle \psi'} \\
I(r)\Omega^\bullet_{D_\bullet} \oplus \Omega^{\geq r}_{\mathfrak{X}_\bullet} & \longrightarrow & p(r)\Omega^\bullet_{\mathfrak{X}_\bullet} \oplus q(r)W_\bullet\Omega^\bullet_{X_1} & \xrightarrow{\text{Step2}} & R\epsilon_*\mathfrak{S}_{X_\bullet}(r)_{\text{ét}}[1]
\end{array} . \tag{14}$$

It suffices to show that ψ' induces isomorphisms in the degree $\leq r$. Notice that [1, Lemma 4.5] implies that $H^{<r}(\psi')$ are isomorphisms, it remains to consider $H^r(\psi')$ in the following diagram

$$\begin{array}{ccccccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & H^r(\widetilde{\mathfrak{S}}_{X_\bullet}(r)) & \longrightarrow & \bullet \oplus W_\bullet\Omega^r_{X_1,\log} & \longrightarrow & \bullet \oplus H^r(q(r)W_\bullet\Omega^\bullet_{X_1}) \\
\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle H^r(\psi')} & & \downarrow & & \downarrow{\scriptstyle 1-F_r} \\
\bullet & \longrightarrow & \bullet & \longrightarrow & H^r(R\epsilon_*\mathfrak{S}_{X_\bullet}(r)_{\text{ét}}) & \longrightarrow & \bullet & \longrightarrow & \bullet \oplus H^r(W_\bullet\Omega^\bullet_{X_1})
\end{array} . \tag{15}$$

Here $\bullet$ stands for the obvious terms. By [1, Lemma 4.5, Corollary 4.6] we obtain that

$$0 \to W_\bullet\Omega^r_{X_1,\log} \to H^r(q(r)W_\bullet\Omega^\bullet_{X_1}) \stackrel{1-F_r}{\to} H^r(W_\bullet\Omega^\bullet_{X_1}) \tag{16}$$

is exact over the Nisnevich topology. Therefore $H^r(\psi')$ is an isomorphism by applying the five lemma to (15).

References

[1] S. Bloch, H. Esnault, and M. Kerz, *p-adic deformation of algebraic cycle classes*, Invent. Math. **195** (2014), no. 3, 673–722, DOI 10.1007/s00222-013-0461-4. MR3166216 ↑1, 2, 3