Homological and homotopical algebras

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1 t-structures

Motivation: $\mathrm{D}_{\mathrm{pro}}(T)$ is not a derived category, but rather a triangulated category with t-structure.

Definition 1.1. [BBD, Section 1.3] Let $\mathcal{D}$ be a triangulated category. A t-structure on $\mathcal{D}$ consists of a pair $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ of full subcategories, closed under isomorphisms, such that

(1) For $A \in \mathcal{D}^{\le 0}$, $B \in \mathcal{D}^{\ge 1}$, $\mathrm{Hom}(A, B) = 0$.

(2) $\mathcal{D}^{\le 0} \subseteq \mathcal{D}^{\le 1}$, $\mathcal{D}^{\ge 0} \supseteq \mathcal{D}^{\ge 1}$.

(3) For every $X \in \mathcal{D}$, there exists a distinguished triangle

$$A \to X \to B \to A[1] \tag{1.1}$$

with $A \in \mathcal{D}^{\le 0}$ and $B \in \mathcal{D}^{\ge 1}$.

Here $\mathcal{D}^{\le r}$ (resp. $\mathcal{D}^{\ge r}$) denotes the full subcategory of $\mathcal{D}$, closed under isomorphisms, spanned by $\mathcal{D}^{\le 0}[-r]$ (resp. $\mathcal{D}^{\ge 0}[-r]$).

Fact 1.2. (1) The distinguished triangle (1.1) is unique up to unique isomorphism. It can be written as

$$\tau^{\le 0}X \to X \to \tau^{\ge 1}X \to (\tau^{\le 0}X)[1],$$

where $\tau^{\le r}\colon \mathcal{D} \to \mathcal{D}^{\le r}$ is a right adjoint of the inclusion functor and $\tau^{\ge r}\colon \mathcal{D} \to \mathcal{D}^{\ge r}$ is a left adjoint of the inclusion functor.

(2) The heart $\mathcal{D}^{\heartsuit} := \mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0}$ is an abelian category. We write $\mathcal{H}^r X \in \mathcal{D}^{\heartsuit}$ for $(\tau^{\le r}\tau^{\ge r}X)[-r] \simeq (\tau^{\ge r}\tau^{\le r}X)[-r]$.

Question 1.3. Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ be a distinguished triangle. When do we have a distinguished triangle

$$\tau^{\le 0}A \xrightarrow{\tau^{\le 0}a} \tau^{\le a}B \xrightarrow{\tau^{\le 0}b} \tau^{\le 0}C \to (\tau^{\le 0}A)[1]? \tag{1.2}$$

Note that the long exact sequence associated to (1.2) implies that $\mathcal{H}^0 b$ is an epimorphism, so that the map $\mathcal{H}^0 c\colon \mathcal{H}^0 C \to \mathcal{H}^1 A$ is zero. The converse also holds:

Lemma 1.4 ([Z, Lemme 2.9], [SZ, Lemma 4.1.9])**.** *Let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ be a distinguished triangle such that $\mathcal{H}^0 c\colon \mathcal{H}^0 C \to \mathcal{H}^1 A$ is zero. Then there exists a unique nine-diagram of the form*

$$\begin{tikzcd}
\tau^{\le 0}A \arrow[r, "\tau^{\le 0}a"] \arrow[d] & \tau^{\le 0}B \arrow[r, "\tau^{\le 0}b"] \arrow[d] & \tau^{\le 0}C \arrow[r, "c_0"] \arrow[d, "u"] \arrow[dr, phantom, "(*)"] & (\tau^{\le 0}A)[1] \arrow[d] \\
A \arrow[r, "a"] \arrow[d] & B \arrow[r, "b"] \arrow[d] & C \arrow[r, "c"] \arrow[d] \arrow[dr, phantom, "(**)"] & A[1] \arrow[d, "v"] \\
\tau^{\ge 1}A \arrow[r, "\tau^{\ge 1}a"] \arrow[d] & \tau^{\ge 1}B \arrow[r, "\tau^{\ge 1}b"] \arrow[d] & \tau^{\ge 1}C \arrow[r, "c_1"] \arrow[d] & (\tau^{\ge 1}A)[1] \arrow[d] \\
(\tau^{\le 0}A)[1] \arrow[r, "(\tau^{\le 0}a)[1]"] & (\tau^{\le 0}B)[1] \arrow[r, "(\tau^{\le 0}b)[1]"] & (\tau^{\le 0}C)[1] \arrow[r, "c_0[1]"] & (\tau^{\le 0}A)[2],
\end{tikzcd} \tag{1.3}$$

where the columns are the canonical distinguished triangles.

By a *nine-diagram* in a triangulated category (cf. [BBD, Proposition 1.1.11]), we mean a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d, dashed] \\
A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & A'[1] \arrow[d, dashed] \\
A'' \arrow[r] \arrow[d] & B'' \arrow[r] \arrow[d] & C'' \arrow[r] \arrow[d] \arrow[dr, phantom, "-"] & A''[1] \arrow[d, dashed] \\
A[1] \arrow[r, dashed] & B[1] \arrow[r, dashed] & C[1] \arrow[r, dashed] & A[2],
\end{tikzcd}$$

in which the square marked with "$-$" is anticommutative and all other squares are commutative, the dashed arrows are induced from the solid ones by translation, and the rows and columns in solid arrows are distinguished triangles.

Proof. First note that vcu is the image of $\mathcal{H}^0 c$ under the isomorphism

$$\mathrm{Hom}(\mathcal{H}^0 C, \mathcal{H}^1 A) \xrightarrow{\sim} \mathrm{Hom}(\tau^{\le 0}C, (\tau^{\ge 1}A)[1]).$$

Hence $vcu = 0$. Moreover, $\mathrm{Hom}(\tau^{\le 0}C, \tau^{\ge 1}A) = 0$. Thus by [BBD, Proposition 1.1.9], there exist a unique c_0 making $(*)$ commutative and a unique c_1 making $(**)$ commutative. This proves the uniqueness of (1.3). It remains to show that (1.3) thus constructed is a nine-diagram. To do this, we extend the upper left square of

(1.3) into a nine-diagram

$$\begin{tikzcd}
\tau^{\leq 0}A \arrow[r, "\tau^{\leq 0}a"] \arrow[d] & \tau^{\leq 0}B \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d] & (\tau^{\leq 0}A)[1] \arrow[d] \\
A \arrow[r, "a"] \arrow[d] & B \arrow[r, "b"] \arrow[d] & C \arrow[r, "c"] \arrow[d] \arrow[dr, phantom, "(***)"] & A[1] \arrow[d] \\
\tau^{\geq 1}A \arrow[r, "\tau^{\geq 1}a"] \arrow[d] & \tau^{\geq 1}B \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & (\tau^{\geq 1}A)[1] \arrow[d] \\
(\tau^{\leq 0}A)[1] \arrow[r, "(\tau^{\leq 0}a)[1]"] & (\tau^{\leq 0}B)[1] \arrow[r] & C_0[1] \arrow[r] & (\tau^{\leq 0}A)[2].
\end{tikzcd} \tag{1.4}$$

By the first and third rows of (1.4), $C_0 \in \mathcal{D}^{\leq 0}$ and $C_1 \in \mathcal{D}^{\geq 0}$. Taking $\mathcal{H}^0$ of $(***)$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{H}^0 C \arrow[r, "0"] \arrow[d, "e"'] & \mathcal{H}^1 A \arrow[d, equal] \\
\mathcal{H}^0 C_1 \arrow[r, "d"] & \mathcal{H}^1 A,
\end{tikzcd}$$

where e is an epimorphism and d is a monomorphism. Thus $\mathcal{H}^0 C_1 = 0$, so that $C_1 \in \mathcal{D}^{\geq 1}$. Further applying [BBD, Proposition 1.1.9], we may identify (1.4) with (1.3). $\square$

Example 1.5. [BEK, Lemma A.1] For a distinguished triangle $A \to B \to C \to A[1]$ with $A \in \mathcal{D}^{\leq r}$, the triangle

$$A \to \tau^{\leq r} B \to \tau^{\leq r} C \to A[1]$$

is distinguished.

2 Continuous cohomology

We let $\mathbb{N}$ denote the ordered set $\{1, 2, \dots\}$. For a category we write $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}$ for the category $\mathrm{Fun}(\mathbb{N}^{\mathrm{op}}, \mathcal{C})$ of systems $\mathcal{F}_\bullet = (\mathcal{F}_1 \leftarrow \mathcal{F}_2 \leftarrow \cdots)$ of objects of $\mathcal{C}$.

Let T be a topos. Then $T^{\mathbb{N}^{\mathrm{op}}}$ is a topos. We write $\mathbf{Ab}(T)$ for the category of abelian sheaves on T, which is a Grothendieck abelian category. Then we have an equivalence $\mathbf{Ab}(T^{\mathbb{N}^{\mathrm{op}}}) \simeq \mathbf{Ab}(T)^{\mathbb{N}^{\mathrm{op}}}$. We write $\mathrm{D}(T)$ for the derived category $\mathrm{D}(\mathbf{Ab}(T))$. The functor $\mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}}) \to \mathrm{D}(T)^{\mathbb{N}^{\mathrm{op}}}$ is *not* an equivalence in general. We will work with $\mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}})$.

Definition 2.1. Let $\mathcal{F}_\cdot \in \mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}})$. The *continuous (hyper)cohomology* is

$$H^i_{\mathrm{cont}}(T, \mathcal{F}_\bullet) := \mathrm{Hom}_{\mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}})}(\underline{\mathbb{Z}}, \mathcal{F}_\bullet[i]).$$

Here $\underline{\mathbb{Z}}$ denotes the constant system of the constant sheaf of value $\mathbb{Z}$.

Lemma 2.2. *(cf.* [BEK, Lemma B.8]*) We have a spectral sequence*

$$E_2^{p,q} = H^p_{\mathrm{cont}}(T, \mathcal{H}^q(\mathcal{F}_\bullet)) \Rightarrow H^{p+q}_{\mathrm{cont}}(T, \mathcal{F}_\bullet).$$

The square of topoi

$$\begin{tikzcd} T^{\mathbb{N}^{\mathrm{op}}} \arrow[r, "\varprojlim"] \arrow[d, "\Gamma^{\mathbb{N}^{\mathrm{op}}}"'] & T \arrow[d, "\Gamma"] \\ \mathbf{Set}^{\mathbb{N}^{\mathrm{op}}} \arrow[r, "\varprojlim"] & \mathbf{Set} \end{tikzcd}$$

induces a square of derived categories

(2.1)
$$\begin{tikzcd} \mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}}) \arrow[r, "R\varprojlim_T"] \arrow[d, "R\Gamma^{\mathbb{N}^{\mathrm{op}}}"'] & \mathrm{D}(T) \arrow[d, "R\Gamma"] \\ \mathrm{D}(\mathbf{Ab}^{\mathbb{N}^{\mathrm{op}}}) \arrow[r, "R\varprojlim_{\mathbf{Ab}}"] & \mathrm{D}(\mathbf{Ab}) \end{tikzcd}$$

Fact 2.3. Let $\mathcal{F}_\bullet \in \mathbf{Ab}^{\mathbb{N}^{\mathrm{op}}}$.

(1) $R\lim_{\mathbf{Ab}}$ has cohomological dimension ≤ 1: $R^i \lim_{\mathbf{Ab}} \mathcal{F}_\bullet = 0$ for $i > 1$.

(2) For $\mathcal{F}_\bullet$ satisfying the Mittag-Leffler condition (for example if $\mathcal{F}_{n+1} \to \mathcal{F}_n$ is surjective for all n), $R\lim_{\mathbf{Ab}} \mathcal{F} \simeq \lim_{\mathbf{Ab}} \mathcal{F}$.

Warning 2.4. The analogue of Fact 2.3 fails for $R\varprojlim_T$ in general, contrary to the claim following [BEK, Notation B.6]. The analogue of Fact 2.3 holds for $R\varprojlim_T$ when T is replete in the sense of [BS, Section 3.1], for example if T is the pro-étale topos.

This gives us a short exact sequence for $\mathcal{F}_\bullet \in \mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}})$ (cf. [BEK, (B.1)])

$$0 \to R^1 \lim_n H^{i-1}(T, \mathcal{F}_n) \to H^i_{\mathrm{cont}}(T, \mathcal{F}_\bullet) \to \varprojlim_n H^i(T, \mathcal{F}_n) \to 0.$$

3 Pro-systems

Definition 3.1. Let $\mathcal{C}$ be a category. The category $\mathcal{C}_{\mathrm{pro}}$ of *pro-systems* in $\mathcal{C}$ is defined as follows. An object of $\mathcal{C}_{\mathrm{pro}}$ is a functor $\mathbb{N}^{\mathrm{op}} \to \mathcal{C}$. For $X_\bullet$, $Y_\bullet$ in $\mathcal{C}_{\mathrm{pro}}$,

$$\mathrm{Hom}_{\mathcal{C}_{\mathrm{pro}}}(X_\bullet, Y_\bullet) := \varprojlim_n \varinjlim_m \mathrm{Hom}_{\mathcal{C}}(X_m, Y_n).$$

Thus a morphism $f \in \mathrm{Hom}_{\mathcal{C}_{\mathrm{pro}}}(X_\bullet, Y_\bullet)$ consists of $f_n \in \varinjlim_m \mathrm{Hom}_{\mathcal{C}}(X_m, Y_n)$.

Lemma 3.2. *There exists a nondecreasing map $\phi\colon \mathbb{N} \to \mathbb{N}$ and $g_n\colon X_{\phi(n)} \to Y_n$ giving rise to f_n such that for all $n \geq m$, the square*

$$\begin{tikzcd} X_{\phi(n)} \arrow[r, "g_n"] \arrow[d] & Y_n \arrow[d] \\ X_{\phi(m)} \arrow[r, "g_m"] & Y_m \end{tikzcd}$$

commutes.

Let $F\colon \mathcal{C}^{\mathbb{N}^{\mathrm{op}}} \to \mathcal{C}_{\mathrm{pro}}$ be the obvious functor. If $\mathcal{C}$ admits sequential limits, then the limit functor $\varprojlim\colon \mathcal{C}^{\mathbb{N}^{\mathrm{op}}} \to \mathcal{C}$ factorizes through F to give $\varprojlim\colon \mathcal{C}_{\mathrm{pro}} \to \mathcal{C}$.

By the lemma, any morphism $f\colon X_\bullet \to Y_\bullet$ in $\mathcal{C}_{\mathrm{pro}}$ can be decomposed into $X_\bullet \simeq X_{\phi(\bullet)} \xrightarrow{F(g_\bullet)} Y_\bullet$. We call $g_\bullet$ a *level representation* of f.

Remark 3.3. The Yoneda embedding $h\colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$ carrying X to $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y)$ can be decomposed into fully faithful functors $\mathcal{C} \to \mathcal{C}_{\mathrm{pro}} \to \mathrm{Pro}(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$, where the first functor carries X to the constant system of value X, and the second functor carries $X_\bullet$ to $Y \mapsto \varinjlim_n \mathrm{Hom}_{\mathcal{C}}(X_n, Y)$. Thus $\mathcal{C}_{\mathrm{pro}}$ can be identified as the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})^{\mathrm{op}}$ spanned by sequential limits of images of h. Recall that the category $\mathrm{Pro}(\mathcal{C})$ of pro-objects of $\mathcal{C}$ is the full subcategory spanned by filtered limits of the image of h.

Definition 3.4. [BEK, Definition A.3] $\mathrm{D}_{\mathrm{pro}}(T) := \mathrm{Ch}(T)_{\mathrm{pro}}[S^{-1}]$, where $\mathrm{Ch}(T) := \mathrm{Ch}(\mathbf{Ab}(T))$ denotes the category of cochain complexes in $\mathbf{Ab}(T)$, and S denotes the class of morphisms represented by levelwise quasi-isomorphisms.

The functor $\mathrm{D}_{\mathrm{pro}}(T) \to \mathrm{D}(T)_{\mathrm{pro}}$ is not an equivalence in general.

Lemma 3.5. [BEK, Lemma A.4] *The triangulated category* $\mathrm{D}_{\mathrm{pro}}(T)$ *has a t-structure* $(\mathrm{D}_{\mathrm{pro}}(T)^{\leq 0}, \mathrm{D}_{\mathrm{pro}}(T)^{\geq 0})$ *with* $\mathcal{F}_\bullet \in \mathrm{D}_{\mathrm{pro}}(T)^{\leq 0}$ *(resp.* $\mathcal{F}_\bullet \in \mathrm{D}_{\mathrm{pro}}(T)^{\geq 0}$*) if and only if* $\mathcal{F}_\bullet \simeq \mathcal{F}'_\bullet$ *in* $\mathrm{D}_{\mathrm{pro}}(T)$ *with* $\mathcal{F}'_n \in \mathrm{D}(T)^{\leq 0}$ *(resp.* $\mathcal{F}'_n \in \mathrm{D}(T)^{\geq 0}$*) for all* $n \in \mathbb{N}$.

Upshot: The diagram (2.1) decomposes into

$$
\begin{tikzcd}
\mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}}) \arrow[r, "\rho"'] \arrow[rr, bend left, "R\varprojlim"] \arrow[d, "R\Gamma^{\mathbb{N}^{\mathrm{op}}}"'] & \mathrm{D}_{\mathrm{pro}}(T) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{D}(T) \arrow[d, "R\Gamma"] \\
\mathrm{D}(\mathbf{Ab}^{\mathbb{N}^{\mathrm{op}}}) \arrow[r] \arrow[rr, bend right, "R\varprojlim"'] & \mathrm{D}_{\mathrm{pro}}(\mathbf{Ab}) \arrow[r, dashed] & \mathrm{D}(\mathbf{Ab})
\end{tikzcd} \tag{3.1}
$$

where the dashed arrows will be constructed using homotopical algebra. The blue arrows are t-exact and induced by the obvious functor $\mathrm{Ch}(\mathcal{A})^{\mathbb{N}^{\mathrm{op}}} \to \mathrm{Ch}(\mathcal{A})_{\mathrm{pro}}$. The black arrows admit left adjoints. For $\mathcal{F}_\bullet \in \mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}})$, it follows from the diagram and adjunctions that

$$\mathrm{Hom}_{\mathrm{D}(T^{\mathbb{N}^{\mathrm{op}}})}(\underline{\mathbb{Z}}, \mathcal{F}_\bullet[i]) \simeq \mathrm{Hom}_{\mathrm{D}_{\mathrm{pro}}(T)}(\underline{\mathbb{Z}}, \rho(\mathcal{F}_\bullet[i])).$$

This leads to the following.

Definition 3.6. [BEK, Definition B.7] Let $\mathcal{F}_\bullet \in \mathrm{D}_{\mathrm{pro}}(T)$. The *continuous (hyper)cohomology* is

$$H^i_{\mathrm{cont}}(T, \mathcal{F}_\bullet) := \mathrm{Hom}_{\mathrm{D}_{\mathrm{pro}}(T)}(\underline{\mathbb{Z}}, \mathcal{F}_\bullet[i]).$$

4 Model categories

Definition 4.1. A *model* category is a category $\mathcal{C}$ equipped with a *model structure*, namely three classes of morphisms (called *fibrations*, *cofibrations*, and *weak equivalences*), such that

(1) ((co)limit) $\mathcal{C}$ admits finite limits and colimits.

(2) (two-out-of-three) Given morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$, if two of $f, g, g \circ f$ are weak equivalences, then so is the third one.

(3) (retract) Fibrations, cofibrations, and weak equivalences are stable under retracts.

(4) (lifting) Let $i\colon A \to B$ be a cofibration and $p\colon X \to Y$ a fibration. Then i has the left lifting property with respect to p if i or p is a weak equivalence.

(5) (factorization) Every morphism in $\mathcal{C}$ admits factorizations $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X \xrightarrow{f'} Y' \xrightarrow{g'} Z$, where f and f' are cofibrations, g and g' are fibrations, g and f' are weak equivalences.

We say that i has the left lifting property with respect to p (or p has the right lifting property with respect to i) if for every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there exists a dashed arrow as indicated making the diagram commutative.

Remark 4.2. Quillen calls the above a *closed* model category [Q, Section I.5], (see Fact 4.4 below). Many authors (Hovey [H2, Section 1.1], Hirschhorn [H1, Section 7.1], Lurie [L1, Section A.2.1]) requires $\mathcal{C}$ to admit *small* limits and colimits. Some authors (Hovey, Hirschhorn, in different ways) require functorial factorizations.

Definition 4.3. A *trivial fibration* is a fibration that is also a weak equivalence. A *trivial cofibration* is a cofibration that is also a weak equivalence.

Fact 4.4 (Closedness)**.** (1) i is a cofibration if and only if it has the left lifting property with respect to trivial fibrations.

(2) i is a trivial cofibration if and only if it has the left lifting property with respect to fibrations.

(3) p is a fibration if and only if it has the right lifting property with respect to trivial cofibrations.

(4) p is a trivial fibration if and only if it has the right lifting property with respect to cofibrations.

In particular, the class of fibrations is determined by that of cofibrations and that of weak equivalences, and the class of cofibrations is determined by that of fibrations and that of weak equivalences.

Definition 4.5. The *homotopy category* is $\mathrm{h}\mathcal{C} := \mathcal{C}[S^{-1}]$, where S denotes the class of weak equivalences.

Fact 4.6. A morphism $X \to Y$ in $\mathcal{C}$ is a weak equivalence if and only if its image in $\mathrm{h}\mathcal{C}$ is an isomorphism.

An object X is said to be *fibrant* if $X \to *$ is a fibration, where $*$ denotes a final object of $\mathcal{C}$. An object X is said to be *cofibrant* if $\emptyset \to X$ is a cofibration, where $\emptyset$ denotes an initial object of $\mathcal{C}$.

Example 4.7. Let $\mathcal{A}$ be a Grothendieck abelian category. The injective model structure [L2, Section 1.3.5] on $\mathrm{Ch}(\mathcal{A})$ is characterized by the following conditions:
(1) The cofibrations are the (degreewise) monomorphisms.
(2) The weak equivalences are the quasi-isomorphisms.
The homotopy category is the derived category $\mathrm{D}(\mathcal{A})$. Under this model structure, every object is cofibrant, and X is fibrant if and only if X^i is injective for all i and X is homotopically injective (namely $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X, Y) = 0$ for every Y acyclic). Contrary to the claim in [BEK, Definition B.1 (b)], this model structure is not simplicial.

Example 4.8. Let $\mathbf{\Delta} = \{[n] \mid n = 0, 1, 2, \dots\}$ be the category of combinatorial simplices, where $[n]$ denotes the ordered set $\{0, \dots, n\}$. The category of simplicial sets is by definition $\mathbf{Set}_\Delta := \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$. The Kan model structure on $\mathbf{Set}_\Delta$ satisfies:
(1) The cofibrations are the monomorphisms.
(2) The weak equivalences are maps $f\colon X \to Y$ such that the geometric realization $|f|\colon |X| \to |Y|$ is a homotopy equivalence of topological spaces.
(3) The fibrations are the maps having the right lifting properties with respect to the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, $0 \le i \le n$, $n \ge 1$. Here Λ^n_i is obtained from Δ^n by removing the interior and the face opposite to the i-th vertex.
(4) The trivial fibrations are the maps having the right lifting properties with respect to $\partial\Delta^n \hookrightarrow \Delta^n$, $n \ge 0$. Here $\partial\Delta^n$ is obtained from Δ^n by removing the interior.

Remark 4.9. Closedness implies that cofibrations and trivial cofibrations are stable under pushout and composition, and that fibrations and trivial fibrations are stable under pullback and composition.

Definition 4.10. We say that a model structure is *left proper* if weak equivalences are stable under pushout by cofibrations, *right proper* if weak equivalences are stable under pullback by fibrations, and *proper* if it is both left and right proper.

The injective model structure on $\mathrm{Ch}(\mathcal{A})$ and the Kan model structure on $\mathbf{Set}_\Delta$ are proper.

Example 4.11. Given a category $\mathcal{C}$ equipped with a proper model structure, Isaksen defined a model structure on $\mathrm{Pro}(\mathcal{C})$ [I, Section 4]. The same argument provides a model structure on $\mathcal{C}_{\mathrm{pro}}$, satisfying:
(1) The cofibrations are the morphisms represented by levelwise cofibrations.
(2) The weak equivalences are the morphisms represented by levelwise weak equivalences.

(3) The fibrations (resp. trivial fibrations) are the retracts of morphisms admitting a level representation $f_\bullet\colon X_\bullet \to Y_\bullet$ such that $f_n\colon X_n \to X_{n-1} \times_{Y_{n-1}} Y_n$ are fibrations (resp. trivial fibrations). Here $X_0 = Y_0$ denotes the final object of $\mathcal{C}$.

For $\mathcal{C} = \mathrm{Ch}(T)$ equipped with the injective model structure, the homotopy category of $\mathrm{Ch}(T)_{\mathrm{pro}}$ is $\mathrm{D}_{\mathrm{pro}}(T)$.

Fact 4.12. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. Given an adjunction (F, G, α) (where $F\colon \mathcal{C} \to \mathcal{D}$ is a left adjoint of $G\colon \mathcal{D} \to \mathcal{C}$), the following conditions are equivalent:
(1) F preserves cofibrations and trivial cofibrations.
(2) G preserves fibrations and trivial fibrations.
(3) F preserves cofibrations and G preserves fibrations.
(4) F preserves trivial cofibrations and G preserves trivial fibrations.

Definition 4.13. An adjunction satisfying the above conditions is called a *Quillen adjunction.*

Fact 4.14. A Quillen adjunction (F, G) induces an adjunction (LF, RG) between homotopy categories. The *left derived functor* $LF\colon \mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ satisfies $LFX \simeq FX$ for X cofibrant. The *right derived functor* $RG\colon \mathrm{h}\mathcal{D} \to \mathrm{h}\mathcal{C}$ satisfies $RGY \simeq GY$ for Y fibrant.

Definition 4.15. We say that (F, G) is a *Quillen equivalence* if LF and RG are equivalences of categories.

We note that this does not imply that F or G is an equivalence of categories.

Example 4.16. Let $f\colon T \to T'$ be a morphism of topoi. Then (f^*, f_*) is a Quillen adjunction between $\mathrm{Ch}(T)$ and $\mathrm{Ch}(T')$, equipped with the injective model structures. Indeed, f^* preserves cofibrations and trivial cofibrations.

Example 4.17. The diagram (3.1) is induced by the diagram

$$\begin{tikzcd}
\mathrm{Ch}(T)^{\mathbb{N}^{\mathrm{op}}} \arrow[r] \arrow[rr, bend left, "\varprojlim"] \arrow[d, "\Gamma^{\mathbb{N}^{\mathrm{op}}}"'] & \mathrm{Ch}(T)_{\mathrm{pro}} \arrow[r, "\varprojlim"'] \arrow[d, "\Gamma_{\mathrm{pro}}"] & \mathrm{Ch}(T) \arrow[d, "\Gamma"] \\
\mathrm{Ch}(\mathbf{Ab})^{\mathbb{N}^{\mathrm{op}}} \arrow[r] \arrow[rr, bend right, "\varprojlim"'] & \mathrm{Ch}(\mathbf{Ab})_{\mathrm{pro}} \arrow[r, "\varprojlim"] & \mathrm{Ch}(\mathbf{Ab})
\end{tikzcd}$$

Here the blue arrows are the obvious functors, and the black arrows are Quillen right adjoints for the model structures described above.

Example 4.18. The Dold-Kan correspondence (see [L2, Section 1.2.3] for a generalization) is an equivalence of categories

$$\mathrm{DK}\colon \mathrm{Ch}(\mathbf{Ab})^{\leq 0} \to \mathbf{Ab}_\Delta,$$

where $\mathrm{Ch}(\mathbf{Ab})^{\leq 0}$ is the full subcategory of $\mathrm{Ch}(\mathbf{Ab})$ spanned by complexes A satisfying $A^i = 0$ for $i > 0$, and $\mathbf{Ab}_\Delta := \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathbf{Ab})$ is the category of simplicial abelian groups. A quasi-inverse is the normalized complex construction N. The functors are defined by

$$\mathrm{DK}_n(A) = \bigoplus_{\alpha\colon [n]\to[k]} A^{-k}, \quad \mathrm{N}^{-n}(X) = \bigcap_{1\le i\le n} \mathrm{Ker}(d_i).$$

The composite functor

$$\mathrm{K}\colon \mathrm{Ch}(\mathbf{Ab}) \xrightarrow{\tau^{\le 0}} \mathrm{Ch}(\mathbf{Ab})^{\le 0} \xrightarrow{\mathrm{DK}} \mathbf{Ab}_\Delta \to \mathbf{Set}_\Delta$$

is a Quillen right adjoint for the injective model structure on $\mathrm{Ch}(T)$ and the Kan model structure on $\mathbf{Set}_\Delta$.

This construction extends to sheaves as follows. Let $\mathbb{S}$ be a small site, namely a small category equipped with a Grothendieck topology. Jardine [J, Seciton 2] defined a global model structure on the category $\mathbf{Set}_\Delta^{\mathbb{S}^{\mathrm{op}}} := \mathrm{Fun}(\mathbb{S}^{\mathrm{op}}, \mathbf{Set}_\Delta)$ of simplicial presheaves, characterized by:

(1) The cofibrations are the monomorphisms.
(2) $f\colon X \to Y$ is a weak equivalence if and only if it induces isomorphisms $\pi_0^{\mathrm{top}}(X) \xrightarrow{\sim} \pi_0^{\mathrm{top}}(Y)$ and

$$\pi_n^{\mathrm{top}}(X|_U, x) \xrightarrow{\sim} \pi_n^{\mathrm{top}}(Y|_U, fx)$$

for $U \in \mathbb{S}$, $x \in X(U)_0$, $n \ge 1$. Here $\pi_0^{\mathrm{top}}(X)$ is the sheaf associated to the presheaf $U \mapsto \pi_0(|U|)$ and $\pi_n^{\mathrm{top}}(X|_U, x)$ is the sheaf on $\mathbb{S}_{/U}$ associated to the presheaf $V \mapsto \pi_n(|V|, x_V)$.

The composite functor

$$\mathrm{K}\colon \mathrm{Ch}(\mathbb{S}^\sim) \to \mathrm{Ch}(\mathbf{Ab})^{\mathbb{S}^{\mathrm{op}}} \xrightarrow{\mathrm{K}^{\mathbb{S}^{\mathrm{op}}}} \mathbf{Set}_\Delta^{\mathbb{S}^{\mathrm{op}}}$$

is a Quillen right adjoint for the injective model structure on $\mathrm{Ch}(\mathbb{S}^\sim)$ and the global model structure on $\mathbf{Set}_\Delta^{\mathbb{S}^{\mathrm{op}}}$. Here $\mathbb{S}^\sim$ denote the topos of sheaves on $\mathbb{S}$.

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