Chern classes in crystalline cohomology

§1: Grothendieck's theory of Chern classes

- k field
- Sm/k : category of smooth k-schemes
- $\mathcal{V} \subseteq Sm/k$ full subcategory, s.t. $\forall X \in \mathcal{V}$, E: v.b. on X

$$\mathbb{P}(E) := \mathrm{Proj}(\mathrm{Sym}^* E^\vee) \in \mathcal{V}$$

Assume that we have the following data:

anticommutative: $xy = (-1)^{\deg x \deg y} yx$ for x, y homogeneous elements

a) a contravariant functor

$$\mathcal{V} \longrightarrow \left(\begin{matrix}\text{graded anticommutative} \\ \text{unitary rings}\end{matrix}\right)$$

$$X \longmapsto A(X) = \bigoplus_{p \in \mathbb{N}} A^p(X)$$

$$(Y \xrightarrow{f} X) \longmapsto (A(X) \xrightarrow{f^*} A(Y))$$

b) a functorial morphism of groups

$$\tilde{c}_1 : \mathrm{Pic}(X) \longrightarrow A^2(X)$$

c) for $Y \overset{i}{\hookrightarrow} X$ a closed immersion in $\mathcal{V}$ of codim c, a morphism of graded abelian groups (= Gysin map)

$$i_* : A(Y) \longrightarrow A(X)[2c]$$

Notation: $\quad cl_X(Y) := i_*(1_{A(Y)}) \in A^{2c}(X)$

assume that these data verify the following axioms

(A1) let $X \in \mathcal{V}$, E a v.b. $\overset{\text{of rk } r}{\text{on}}$ X, $P := \mathbb{P}(E) \xrightarrow{f} X$

let $\xi_E := \tilde{c}_1(\mathcal{O}_P(1)) \in A^2(P)$

then $A(P)$ is a free $\overset{\text{left}}{A(X)}$-module (relative to

$f^*: A(X) \to A(Y)$), with a basis given by

$1, \xi_E, \cdots, \xi_E^{r-1}$

(A2): let $X \in \mathcal{V}$, $L = \mathcal{O}_X(D)$ a line bundle with $D \hookrightarrow X$

a ~~non-trivial~~ smooth effective divisor. Then

$$\tilde{c}_1(\mathcal{O}_X(D)) = cl_X(D)$$

if $D = \phi$, we have obviously $\tilde{c}_1(\mathcal{O}_X) = cl_X(\phi) = 0 \in A^2(X)$

(A3): let $Z \overset{i}{\hookrightarrow} Y \overset{i'}{\hookrightarrow} X$ be two closed immersions in $\mathcal{V}$, then

$$(i' \circ i)_* = i'_* \circ i_*$$

(A4) Projection formule: $\forall\ Y \overset{i}{\hookrightarrow} X$ closed immersion in $\mathcal{V}$,

then $i_*(b\, i^*(a)) = (i_* b) \cdot a \quad \forall\ a \in A(X), b \in A(Y)$

one deduces immediately two consequences from these axioms

<u>lemma 1</u>: let $X \in \mathcal{V}$, E a v.b. of rk r on X. let $Fl(E)$ be the flag variety.

$$Fl(E) = \{ 0 = E_r \subset E_{r-1} \subset \cdots \subset E_1 \subset E_0 = E \mid E_{i-1}/E_i \text{ line bundles} \}$$

$$\downarrow f$$

$$X$$

then: $f^* : A(X) \longrightarrow A(Fl(E))$ is injective

<u>Pf</u>: one uses (A1).

<u>lemma 2</u>: let $X \in \mathcal{V}$, E a v.b. of rk r on X. $s \in E(X)$ a global section.

let $0 = E_r \subset E_{r-1} \subset \cdots \subset E_1 \subset E_0 = E$

be a filtration of E with E_{i-1}/E_i line bundles $(1 \le i \le r)$

for $1 \le i \le r$, let Y_i be the locus of X where $s \in E_i \subseteq E$.

Assume $Y_i \in \mathcal{V}$, and the section s_i of $\left(E_i/E_{i+1}\right)\big|_{Y_i}$ induced from s

is transversal to the zero section. (so $\left(E_i/E_{i+1}\right)\big|_{Y_i} = \mathcal{O}_{Y_i}(Y_{i+1})$)

let $\xi_i = \tilde{c}_1\left(E_{i-1}/E_i\right)$ $\quad 1 \le i \le r$

<u>then</u> $$Cl_X(Y_r) = \prod_{1 \le i \le r} \xi_i \in A^{2r}(X).$$

<u>Pf</u>: We shall prove by induction that

$$(*) \qquad Cl_X(Y_j) = \prod_{1 \le i \le j} \xi_i \qquad 1 \le j \le r$$

$\boxed{j=1}$ i.e. $cl_X(Y_1) = \tilde{c}_1(E_0/E_1)$?

in fact, $E_0/E_1 = \mathcal{O}_X(Y_1)$, so our assertion follows from (A1).

$\boxed{\text{Induction}}$ assume that we have proved. $(*)$ for j $(1 \le j < r)$

we claim that $(*)$ holds for $j+1$:

consider the line bundle $L_j := (E_j/E_{j+1})|_{Y_j} = u_j^*(E_j/E_{j+1})$

then $L_j = \mathcal{O}_{Y_j}(Y_{j+1})$

$$\begin{tikzcd} Y_j \arrow[r, hook, "u_j"] & X \\ Y_{j+1} \arrow[u, hook, "\tau"] \arrow[ur, "u_{j+1}"'] & \end{tikzcd}$$

so (A2) $\Rightarrow$ $\tilde{c}_1(L_j) = cl_{Y_j}(Y_{j+1}) \in A^2(Y_j)$

but functoriality $\Rightarrow$ $\tilde{c}_1(L_j) = u_j^* \xi_{j+1} \in A^2(Y_j)$

& (A3) $\Rightarrow$ $cl_X(Y_{j+1}) = u_{j*}(cl_{Y_j}(Y_{j+1}))$

$$= u_{j*}\, u_j^*\, \xi_{j+1}$$

$$\overset{(A4)}{=} u_{j*}(1_{A(Y_j)}) \cdot \xi_{j+1}$$

$$= cl_X(Y_j) \cdot \xi_{j+1}$$

$$\overset{\text{Induction}}{=} \prod_{i=1}^{j+1} \xi_i$$

$\square$.

<u>Cor</u>: under the condition of the previous lemmas and assume that $s \in E(X)$ is everywhere non-zero (so $Y_r = \phi$), then $\prod_{i=1}^{r} \xi_i = 0$

Examples:

(1) $\mathcal{V} = \left\{ \begin{array}{l} \text{smooth } k\text{-schemes} \\ \text{of finite type} \end{array} \right\}$

- for $X \in \mathcal{V}$, $A(X) := \bigoplus_{p \in \mathbb{N}} \underbrace{\bigoplus_{a+b=p} H^a(X, \Omega_X^b)}_{A^p(X)}$ (Hodge cohomology)

- $\tilde{c}_1 : \mathrm{Pic}(X) \longrightarrow H^1(X, \Omega_X^1) \subseteq A^2(X)$

induced by $\quad d\log : \mathcal{O}_X^\times \longrightarrow \Omega_X^1$

$\qquad f \longmapsto \frac{df}{f}$

- $\forall\ Y \xhookrightarrow{i} X$ closed immersion in $\mathcal{V}$, of codim. c

$$\begin{array}{ccc}
H^a(Y, \Omega_Y^b) & \xrightarrow{\quad i_* \quad} & H^{a+c}(X, \Omega_X^{b+c}) \\
\wr\wr & & \\
H^a(X, i_* R\underline{\mathcal{H}om}(\Omega_Y^{d_Y-b}, \Omega_Y^{d_Y})) & & \\
\wr\wr & & \\
H^a(X, i_* R\underline{\mathcal{H}om}(\Omega_Y^{d_Y-b}, i^! \Omega_X^{d_X})[c]) & & \\
\wr\wr & & \\
H^{a+c}(X, R\underline{\mathcal{H}om}(i_*\Omega_Y^{d_Y-b}, \Omega_X^{d_X})) & \longrightarrow & H^{a+c}(X, R\underline{\mathcal{H}om}(\Omega_X^{d_Y-b}, \Omega_X^{d_X}))
\end{array}$$

induced from

$i^*\Omega_X^1 \to \Omega_Y^1$

or equiv.

$\Omega_X^1 \to i_*\Omega_Y^1$

One checks that the axioms (A1)–(A4) hold.

(2) $\mathcal{V} = \left\{ \begin{array}{l} \text{smooth } k\text{-schemes} \\ \text{(quasi-projective)} \end{array} \right\}$

- for $X \in \mathcal{V}$, set $A(X) = \bigoplus\limits_{p \geq 0} A^p(X)$ with

$$A^p(X) = \begin{cases} 0 & p \text{ odd} \\ CH^{p/2}_{rat}(X) & p \text{ even} \end{cases}$$

intersection theory $\Rightarrow$ $A(X)$ unitary commutative ring

- $\forall\, f: Y \to X$, $\xrightarrow{\text{moving lemma}}$ $f^*: A(X) \to A(Y)$

- $\tilde{c}_1 : \mathrm{Pic}(X) \xrightarrow{\sim} A^2(X) = CH^1(X)$

$\qquad \mathcal{O}_X(D) \longmapsto [D]$

- $i: Y \hookrightarrow X$ in $\mathcal{V}$, i_* is pushforward of cycles.

one checks that the axioms (A1) ~ (A4) hold.

now, assume we are given the data $\mathcal{V}, \tilde{c}_1$, & Gysin maps verifying (A1) - (A4)

let $X \in \mathcal{V}$, E: v.b. of rk r on X, $P = \mathbb{P}(E) \xrightarrow{f} X$, $\xi_E = \tilde{c}_1(\mathcal{O}_P(1))$

(A1) $\Rightarrow$ $A(P) = \bigoplus\limits_{i=0}^{r-1} A(X) \cdot \xi_E^i$ is a free $A(X)$-module

consider $\xi_E^r \in A^{2r}(P)$ $\Rightarrow$ $\exists!$ $c_i(E) \in A^{2i}(X)$ with

$$\begin{cases} \sum_{i=0}^{r} c_i(E)\cdot \xi_E^{r-i} = 0. \\ c_0(E) = 0 \qquad c_i(E)=0 \quad \forall i > r \end{cases}$$

<u>Def</u>: $C(E) = \bigoplus_{i=0}^{\infty} c_i(E) \in A(X)$ <u>total chern class</u> of E

$c_i(E)$: the <u>i-th chern class</u> of E

<u>Theorem</u> (Grothendieck) the chern classes constructed above satisfy the following properties:

(1) functoriality: $\forall\ Y \xrightarrow{f} X$ in $\mathcal{V}$, we have $c(f^*(E)) = f^* c(E)$

(2) normalization: if E is a line bundle on $X \in \mathcal{V}$, we have

$$c(E) = 1 + \tilde{c}_1(E) \quad \text{i.e.} \quad c_1(E) = \tilde{c}_1(E).$$

(3) additivity: $\forall\ X \in \mathcal{V}$, and any short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0$$

we have $c(E) = c(E')c(E'')$

Moreover, the properties (1) ~ (3) above characterize the Chern classes.

<u>Pf</u>: "uniqueness", "functoriality" "normalization" are easy to check.

$\boxed{\text{additivity}}$ one reduces to the following statement: (by splitting principal)

let E be a v. b. of rk r on $X \in \mathcal{V}$, together with a filtration

$$0 = E_r \subset E_{r-1} \subset \cdots \subset E_1 \subset E_0 = E$$

such that E_{i-1}/E_i are line bundles $1 \le i \le r$

then $c(E) = \prod_{i=1}^{r} c(E_{i-1}/E_i)$

let $X' = \mathbb{P}(E) \xrightarrow{f} X$

$E' := f^* E, \quad E_i' := f^* E_i, \quad \zeta_i := c_1(E_{i-1}/E_i), \quad \zeta_i' = c_1(E_{i-1}'/E_i') = f^*\zeta_i$

by the construction of $\mathbb{P}(E) = X'$, $\rightsquigarrow \exists$ canonical surjection

$$f^* E^\vee \longrightarrow \mathcal{O}_{X'}(1)$$

$$\Rightarrow \quad \mathcal{O}_{X'}(-1) \hookrightarrow f^* E = E'$$

$\rightsquigarrow$ get a global section s of $E'(1) =: F'$

set $F_i' := E_i'(1) \rightsquigarrow 0 = F_r' \subset F_{r-1}' \subset \cdots \subseteq F_1' \subset F_0' = F'$

and $c_1(F_{i-1}'/F_i') = c_1((E_{i-1}'/E_i')(1)) = f^*\zeta_i + \zeta_E$

let Y_i be the locus of $x \in X'$ such that $s \in F_i' \subseteq F_i$

$=$ ―――――――――――― $\mathcal{O}_P(-1) \subseteq E_i' \subset E'$

so $Y_i = \mathbb{P}(E_i') \subseteq \mathbb{P}(E) = X'$, and $Y_i \in \mathcal{V}$

<u>Claim</u>: let s_i be the section of $(F_{i-1}'/F_i')\big|_{Y_{i-1}}$ induced from s, then s_i is transversal to the zero section of F_{i-1}'/F_i'

<u>Pf of claim</u>: this is a local question on X.

$\rightsquigarrow$ We may assume $E = \mathcal{O}_X^r$, and the filtration $E_\bullet$ of E is the standard one

(so E_i is the sum of the first $r-i$ factors.)

$\Rightarrow$ the map $E' \to \mathcal{O}_P(1)$ becomes

$\mathcal{O}_{X'}^r \longrightarrow \mathcal{O}_{X'}(1)$, and for each i,

the composite $\mathcal{O}_{X'} \longrightarrow \mathcal{O}_{X'}^r \longrightarrow \mathcal{O}_{X'}(1)$ is non-zero.

$$x \longmapsto (\underset{\substack{\uparrow \\ i\text{th}}}{0, \cdots 0, x, 0 \cdots 0})$$

$$\Rightarrow \quad \left(F'_{i-1}/F'_i\right)\Big|_{Y_{i-1}} \cong \mathcal{O}_{Y_{i-1}}(1)$$

and each s_i is non-zero section of $\left(F'_{i-1}/F'_i\right)\Big|_{Y_{i-1}} = \mathcal{O}_{Y_{i-1}}(1)$

hence s_i is transversal to the zero section of $\left(F'_{i-1}/F'_i\right)\Big|_{Y_{i-1}}$

(as this is the case for any non-zero section of $\mathcal{O}_{\mathbb{P}^N_k}(1)$)

$\Rightarrow$ claim.

As $\mathcal{O}_{X'}(-1) \longrightarrow E'$ is a subbundle

$$\Rightarrow \quad Y_r = \emptyset$$

Cor to lemma 2.

$$\Rightarrow \quad 0 = \prod_{i=1}^{r} c_1\left(F'_{i-1}/F'_i\right) = \prod_{i=1}^{r} (\xi_i + \xi_E)$$

So, $c_i(E')$ is the i-th elementary function of the ξ'_j's

$$\Rightarrow \quad c(E') = \prod_{i=1}^{r} (1+\xi'_i) = \prod_{i=1}^{r} c\left(E'_{i-1}/E'_i\right)$$

$$\Rightarrow \quad c(E) = \prod_{i=1}^{n} c\left(E_{i-1}/E_i\right)$$

$\square$

<u>Application</u>:

① chern classes for Hodge cohomology

② chern classes for chow group:

for E v.b. on smooth X/k. define its <u>chern character</u> to be

$$ch(E) := \sum_{i=1}^{\infty} \exp(c_i(E)) \in CH^*(X)_{\mathbb{Q}}$$

$\rightsquigarrow \quad ch(E) = ch(E') + ch(E'')$ whenever we have a short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0$$

$\rightsquigarrow$ additive map $ch: K_0(X)_{\mathbb{Q}} \longrightarrow CH^*(X)_{\mathbb{Q}}$.

<u>Fact</u>: this is bijective

$\square$

§2 Chern classes in crystalline cohomology (after Berthelot-Illusie)

- (S, I, γ) : PD-scheme
- $X \xrightarrow{f} S$ morphism of schemes
- $\mathrm{Cris}(X/S, I, \gamma)$: objects: triples (U, T, J) where
  - $U \overset{\circ}{\hookrightarrow} X$
  - $U \hookrightarrow T$ closed immersion of S-schemes, with defining ideal J

  Such that:

  * J is a PD-ideal, with PD-structure compatible with γ
  * $\exists\, m, q \in \mathbb{Z}_{>0}$, with $m \cdot J^{[q]} = 0$
- $\mathcal{O}_{X/S}$ structure sheaf of $\mathrm{Cris}(X/S, I, \gamma)$
- $J_{X/S} \subseteq \mathcal{O}_{X/S}$ defined by

$$0 \to J_{X/S} \to \mathcal{O}_{X/S} \to i_{X/S *}\mathcal{O}_X \to 0$$

multiplicative analogue of this sequence:

$$0 \to 1 + J_{X/S} \to \mathcal{O}_{X/S}^* \to i_{X/S *}\mathcal{O}_X^* \to 0$$

$$\Rightarrow \quad \begin{array}{ccc} H^1(X, \mathcal{O}_X^*) = H^1(X/S, i_{X/S *}\mathcal{O}_X^*) & \longrightarrow & H^2(X/S, 1 + J_{X/S}) \\ & \searrow^{\tilde{c}_1} & \downarrow \\ & & H^2(X/S, J_{X/S}) \end{array}$$

induced by $\log : 1 + J_{X/S} \to J_{X/S}$ sending $1+x$ to $\sum_{n=1}^{\infty} (-1)^{n-1}(n-1)!\, x^{[n]}$

first chern class of line bundles

<u>Theorem</u> (Berthelot - Illusie) there exists a theory of chern classes attached to all vector bundles E of finite rank on X,

$$c_i(E) \in H^{2i}(X/S, J_{X/S}^{[i]}) \qquad (i=0,1,\dots rk(E))$$

satisfying the following conditions

- functoriality: $\forall$ $\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & \circlearrowright & \downarrow \\ (S', I', \gamma') & \to & (S, I, \gamma) \end{array}$, we have $f^* c_i(E) = c_i(f^* E)$

- normalization: if E = line bundle, $c_0(E) = 1$, $c_1(E) = \tilde{c}_1(E)$.

- additivity: for all short exact sequence of vector bundles:

$$0 \to E' \to E \to E'' \to 0$$

we have $\quad C(E) = C(E') \cdot C(E'')$

where $\quad C(E) = \sum_{i=0}^{rk(E)} C_i(E)$ is the <u>total chern class</u>.

In order to prove this results, one needs the analogue of axiom (A1) in this setting:

<u>lemma</u>: let E be a vector bundle of rk r on X, $P = \mathbb{P}(E) \xrightarrow{f} X$

let $\xi_E = \tilde{c}_1(\mathcal{O}_P(1)) \in H^2(P, J_{P/S}^{[1]})$

the cup product gives

$$\cdot \xi_E^n : J_{X/S}^{[k-n]}[-2n] \longrightarrow Rf_{cris*} J_{P/S}^{[k]}$$

giving a map

$$\bigoplus_{i=0}^{r-1} J_{X/S}^{[k-i]}[-2i] \longrightarrow Rf_{cris*} J_{P/S}^{[k]}$$

Then, this is an isomorphism in the derived category.

Cor: $$H^n(P/S, J_{P/S}^{[k]}) \xleftarrow[\Sigma \cdot \xi_E^i]{\sim} \bigoplus_{i=0}^{r-1} H^{n-2i}(X/S, J_{X/S}^{[k-i]})$$

take $n=2r$, $k=r$ $\Rightarrow$ $\exists!$ $C_i(E) \in H^{2i}(X/S, J_{X/S}^{[i]})$

with $$\xi_E^r = -\sum_{i=0}^{r-1} C_{r-i}(E) \cdot \xi_E^i$$

set $C_0(E)=1$, $C_i(E)=0$ if $i>r$

$$C(E) := \sum_{i=0}^{\infty} C_i(E) \in \bigoplus_{i=0}^{\infty} H^{2i}(X/S, J_{X/S}^{[i]})$$

- the "uniqueness", "functoriality" & "normalization" can be shown as before. (in Grothendieck's theory)
- for the additivity, it seems that for the lack of a good duality theory in crystalline cohomology, one can not argue as in Grothendieck's theory.

personal point of view!

In the paper of Berthelot - Illusie, they used a different approach.

We omit the detail here.

§3: Review of de Rham - Witt complex

① definitions

- X topos
- $\mathbb{Z}$ - dga on X = $\mathbb{Z}$ - differential graded algebra on X

  = graded $\mathbb{Z}$-algebra $M = \bigoplus_{i=0}^{\infty} M^i$, endowed with an additive map of degree 1

$$d : M^i \longrightarrow M^{i+1}$$

such that

- M is strictly anticommutative: for $x, y \in M$ homogeneous

$$x \cdot y = (-1)^{\deg x \deg y} y x$$

and $x^2 = 0$ if x is moreover of odd degree.

- $d(xy) = dx\, y + (-1)^{\deg x} x\, dy$

  for $x, y \in M$ homogeneous,

  and $d^2 = 0$

Example: A unitary commutative ring on X $\rightsquigarrow$ $\Omega^{\bullet}_A$ is a $\mathbb{Z}$-dga.

Moreover, for M a $\mathbb{Z}$-dga, we have natural bijection

$$\mathrm{Hom}_{\mathrm{ring}(X)}(A, M^0) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}\text{-dga}(X)}(\Omega^{\bullet}_A, M)$$

<u>Def</u> a <u>V-pro-DR complex</u> on X = a projective system $M_\cdot = ((M_n)_{n\geq 1}, M_{n+1} \xrightarrow{R} M_n)$ of $\mathbb{Z}$-dga on X + additive map of degree 0

$$V : M_n \longrightarrow M_{n+1}$$

such that $RV = VR$. Moreover the following conditions hold:

<u>(V1)</u>: M_1° = $\mathbb{F}_p$-algebra on X

$M_n^\circ = W_n(M_1^\circ)$, & $R : M_{n+1}^\circ \to M_n^\circ$ & $V : M_n^\circ \to M_{n+1}^\circ$ are the natural maps in the theory of Witt rings

<u>(V2)</u>: $V(x\,dy) = V(x)\,dV(y)$, with $x, y \in M_n$ homogeneous.

<u>(V3)</u>: $(Vy)\,d\underline{x} = V(\underline{x}^{p-1} y)\,dV\underline{x}$, for any $x \in M_1^\circ$, $y \in M_n^\circ$

$\underline{x}$:= Teichmüller lift of x. □

$\rightsquigarrow$ $VDR(X)$ = category of V-pro-DR complexes

$\rightsquigarrow$ forgetful functor

$$VDR(X) \longrightarrow \mathbb{F}_p\text{-alg}(X)$$
$$M_\cdot \longmapsto M_1^\circ$$

<u>Thm</u>: the forgetful functor above has a left adjoint

$$\mathbb{F}_p\text{-alg}(X) \ni A \longmapsto W_\cdot\Omega_A^\cdot \in VDR(X)$$

Moreover the canonical map of $\mathbb{Z}$-dga

$$\pi_n : \Omega^\cdot_{W_n(A)} \longrightarrow W_n\Omega_A^\cdot$$

is surjective, and

$$\pi_1 : \Omega^\cdot_{W_1(A)} = \Omega_A^\cdot \longrightarrow W_1\Omega_A^\cdot$$

is an isomorphism.

② Frobenius on $W.\Omega_A^\bullet$

$W.\Omega_A^\bullet$ depends functorially on A: $\forall\ A \xrightarrow{f} B \Rightarrow W.\Omega_A^\bullet \to W.\Omega_B^\bullet$.

$\Rightarrow$ the Frobenius $\begin{array}{ccc} A & \longrightarrow & A \\ x & \longmapsto & x^p \end{array}$ induces a morphism in $VDR(X)$:

$$\underline{F} : W.\Omega_A^\bullet \longrightarrow W.\Omega_A^\bullet$$

Fact: $\underline{F} : W.\Omega_A^i \to W.\Omega_A^i$ is p^i-divisible.

More precisely, we have

Theorem: the morphism of projective systems

$$\begin{tikzcd} W.(A) \arrow[rr] \arrow[dr, "\underline{F}"'] & & W_{\cdot-1}(A) \\ & W.(A) \arrow[ur, "R"'] & \end{tikzcd}$$

extends uniquely to a morphism of projective system of graded algebras

$$F : W.\Omega_A^\bullet \longrightarrow W_{\cdot-1}\Omega_A^\bullet$$

such that

(i) $F\, d\underline{x} = \underline{x}^{p-1}\, d\underline{x}$, $\forall\ \underline{x} \in A$

(ii) $FdV = d : W.A \to W.\Omega_A^1$

moreover, we have

$$FV = VF = p,\quad FdV = d \quad (\Rightarrow dF = pFd,\ Vd = pdV)$$

$$x \cdot Vy = V(Fx \cdot y) \quad \forall\ x, y \text{ homogeneous elements}$$

③ DRW of schemes

X scheme of characteristic p, consider $\widetilde{X_{Zar}}$ (= the topos X in previous §).

$\mathcal{O}_X$: $\mathbb{F}_p$-algebra in $\widetilde{X_{Zar}}$

Def. $W.\Omega^{\bullet}_X := W.\Omega^{\bullet}_{\mathcal{O}_X}$ pro-de Rham-Witt complex of X

$W\Omega^{\bullet}_X := \varprojlim_n W_n\Omega^{\bullet}_X$ de Rham-Witt complex of X

- functoriality: $Y \xrightarrow{f} X$ morphism of $\mathbb{F}_p$-schemes $\rightsquigarrow$ canonical maps

$$f^{-1}W.\Omega^{\bullet}_X \longrightarrow W.\Omega^{\bullet}_Y \quad \& \quad W.\Omega^{\bullet}_X \longrightarrow f_* W.\Omega^{\bullet}_Y$$

- $\forall x \in X$, $\left(W_n\Omega^{\bullet}_X\right)_x = W_n\Omega^{\bullet}_{\mathcal{O}_{X,x}}$

- for $X/\mathbb{F}_p$ scheme, $(|X|, W_n(\mathcal{O}_X))$ is again a scheme, denoted by $W_n(X)$

$\rightsquigarrow$ $W_n\Omega^i_X$ quasi-coherent sheaf on $W_n(X)$.

Moreover, for any $U = \operatorname{Spec}(A) \hookrightarrow X$ affine, $\Gamma(U, W_n\Omega^i_X) = W_n\Omega^i_A$

- if
$$\begin{tikzcd} Y \arrow[rr, "f", "\text{étale}"'] \arrow[dr] & & X \arrow[dl] \\ & \operatorname{Spec}\mathbb{F}_p & \end{tikzcd}$$
$\rightsquigarrow$ $W_n(f) : W_n(Y) \longrightarrow W_n(X)$ étale

moreover, $W_n(f)^* W_n\Omega^i_X \xrightarrow{\sim} W_n\Omega^i_Y$

- $X = \operatorname{Spec} k \longrightarrow \operatorname{Spec}\mathbb{F}_p$ with k perfect. $\Rightarrow$ the Frobenius on $W_n(k)$ is bijective

$\Rightarrow$ $\Omega^i_{W_n(k)} = 0$ $\forall i \geq 1$ $\Rightarrow$ $W_n\Omega^{\bullet}_{\operatorname{Spec} k} = W_n\Omega^0_{\operatorname{Spec} k} = W_n(k)$

In particular, for <u>any</u> k-scheme X, $W_n\Omega^i_X$ is naturally a W_n-module

④ An example

- $X = \mathbb{G}_{m,k}^{d} = \operatorname{Spec} \overbrace{k[T_1^{\pm 1}, \cdots T_d^{\pm 1}]}^{A}$ k perfect ring of char. $p > 0$

- $W = W(k)$ $K = W[\frac{1}{p}]$

- $B = W[T_1^{\pm 1}, \cdots T_d^{\pm 1}]$

  $C = k[T_1^{\pm 1/p^\infty}, \cdots, T_d^{\pm 1/p^\infty}] = \bigcup_{r \geq 0} k[T_1^{\pm 1/p^r} \cdots T_d^{\pm 1/p^r}]$

- $\Omega_C^m = \bigoplus_{1 \leq i_1 < \cdots < i_m \leq d} C \cdot d\log T_{i_1} \cdots d\log T_{i_m}$

- Def: call $x \in \Omega_C^m$ integral, if $x \in \bigoplus_{1 \leq i_1 < \cdots < i_m = d} W[T_1^{\pm 1/p^\infty}, \cdots T_d^{\pm 1/p^\infty}] \, d\log T_{i_1} \cdots d\log T_{i_m}$

  define $E_A^m := \{ x \in \Omega_C^m \mid x \,\&\, dx \text{ are integral.}$

  Example | $\omega = T_1^{1/p}$ is integral, but $d(T_1^{1/p}) = \frac{1}{p} T_1^{\frac{1-p}{p}} dT_1 = \frac{1}{p} T_1^{\frac{1}{p}} d\log T_1$
  so $d\omega$ is not integral. but $p\omega \in E_A^0$.

  $\rightsquigarrow$ inclusions of $\mathbb{Z}$-dga's:

$$\Omega_B^\bullet \subset E_A^\bullet \subset \Omega_C^\bullet$$

- consider the Frobenius σ on C, which is induced from the Frobenius on C,
  such that $T_i \longmapsto T_i^p$: this is an automorphism of C

  σ induces a morphism of $\mathbb{Z}$-dga $\underline{\sigma}: \Omega_C^\bullet \longrightarrow \Omega_C^\bullet$

  in particular, $\underline{\sigma}(d\log T_i) = d\log T_i^p = p\, d\log T_i$

  let $F: \Omega_C^\bullet \to \Omega_C^\bullet$ such that $F(\omega) = \frac{\underline{\sigma}(\omega)}{p^i}$ if $\omega \in \Omega_C^i$

  $\longrightarrow$ a graded morphism, which is an isomorphism.

$$F: \Omega_C^\bullet \longrightarrow \Omega_C^\bullet$$

Set $\quad V = p \cdot F^{-1}$

<u>Fact</u>: $E_A^m \subseteq \Omega_C^m$ stable under F and V. Moreover

$$\begin{cases} x \cdot Vy = V(Fx \cdot y) \\ V(x\,dy) = Vx \cdot dVy \end{cases}$$

- $\exists$ commutative diagram.

$$\begin{tikzcd}
W[T_1^{\pm 1}, \cdots T_d^{\pm 1}] \arrow[rr, "T_i \mapsto \underline{T_i}"] \arrow[d] & & W(A) \arrow[dd] \\
E_A^\circ \arrow[urr, dashed, "\varphi"] \arrow[d] & & \\
W[T_1^{\pm 1/p^\infty}, \dots T_d^{\pm 1/p^\infty}] \arrow[rr] & & W(k[T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}])
\end{tikzcd}$$

<u>Fact</u>: $\exists$ dotted map, which is compatible with V

$\left(\text{for example,} \quad p\,T_1^{1/p} \in E_A^\circ, \text{ and } \quad p \cdot \underline{T_1}^{1/p} = p \cdot (T_1^{1/p}, 0, 0 \cdots) = (0, T_1, 0, \cdots) \in W(A) \right)$

φ gives $\quad E_A^\circ / V^r E_A^\circ \xrightarrow{\sim} W(A) / V^r W(A) = W_r(A)$

Let $\quad \mathrm{Fil}^r E_A^i := V^r E^i + d V^r E^{i-1}$

Fact: $\bigoplus_{r=0}^{\infty} \mathrm{Fil}^r E_A^i \subseteq E_A^\bullet$ is a graded differential ideal

$\Rightarrow$ $E_r^\bullet := E_A^\bullet / \mathrm{Fil}^r E_A^\bullet$ is a $\mathbb{Z}$-dga

$\Rightarrow$ $E_\bullet^\bullet = (E_r^\bullet)_{r \geq 1}$ together with $V: E_\bullet^\bullet \to E_{\bullet-1}^\bullet$ is a pro-V DR complex

$\rightarrow$ canonical map

$$W_\bullet \Omega_A^\bullet \longrightarrow E_\bullet$$

Theorem (Deligne) the canonical map above is an iso.

⑤ structure of DRW for X smooth over a perfect base.

- S perfect scheme of char $p > 0$, $X \to S$ smooth

Prop: a) $\exists$ commutative diagram

$$\begin{tikzcd} W_{n+1}\Omega_X^i \arrow[r, "F"] \arrow[d, "\text{canonical}"'] \arrow[dr, phantom, "\circlearrowright"] & W_n \Omega_X^i \arrow[d, "\text{canonical}"] \\ \Omega_X^i \arrow[r, "C^{-1}"] & \Omega_X^i / d\Omega_X^{i-1} \end{tikzcd}$$

b) the endomorphisms p, F, V of the pro-object $W_\bullet \Omega_X^\bullet$ are injective. in particular $W\Omega_X^\bullet = \varprojlim_n W_n \Omega_X^\bullet$ has no p-torsion

(c) if X has relative dim. N, then $W.\Omega_X^i = 0$ for $i > N$

(d) $W_n\Omega_X^i$ is coherent over W_nX

<u>Prop</u> (fixed pts of F) $\exists$ short exact sequences in $\tilde{X_{\acute{e}t}}$

(1) $0 \to (\mathbb{Z}/p^n\mathbb{Z})_X \longrightarrow W_n(\mathcal{O}_X) \xrightarrow{1-F} W_n(\mathcal{O}_X) \to 0$

(this is classical)

(2) exact sequence of pro-sheaves

$$0 \to \mathcal{O}_X^* / \mathcal{O}_X^{*p^\cdot} \xrightarrow{d\log} W.\Omega_X^1 \xrightarrow{1-F} W.\Omega_X^1 \to 0$$

⚠ this does <u>not</u> mean that the sequence below is exact

$$0 \to \mathcal{O}_X^* / \mathcal{O}_X^{*p^n} \xrightarrow{d\log} W_n\Omega_X^1 \xrightarrow{1-F} W_{n-1}\Omega_X^1 \to 0 .$$

⑥ <u>Comparison theorem</u>

k perfect field of char. $p > 0$

$W = W(k)$

$X \longrightarrow \operatorname{Spec} k$ of finite type

$(X/W_n)^\sim_{\mathrm{crys}}$ crystalline topos, $u = u_{X/W_n} : (X/W_n)^\sim_{\mathrm{crys}} \to \tilde{X}_{\mathrm{Zar}}$

the natural morphism of topoi

Fact. $\exists$ canonical map in $D(f^{-1}(\mathcal{O}_{\operatorname{Spec} k}))$

$$R u_{X/W_n *} \mathcal{O}_{X/S} \longrightarrow W_n \Omega_X^{\bullet} \qquad (*)$$

We shall construct $(*)$ in the following situation: X/k smooth.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook] & \operatorname{Spf} W \end{tikzcd}$$

$\leftarrow$ smooth

$X = \mathfrak{X} \otimes_W k$, and the Frobenius of X lifts to $F: \mathfrak{X} \to \mathfrak{X}$, compatible with the Frobenius on W.

Let $X_n = \mathfrak{X} \otimes_W W_n$

$$\Rightarrow \quad R u_* \mathcal{O}_{X/S} = \Omega^{\bullet}_{X_n/W_n}.$$

on the other hand, F gives a morphism of sheaves of rings $\mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}}$

($|F|: |\mathfrak{X}| = |X| \to |\mathfrak{X}| = |X|$ is identity)

$\rightsquigarrow \exists$ morphism $\mathcal{O}_{\mathfrak{X}} \xrightarrow{s_F} W(\mathcal{O}_{\mathfrak{X}})$ such that

$$\begin{tikzcd} \mathcal{O}_{\mathfrak{X}} \arrow[r, "s_F"] \arrow[dr] & W(\mathcal{O}_{\mathfrak{X}}) \arrow[d, hook] \\ & \mathcal{O}_{\mathfrak{X}}^{\mathbb{N}} \end{tikzcd}$$

$(a_0, a_1, a_2, \dots) \mapsto (a_0, a_0^p + p a_1, a_0^{p^2} + p a_1^p + p^2 a_2, \dots)$

$x \mapsto (x, F(x), F^2(x), \dots)$

$\Rightarrow$ morphism of sheaves of rings

$$\mathcal{O}_{\mathfrak{X}} \xrightarrow{s_F} W(\mathcal{O}_{\mathfrak{X}}) \to W(\mathcal{O}_X)$$

giving $\quad \mathcal{O}_{X_n} \longrightarrow W_n(\mathcal{O}_X)$

So universal property of $\Omega^\bullet_{\mathfrak{X}_n/W_n}$ gives a morphism of $\mathbb{Z}$-dga's

$$\Omega^\bullet_{\mathfrak{X}_n/W_n} \longrightarrow W_n\Omega^\bullet_X$$

$\rightsquigarrow$ natural map $Ru_{X/W_n *}\mathcal{O}_{X/S} \longrightarrow \Omega^\bullet_{\mathfrak{X}_n/W_n} \longrightarrow W_n\Omega^\bullet_X$

Rmk: one checks that the map $Ru_{X/W_n *}\mathcal{O}_{X/S} \to W_n\Omega^\bullet_X$ does not depend on the choice of smooth lift $\mathfrak{X}$, and the lift of Frobenius F on $\mathfrak{X}$

(to do so, need to consider similar situations slightly more general than the one considered above)

Thm (1) Assume X/k smooth. Then $Ru_{X/W_n}\mathcal{O}_{X/S} \xrightarrow{\sim} W_n\Omega^\bullet_X$

(2) If X/k proper smooth, we have

$$H^*(X/W, \mathcal{O}_{X/W}) := \varprojlim_n H^*(X/W_n, \mathcal{O}_{X/W_n}) \overset{(1)}{\simeq} \varprojlim \mathbb{H}^*(X, W_n\Omega^\bullet_X)$$

$$\simeq \mathbb{H}^*(X, R\varprojlim W_n\Omega^\bullet_X)$$

need some finiteness statements. □

Rem: the morphism $Ru_{X/W_n}\mathcal{O}_{X/S} \longrightarrow W_n\Omega^\bullet_X$ (constructed in the special situation above) is merely a quasi-isomorphism.

nevertheless, with more work, one checks that there exists an iso.

$$W_n\Omega^i_X \xrightarrow{\sim} \sigma^n_* R^i u_{X/W_n}\mathcal{O}_{X/W_n}.$$

where σ is the Frobenius automorphism of W.

$\rightsquigarrow$ one can take this as the crystalline definition of DRW complex.

§4: logarithmic Chern classes of M. Gros

- k perfect field of char. $p>0$, we shall work in the étale topology
- X/k smooth $\rightsquigarrow$ $W_\cdot\Omega_X^\bullet = \{W_n\Omega_X^\bullet\}_{n\geq 1}$ pro-de Rham-Witt of X.

  considered as a pro-system of $\mathbb{Z}$-dga's, endowed with V.

- Def: $$d\log : \mathcal{O}_X^* \longrightarrow W_n\Omega_X^1$$ $$x \longmapsto d\log x := \frac{d\underline{x}}{\underline{x}}$$

  where $\underline{x} \in W_n(\mathcal{O}_X)$ is the Teichmüller of x.

  varying n $\rightsquigarrow$ $d\log : \mathcal{O}_X^* \longrightarrow W_\cdot\Omega_X^1$

- Def (the sheaves $W_n\Omega^i_{X,\log}$)

$$W_n\Omega^i_{X,\log} := \begin{cases} \mathbb{Z}/p^n\mathbb{Z} & i=0 \\ \text{abelian subsheaf of } W_n\Omega_X^i \text{ locally generated by } d\log x_1 \cdots d\log x_i, \text{ with } x_1 \cdots x_i \in \mathcal{O}_X^* & i\geq 1 \end{cases}$$

for $n=1$, also write $\Omega^i_{X,\log} := W_1\Omega^i_{X,\log}$

- functoriality of $W_n\Omega^i_{X\log}$: $\forall$ $Y \xrightarrow{f} X$ over $\operatorname{Spec} k$

  $\rightsquigarrow$ $f^{-1}W_n\Omega^i_{X,\log} \longrightarrow W_n\Omega^i_{Y,\log}$ & $W_n\Omega^i_{X,\log} \to f_*W_n\Omega^i_{Y,\log}$

- $W_n\Omega^\bullet_{X,\log} := \bigoplus_{i\geq 0} W_n\Omega^i_{X,\log}$ is a unitary strictly anticommutative graded ring

- $\exists$ short exact sequence of pro-sheaves for the étale topology:

$$0 \to W.\Omega^{\cdot}_{X,\log} \to W.\Omega^{\cdot}_X \xrightarrow{1-F} W.\Omega^{\cdot}_X \to 0$$

- for X/k smooth, define $A(X) = \bigoplus_{n \geq 0} A^n(X)$, with

$$A^n(X) = \begin{cases} 0 & , n \text{ odd} \\ \mathbb{H}^{\frac{n}{2}}(X, R\varprojlim_n W_n\Omega^{n/2}_{X,\log}) & , n \text{ odd} \end{cases}$$

aim: for E v.b. on X, define $C_i(E) \in A^{2i}(X)$ by using Grothendieck's approach.

① logarithmic chern class

- X/k smooth k perfect of char. $p > 0$

$$d\log : \mathcal{O}_X^* \to W.\Omega^1_{X,\log}$$

$$\rightsquigarrow \quad \tilde{c}_1 : H^1(X, \mathcal{O}_X^*) \to H^1(X, R\varprojlim W.\Omega^1_{X,\log}) = A^2(X)$$

$$L \longmapsto \tilde{c}_1(L)$$

② Gysin map:

- Autoduality of DRW complex of Ekedhal (and Illusie)

Def (Cartier-Dieudonné-Raynaud ring)

$R :=$ the graded W-algebra generated by

- two operators F and V of degree 0, and
- one generator d of degree 1

with respect to the following conditions

* $\underline{F}\, a = F(a)\, \underline{F} \qquad \underline{F}\, \underline{d}\, \underline{V} = \underline{d}$

* $a\, \underline{V} = \underline{V}\, F(a)$

* $\underline{V}\, a\, \underline{F} = V(a)$

* $\underline{V}\, \underline{F} = \underline{F}\, \underline{V} = p$

* $\underline{d} a = a \underline{d}$

for every $a \in W$

($\Rightarrow$ $R = R^0 \oplus R^1$, and R^0 = the usual Dieudonné algebra $W_\sigma[F, V]$)

$$R_n := \frac{R}{\underline{d}\,\underline{V}^n R + \underline{V}^n R}$$: this is a left $W[\underline{d}]$-module and a right R-module

$\rightsquigarrow$ (graded left) R-module $= \cdots$

<u>Example</u>: $W_n \Omega_X^\bullet$, $W\Omega_X^\bullet$ are (graded left) R-modules

<u>Fact</u>: $R_n \overset{L}{\otimes}_R W\Omega_X^\bullet \xrightarrow{\sim} W_n \Omega_X^*$ as $W_n[d]$-modules

$\Rightarrow$ $R_\bullet \overset{L}{\otimes}_R W\Omega_X^\bullet \xrightarrow{\sim} W_\bullet \Omega_X^\bullet$ as pro-sheaves of R-modules, compatible with F & V.

<u>Def</u>. For all $M \in D^b_c(R, X)$ (= derived category of sheaves of R-modules on X, with <u>coherent</u> cohomology)

$$D_X(M) := R\varprojlim \underline{R\mathrm{Hom}}_R\left(M, W_n \Omega_X^{d_X}[d_X]\right), \text{ with } d_X = \dim X$$

Theorem let $f: X \to k$ smooth.

(1) (Illusie - Ekedhal) $\exists$ natural iso. $W_n(f)^! \mathcal{O}_{\operatorname{Spec} W_n} \simeq W_n\Omega_X^{d_X}[d_X]$

Ekedhal:

(2) $W\Omega_X^\bullet \xrightarrow{\simeq} D_X(\,W\Omega_X^\bullet\,)\underbrace{(-d_X)}_{\text{twist as graded object}}\underbrace{[-d_X]}_{\text{shift in derived category}}$ in $D_c^b(R, X)$

(3) $Y \xrightarrow{f} X$, $Y \to \operatorname{Spec} k$ smooth, $X \to \operatorname{Spec} k$ smooth, $\forall\, M \in D_c^b(R, Y) \rightsquigarrow$ canonical duality isomorphism

$$Rf_* D_Y M \xrightarrow{\simeq} D_X Rf_* M.$$

In particular for $Y \xrightarrow{f} X$ as in (3) $\rightsquigarrow$ <u>Gysin morphism</u>

$$(**) \qquad f_*: Rf_* W\Omega_Y^\bullet \longrightarrow W\Omega_X^*(c)[c] \quad , c := d_X - d_Y ,$$

obtained as the transposition of

$$f^*: W\Omega_X^\bullet \longrightarrow Rf_* W\Omega_Y^\bullet$$

Applying $R_\cdot \overset{L}{\otimes}_R -$ to $(**)$, get

$$f_*: Rf_* W_\cdot\Omega_Y^\bullet \longrightarrow W_\cdot\Omega_X^\bullet(c)[c], \text{ compatible with } F, V.$$

taking Frobenius invariant $\rightsquigarrow$ morphism of pro-sheaves:

$$f_*: Rf_* W_\cdot\Omega_{Y,\log}^\bullet \longrightarrow W_\cdot\Omega_{X,\log}^\bullet(c)[c]$$

taking cohomology, $\rightsquigarrow$

$$f_*: H^i(Y, R\varprojlim W_\cdot\Omega_{Y,\log}^j) \longrightarrow H^{i+c}(X, R\varprojlim W_\cdot\Omega_{X,\log}^{j+c})$$

$\rightsquigarrow$ the desired Gysin map

$$f_*: \quad A(Y) \longrightarrow A(X)[2c]$$

Prop: the axioms (A2) ~ (A4) in Grothendieck's theory hold.

Pf: for (A2), one reduces to an explicit description of the Gysin map when X/k is liftable (then do explicit calculations).

for (A2) & (A3), one reduces to the corresponding statements for the Gysin map (☆☆) □.

③ logarithmic cohomology of projective bundles

- X/k smooth. E. v.b. of rk r on X
- $P = \mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}^* E^\vee) \xrightarrow{f} X$
- $d\log: \mathcal{O}_P^* \longrightarrow W.\Omega^1_{P,\log}$
- $\xi_E := \tilde{c}_1(\mathcal{O}_P(1)) \in H^1(P, R\varprojlim W.\Omega^1_{P,\log})$

Cup product $\rightsquigarrow$

$$W.\Omega^\bullet_{X,\log}(-i)[-i] \xrightarrow{\cdot \xi_E^i} Rf_* W.\Omega^\bullet_{P,\log}$$

$$\rightsquigarrow \bigoplus_{i=0}^{r-1} \cdot \xi_E^i : \bigoplus_{i=0}^{r-1} W.\Omega^\bullet_{X,\log}(-i)[-i] \longrightarrow Rf_* W.\Omega^\bullet_{P,\log}$$

morphism of projective system

Prop: this is an isomorphism

Prop $\Rightarrow$ axiom (A1) in Grothendieck's theory.

<u>Pf of prop</u> : one reduces to the coherent cohomology of projective spaces.

more precisely, it suffices to check that the similar map

$$\bigoplus_{i=0}^{r-1} W_n\Omega^{\bullet}_{X,\log}(-i)[-i] \longrightarrow W_n\Omega^{\bullet}_{X,\log}$$

is an isomorphism.

then, one reduces further to the case $n=1$. i.e,

$$\bigoplus \Omega^{\bullet}_{X,\log}(-i)[-i] \xrightarrow{\sim} Rf_* \Omega^{\bullet}_{P,\log}$$

on the other hand, $\exists$ short exact sequence

$$0 \to \Omega^i_{X,\log} \to Z\Omega^i_X \xrightarrow{1-C_X} \Omega^i_X \to 0$$

so one reduces to show

$$\bigoplus_{i=0}^{r-1} Z\Omega^{\bullet}_X(-i)[-i] \xrightarrow{\sim} Rf_* Z\Omega^{\bullet}_P \quad , \text{ and}$$

$$\bigoplus_{i=0}^{r-1} \Omega^{\bullet}_X(-i)[-i] \xrightarrow{\sim} Rf_* \Omega^{\bullet}_P$$

by some dévissage, it suffices to check the second isomorphism.

this is a local question, so may assume $E = \mathcal{O}_X^r$

$$\rightsquigarrow P = \mathbb{P}^{r-1}_X = \mathbb{P}^{r-1}_k \times_{\operatorname{Spec} k} X$$

$$\begin{tikzcd} \mathbb{P}_k^{r-1} \arrow[d] & P \arrow[l, "g"'] \arrow[d, "f"] \\ \operatorname{Spec} k & X \arrow[l] \end{tikzcd}$$

$$\Rightarrow \Omega_P^1 = g^*\Omega_{\mathbb{P}_k^{r-1}/k} \oplus f^*\Omega^1_{X/k}$$

$$\Rightarrow \Omega_P^i = \bigoplus_{j=0}^{i} (f^*\Omega^j_{X/k}) \otimes g^*\Omega^{i-j}_{\mathbb{P}_k^{r-1}/k})$$

$$\Rightarrow Rf_*\Omega_P^i \cong \bigoplus_{j=0}^{i} \Omega_X^i \otimes_k R\Gamma(\mathbb{P}_k^{r-1}, \Omega^{i-j}_{\mathbb{P}_k^{r-1}/k}).$$

So only need to check

$$H^m(\mathbb{P}_k^n, \Omega^j_{\mathbb{P}_k^n}) = \begin{cases} k\cdot \tilde{c}_1(\mathcal{O}_{\mathbb{P}_k^n}(1))^j & j=m \\ 0 & m\neq j \end{cases}$$

__Pf__: one uses the exact sequence

$$0 \to \Omega_{\mathbb{P}_k^n} \to \mathcal{O}_{\mathbb{P}_k^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}_k^n} \to 0$$

& cohomology of projective spaces $\square$.

__Conclusion__: Chern classes for logarithmic de Rham-Witt cohomology

__properties__

- compatibility with crystalline chern classes of Berthelot-Illusie

i.e. $H^i(X, W_n\Omega^i_{X,\log}) \xrightarrow{} \mathbb{H}^{2i}(X, W_n\Omega_X^\bullet) \xrightarrow{\text{can}} H^{2i}(X/W_n)$

induced by $W_n\Omega^i_{X,\log}[-i] \to W_n\Omega_X^\bullet$

$$c_1^{\log}(E) \longmapsto c_i^{\text{cris}}(E)$$

- X/k smooth $Y \overset{f}{\hookrightarrow} X$ of codim c.

Gros $\rightsquigarrow$ can define logarithmic cycle class of Y, denoted by

$$cl_X(Y) \in H^c(X, R\varprojlim W.\Omega^c_{X,\log})$$

if Y/k is smooth, this is just $f_*(1)$.

<u>Prop</u>: $C_{c+1}(f_* \mathcal{O}_Y) = (-1)^{\delta-1}(\delta-1)!\, cl_X(Y) \in H^c(X, R\varprojlim W.\Omega^c_{X,\log})$

$\boxed{\text{Final remarks}}$

- should be able to develop exactly in the same way a theory of chern classes in de Rham-Witt cohomology (or equivalently, in crystalline cohomology)
- the more important thing in Gros' work is the construction of <u>cycle classes</u> in the general case: Berthelot's original approach only work for closed subvariety $Y \hookrightarrow X$ which is smooth $/k$.