§ p-adic V.H.C for line bundles.

Part I.
- Connections
- H.P.D stratification
  crystal. (pull-back) (morphism)
- Linearization.
- Poincare lemma.
- comparison (following Berthlot).

— Connection X/S Alg scheme $\mathcal{V}$ - $\mathcal{O}_X$-mod.

§. $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{X/S}$. $\nabla(fm) = df \otimes m + f \nabla m$
$f \in \mathcal{O}_X$ $m \in \mathcal{V}$

p'-linear.

$\Leftrightarrow$. $P^1 \otimes_X \mathcal{V} \xrightarrow{\varepsilon} \mathcal{V} \otimes_X P^1$ where $P^1 = \mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X / I^2$

s.t $\varepsilon = \mathrm{Id} \mod \Omega^1_{X/S}$ $I/I^2 = \Omega^1_{X/S}$

$P^1 = \mathcal{O}_X \oplus \Omega^1_{X/S}$

§ G.M connection

$\boxed{\nabla \text{ connection}}$ with $\nabla^2 = 0$ $\Rightarrow$ stratification / $\mathbb{Q}$

$\Downarrow$ $p! = 0$

Fact: $P^1(2) \overset{q_0}{\underset{q_1}{\rightrightarrows}} P^1$ $\Rightarrow$ $q_0^*(\varepsilon) \circ q_1^*(\varepsilon) = \varepsilon$.

$X \times_S X \times_S X \rightrightarrows X \times_S X$

P.D - stratification.
pro-obj
indexed by n-th infinitesimal information

H.P.D stratification $= \left(\nabla^2 = 0\right) +$ ∇ quasi-nilpotent.

if $p^m . \mathcal{O}_S = 0$. $\left(\frac{\partial}{\partial x_i}\right)^p = 0$ if $p\mathcal{O}_S = 0$ $\boxed{\text{For a local section } s, \left(\nabla_{\frac{\partial}{\partial x_i}}\right)^e s = 0 \text{ locally}}$ (x_i coordinates on X/S)

- H.P.D stratification and Crystal.

$$\begin{array}{c} Y \\ \downarrow \text{smooth} \\ (S, I, \gamma) \end{array} \qquad \gamma \text{ extends to } Y. \qquad p^n \cdot \mathcal{O}_S = 0.$$

$D_{Y/S}(\nu)$ = divided power envelope of $Y \subseteq \underbrace{Y \times_S Y \times_S \cdots \times_S Y}_{\nu+1}$

$= D_Y(Y/S^{\nu+1}) = D(\nu)$.

e.g.:

$$\begin{tikzcd} Y = V(J_{Y/S}) \arrow[d] \arrow[r] & D_{Y/S}(1) \arrow[d] \\ Y \arrow[r] & Y \times_S Y \end{tikzcd}$$

$\mathcal{O}_Y$-mod $\mathcal{E}$

Def: A hyper-PD-stratification is an $D(1)$-linear.

- isomorphism

$$\varepsilon : D_{Y/S}(1) \otimes_Y \mathcal{E} \xrightarrow{\simeq} \mathcal{E} \otimes_Y D_{Y/S}(1)$$

satisfying ① $\varepsilon \equiv \mathrm{Id} \mod J_{Y/S}$.

② cocycle condition for

$$D(2) \rightrightarrows D(1).$$

$$Y \times_S Y \times_S Y \rightrightarrows Y \times_S Y.$$

- $\mathcal{E}, \mathcal{F}$ hyper-PD-stratification with $\mathcal{E} \xrightarrow[\mathcal{O}_Y\text{-linear}]{h} \mathcal{F}$ is called horizontal if

$$\begin{tikzcd} D(1) \otimes_Y \mathcal{E} \arrow[r, "\varepsilon"] \arrow[d, "\mathrm{Id} \otimes h"] & \mathcal{E} \otimes_Y D_{Y/S}(1) \arrow[d, "h \otimes \mathrm{Id}"] \\ D_Y(1) \otimes_Y \mathcal{E} \arrow[r, "\varepsilon"] & \mathcal{E} \otimes_Y D_{Y/S}(1) \end{tikzcd}$$

H.P.D stra/Y.

Def: A crystal is a $\mathcal{O}_{\cancel{X}/S}$-mod on $(X/S)_{cris}$

for any morphis $(U', T', \delta') \longrightarrow (U, T, \delta)$ in $(X/S)_{cris}$.

the transition map is an isomorphism.

<u>Lemma</u>: (pull-back) $\mathcal{E}$ is a crystal on $(X/S)_{cris}$

$$\begin{array}{ccc} f: X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ (S',I',\gamma') & \longrightarrow & (S,I,\gamma) \end{array}$$

$f^*_{cris}\mathcal{E}$ as pull-back of ($\mathcal{O}_{X/S}$-mod.) ringed topoi

$$\Rightarrow f^*_{cris}\mathcal{E}\big|_{T'} = h^*\mathcal{E}|_T \quad \text{if} \quad \begin{array}{ccc} T' & \xrightarrow{h} & T \in Cris X/S \\ \updownarrow & & \cup \\ U' & \xrightarrow{f|_{U'}} & U \end{array}$$

$Cris X'/S' \ni T'$

<u>Lemma</u>: (equivalence).

$$H.P.D_{stra}/Y \longleftrightarrow Crystal(Y/S)$$

— morphism $\mathcal{O}_{Y/S}$-linear.

why HPD stratification

$p_1^*E \simeq \mathcal{E}_{D(1)} \simeq p_2^*E$

$$E = \mathcal{E}|_{(Y,Y,0)} \longleftarrow \mathcal{E}$$

$D(1) \xrightarrow{\varepsilon} Y\times_S Y$, $p_1, p_2: D(1) \to Y$

using $D(2) \rightrightarrows D(1)$ to check cocycle condition

inverse functor:

$T \xrightarrow{g} S$, $h: T \dashrightarrow Y$, $U \subseteq Y$

$$\mathcal{E}_T \overset{def}{=} h^*E \quad \text{why well-defined?}$$

$= g^*E$

$$\begin{array}{ccccc} T & \xrightarrow{(h,g)} & D(1) & \longrightarrow & Y\times_S Y \\ \updownarrow & \overset{g}{\searrow}\ \overset{h}{\searrow} & p_1\downarrow\downarrow p_2 & \swarrow & \\ U & \subseteq & Y & & \end{array}$$

$$h^*E = (h,g)^*p_1^*E$$

$\wr\wr$ using HPD stratification

$$(h,g)^*p_2^*E$$

$$\| $$

$$g^*E$$

+ cocycles condition

$\Downarrow$

$\mathcal{E}_T$ is a crystal.

- Linearization

$$\mathcal{O}_Y\text{-mod} \xrightarrow{L_Y} \text{H.P.D}_{\text{stra}}/Y.$$

with HPD-diff operators

horizontal morphism (in particular $\mathcal{O}_Y$-linear).

recall: $D(1)\otimes_Y \mathcal{F} \xrightarrow{h} \mathcal{G}$ HPD diff operator

$\mathcal{O}_Y$-linear.

$$E \longmapsto D(1)\otimes_Y E. = L_Y(E)$$

$$E \xrightarrow{u} F \longmapsto$$

HPD stratification

$$\begin{array}{ccc} L_Y(E) = D(1)\otimes_Y E & \xrightarrow{L_Y(u).} & L_Y(F) \\ \downarrow \delta\otimes_Y \mathrm{Id} & & \| \\ D(1)\otimes_Y D(1)\otimes_Y E & \xrightarrow{\mathrm{Id}\otimes u} & D(1)\otimes_Y F \end{array}$$

$$D(1)\otimes_Y L_Y(E) = D(1)\otimes_Y D(1)\otimes E \xrightarrow{\mathrm{Id}\otimes\delta\otimes\mathrm{Id}} D\otimes D\otimes D\otimes E$$

$$\downarrow (\mathrm{Id},\sigma)\otimes\mathrm{Id}\otimes\mathrm{Id}$$

$$= D_2\otimes(D\otimes E)$$

$$\| $$

$$L_Y(E)\otimes D(1)$$

lemma: L_Y is a functor.

upshot: $\Omega^k_{Y/S} \xrightarrow{d} \Omega^{k+1}_{Y/S}$ d is not linear.

but a H.P.D stratification

$$\Omega^k_{Y/S} \to D(1)\otimes_Y \Omega^k_{Y/S} \to \Omega^{k+1}_{Y/S}$$

"not trivial."

it factors through

$$P^1\otimes_Y \Omega^k_{Y/S} \longrightarrow \Omega^{k+1}_{Y/S}$$

$$\| $$

$$\mathcal{O}_Y\oplus\Omega^1_{Y/S} \ni (a,\omega)\otimes y \longmapsto \omega\wedge y$$

but it is not naively.

- Poincaré Lemma.

$$\begin{tikzcd} X \arrow[r, hook, "i", "\text{closed immersion}"'] \arrow[dr] & Y \arrow[d, "\text{smooth}"] \\ & (S, I, \gamma) \end{tikzcd}$$

γ extends to X.

$L(\Omega^{\cdot}_{Y/S})$

Poincaré Lemma. $\mathcal{O}_{X/S} \to i^*_{cris} L_Y(\underline{\Omega^{\cdot}_{Y/S}})$ qis-iso.

complex.

<u>sub lemma</u>: $L_Y(d): D(1) \otimes \Omega^k_{Y/S} \xrightarrow{\mathcal{O}_Y\text{-linear}} D(1) \otimes \Omega^{k+1}_{Y/S}$

$\xi_i \stackrel{\text{def}}{=} x_i \otimes 1 - 1 \otimes x_i$

$\in D_Y(Y \times_S Y)$

$\mathcal{O}_Y\langle \xi_1, \dots \xi_n \rangle$

$x_1, \dots x_n$ coordinates of Y/S.

$$\underset{\mathcal{O}_Y}{a}\, \xi_1^{[k_1]} \cdots \xi_n^{[k_n]} \otimes \omega \mapsto a\left(\sum_{i=1}^{n} \xi_1^{[k_1]} \cdots \xi_i^{[k_i - 1]} \cdots \xi_n^{[k_n]} \otimes dx_i \wedge \omega + \prod_{i=1}^{n} \xi_i^{[k_i]} \otimes d\omega \right).$$

$$\boxed{\text{“} \xi_j^{[-1]} = 0 \text{”.} \quad \text{“} \xi_j^{[0]} = 1. \text{”}}$$

“<u>exact</u>: $A \to A\langle t_1, \dots t_n \rangle \xrightarrow{d} \bigoplus_{i=1}^{n} A\langle t_1 \dots t_n \rangle \cdot dt_i \to \bigoplus_{i<j} A\langle t_1, \dots t_n \rangle\, dt_i \wedge dt_j \to \cdots \cdots$ ”

A-linear

$d(t_i^{[a]}) = t_i^{[a-1]} dt_i$.

$\wr\wr$

$(A\langle t_1 \rangle \xrightarrow{d} A\langle t_1 \rangle dt_1) \otimes_A \cdots \otimes_A (A\langle t_n \rangle \xrightarrow{d} A\langle t_n \rangle dt_n)$

<u>Pf</u>: sub lemma.

$$\begin{tikzcd} L_Y(d): D(1) \otimes \Omega^k \arrow[r, "\delta \otimes_Y \mathrm{id}_{\Omega^k}"] & D \otimes D \otimes \Omega^k \arrow[r] \arrow[d, "\mathrm{id}_D \otimes \pi \otimes 1 \mid d_\Omega"] & D \otimes \Omega^{k+1} \\ & D \otimes P' \otimes \Omega^k \arrow[ur, "\mathrm{id} \otimes \tilde{d}"'] & \end{tikzcd}$$

P.D-ring hom

$\delta(\xi_i) = 1 \otimes \xi_i + \xi_i \otimes 1$.

$D \xrightarrow{\delta} D \otimes_Y D$

$$\delta \otimes \mathrm{id}(a \xi^{[k]} \otimes \omega) = a(1 \otimes \xi + \xi \otimes 1)^{[k]} \otimes \omega$$
$$= a \sum_{i+j=k} \xi^{[i]} \otimes \xi^{[j]} \otimes \omega.$$

since π kills $\xi^{[l]}$ $|l| \geq 2$, ..

$id_D \otimes \pi \otimes id_{\Omega^k}$ maps to $a \cdot \left(\sum_{i=1}^{n} \xi^{[k-1_i]} \otimes \xi_i \otimes \omega + \xi^{[k]} \otimes \omega \right)$

using. • $\tilde{d}(\xi_i \otimes \omega) = dx \wedge \omega$

• $\tilde{d}(1 \otimes \omega) = d\omega$.

$$\begin{tikzcd} \Omega^k \arrow[r, "d"] \arrow[d] & \Omega^{k+1} \\ P^1 \otimes \Omega^k \arrow[ur, "\tilde{d}"'] & \end{tikzcd}$$

$\Rightarrow \square$.

~~Comparison.~~ **Rmk**:

$$\begin{tikzcd} \mathcal{O}_Y \arrow[r] & L_Y(\mathcal{O}_Y) \arrow[r] \arrow[d, phantom, "\cong"] & L_Y(\Omega^1) \cdots \\ & D(1) & \\ x \arrow[r, mapsto] & x \otimes 1 & \end{tikzcd}$$

HPD-stra $D(1) \overset{id}{=} D(1)$

~~HPD-stra~~ has natural HPD-stra.

need to check it is horizontal.

Comparison.

Thm 1

① • $Ru_{X/S*}(\mathcal{F}^\cdot) \xrightarrow{\sim} \check{C}A_Y^\cdot(\mathcal{F}^\cdot)$.

② • $\check{C}A_Y^\cdot(L(E)) \xleftarrow{q.i.s} D_X(Y) \otimes_Y E$ $\qquad E$ $\mathcal{O}_Y$-mod

using the natural HPD-stra.

functorially for H.P.D.-differential operator

HPD-diff operator $E \xrightarrow{h} F$

$$\begin{tikzcd} \check{C}A_Y^\cdot(L_Y(E)) \arrow[d, "\check{C}A_Y(L_Y(h))"'] & & D_X(Y) \otimes_Y E \arrow[ll, "q.i.s"'] \arrow[d, "\theta \otimes id_E"] \\ & \arrow[phantom, "\circlearrowleft"] & D_X(Y) \otimes_Y D(1) \otimes_Y E \arrow[d, "id_{D_X(Y)} \otimes h"] \\ \check{C}A_Y^\cdot(L_Y(F)) & & D_X(Y) \otimes_Y F \arrow[ll, "q.i.s"'] \end{tikzcd}$$

Lemma: $u_{X/S*}(E)|_U$ is horizontal sections, i.e,
(e.g $\nabla^{\nabla=0}$).

exact. $u_{X/S*}(E)|_U \to E_{(U, D_U(Y))} \rightrightarrows E_{(U, D_U(Y\times_S Y))}$

$U \subseteq X_{zar}$

Def: $$\check{C}A^{\bullet}_Y(E) = \left(\overset{0}{E_{(X, D_X(Y))}} \rightrightarrows \overset{1}{E_{(X, D_X(Y^2))}} \overset{\to}{\rightrightarrows} \overset{2}{E_{(X, D_X(Y^3))}} \cdots \right).$$

To show ①, claim: $\boxed{\check{C}A^{\bullet}_Y(I) \text{ is acyclic in degree} \geq 1.}$

(by take section $\Gamma(U, -)$ $U \subseteq X$ since they are sheaf in X_{zar}).

Pf of claim:
take $\tilde{Y}$ a sheaf cris X/S $\quad \tilde{Y} \to e$ covering.
$\quad\ \ \shortparallel$
$\mathrm{Hom}(-, D_X(Y))$.

Cěch-Cartan-Leray S.S $\quad E_2^{p,q} = H^p(\nu \mapsto H^q(\tilde{Y}^{\nu+1}, \mathcal{F}))$
$$\Downarrow$$
$$H^n((X/S)_{cris}, \mathcal{F})$$

take $\mathcal{F} = I^{\bullet}_{inj}$, $\Rightarrow H^p(\nu \longmapsto \Gamma(\tilde{Y}^{\nu+1}, \mathcal{F})) = 0 \quad p > 0.$
$$\shortparallel$$
$$\mathrm{Hom}(h_{D_X(Y^{\nu+1})}, \mathcal{F})$$
$$\shortparallel$$
$$\Gamma(X, \mathcal{F}_{(X, D_X(Y^{\nu+1}))}) \Rightarrow \text{claim}$$

For ②, we only show. exact

$$0 \to D_X(Y) \otimes_Y E \to \underset{\underset{D_X(Y^2)}{\cong}}{\underline{D_X(Y) \otimes D(1) \otimes_Y E}} \to \underset{\underset{D_X(Y^3)}{\cong}}{\underline{D_X(Y^2) \otimes D(1) \otimes_Y E}} \to \cdots$$

(Lemma 2.2.1)

Simplicial method. $E^\cdot$ cosimplicial obj.

$0 \to E^0 \underset{d_0}{\rightrightarrows} E^1 \underset{d_0 - d_1}{\rightrightarrows} E^2 \underset{d_0-d_1+d_2}{\cdots} \cdots$ exact.

$\Rightarrow$. $\mathcal{D}_X(Y) \otimes_Y E \xrightarrow[\simeq]{qis} \check{C}A_Y^\cdot(L(E))$ $\square$.

Thm: $\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & \# & \downarrow \\ S/I & \rightarrow & (S, I, \gamma) \end{array}$ $\qquad X$ smooth $/S$.

γ extends to X_0 $\qquad H^*_{cris}(X_0/S) \cong H^*_{DR}(X/S)$.

Pf: $R\Gamma_{cris}(X_0, \mathcal{O}_{X_0/S}) \cong R\Gamma_{zar}(X_0, Ru_{X_0/S*}\mathcal{O}_{X_0/S})$

$\underset{\text{Poincare lemma}}{\cong} R\Gamma_{zar}(X_0, Ru_{0/S*}(L\Omega^\cdot_{X/S}))$

Thm 1

$\cong R\Gamma_{zar}(X_0, \underset{=}{\overset{\mathcal{O}_X}{\mathcal{D}_{X_0}(X)}} \otimes_X \Omega^\cdot_{X/S})$

$\cong R\Gamma_{zar}(X, \Omega^\cdot_{X/S})$

$\square$

Part II.

- Def of $c^1_{cris}(\mathcal{L})$
- In terms of de Rham Theory.
- Commutative diagrams and obstruction.

- Def of c^1_{cris} $\quad$ $X \xrightarrow{f} /(T, I, \gamma)$ $\quad$ γ extends to X.

$i: X_{Zar} \to (X/T)_{cris}$. $\quad i_* = u^*_{X/S}$ $\quad i_* \mathcal{F}(U, T, \delta) = \mathcal{F}(U)$.

$0 \to \mathcal{J}_{X/T} \to \mathcal{O}^*_{X/T} \longrightarrow i_* \mathcal{O}_X \to 0.$

Note: $\mathcal{J}_{X/T}$ is a sheaf of nil ideals, $\left(x^{p^n} = (p^n)! \cdot \gamma_{p^n}(x^n) = 0\right)$.

$\Downarrow$

$1 + \mathcal{J}_{X/T}$ sheaf of Ab gp.

$\Rightarrow \ 0 \to 1 + \mathcal{J}_{X/T} \to \mathcal{O}^*_{X/T} \to i_* \mathcal{O}^*_X \to 0$

- $\log: \ 1 + \mathcal{J}_{X/T} \longrightarrow \mathcal{J}_{X/T}$

$1 + x \longmapsto \log(1+x) :\overset{def}{=} \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \, \delta_n(x)$

$\Rightarrow \quad R^1 f_{X/T} \, i_* \mathcal{O}^*_X \xrightarrow{\partial} R^2 f_{X/T} (1 + \mathcal{J}_{X/T}) \xrightarrow{\log} R^2 f_{X/T} \mathcal{J}_{X/T} \to R^2 f_{X/T} \mathcal{O}_{X/T}$.

using $u_{X/T*} \circ i_* = i_* \circ u_{X/T*}$.

$H^1(X, \mathcal{O}^*_X) \xrightarrow{c^1_{cris}} H^2_{cris}(X/T)$.

$\quad \parallel$

$\mathrm{Pic}(X)$

- Recall Y/S smooth de Rham c^1_{dR}.

$\Omega^{\times}_{Y/S} = \mathcal{O}^*_Y \xrightarrow{d\log} \Omega^1_{Y/S} \to \Omega^2_{Y/S} \to \cdots$.

$0 \to F^1 \Omega^\bullet_{Y/S} \to \Omega^{\times}_{Y/S} \to \mathcal{O}^*_Y[0] \to 1$

$$\begin{array}{ccc} H^1(Y, \mathcal{O}^*_Y) & \to & H^2(Y, F^1\Omega) \\ & \searrow^{c^1_{dR}} & \downarrow \\ & & F^1 H^2_{dR}(Y) \end{array}$$

- In terms of de Rham

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & X \arrow[d, "\text{smooth}"] \\ T/I \arrow[r] & T \end{tikzcd}$$

$$\mathcal{O}_{X_0/T} \longrightarrow L(\Omega^{\cdot}_{X/T}).$$

$$0 \to \left(\begin{array}{c} \vdots \\ \uparrow \\ \Omega^1_{X/T} \\ \uparrow \, d\log \times \\ 1 + I\mathcal{O}_X \\ \parallel \\ \mathcal{X}^{\times}_{X/T} \end{array} \right) \to \Omega^{\times}_{X/T} \longrightarrow \mathcal{O}^{*}_{X_0} \to 0.$$

$$0 \to 1 + \mathcal{J}_{X/T} \to \mathcal{O}^{\times}_{X/T} \to i_* \mathcal{O}^{\times}_{X_0} \to 0.$$

$$\begin{tikzcd} \Rightarrow \quad H^1(X_0, \mathcal{O}^{*}_{X_0}) \arrow[r] \arrow[rr, bend left, "C^1_{dR}"] & H^2(X_0, \mathcal{X}^{\times}_{X/T}) \arrow[r, "\log"] & H^2(X, \mathcal{X}_{X/T}) \arrow[r] & H^2(X, \Omega^{\cdot}_{X/T}) \end{tikzcd}$$

$$\mathcal{X}_{X/T} = \left(I\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/T} \to \Omega^2_{X/T} \cdots \right)$$

lemma: $C^1_{dR} = C^1_{cris}$ under identification $H^2_{cris}(X_0/T) \cong H^2(X, \Omega^{\cdot}_{X/T})$

Pf:

$$\mathcal{O}_{X_0/T} \to L\Omega^{\cdot}_{X/T} \xrightarrow{Ru_{X/T}} \Omega^{\cdot}_{X/T}$$

$$\begin{tikzcd} 0 \arrow[r] & 1 + \mathcal{J}_{X/T} \arrow[r] \arrow[d] & \mathcal{O}^{*}_{X/T} \arrow[r] \arrow[d] & i_* \mathcal{O}^{*}_{X_0} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ker}^{\times} \arrow[r] & (L\Omega_{X/T})^{\times} \arrow[r] & i_* \mathcal{O}^{*}_{X_0} \arrow[r] & 0 \end{tikzcd}$$

Kernel with $1+K$

$$\begin{tikzcd} (L\mathcal{O}_X)^{*} \arrow[r, "d\log"] & L\Omega^1_{X/T} \cdots & (L\mathcal{O}_{X_0})^{*} \arrow[r] & i_* \mathcal{O}^{*}_{X_0} \\ \mathrm{Ker}^{\times} := 1+K \arrow[u, hook] \arrow[r, "d\log"] & L\Omega^1_{X/T} \cdots \arrow[u] & L\mathcal{O}_{X_0} \arrow[r] & i_* \mathcal{O}_{X_0} \arrow[u, hook] \\ 1 + \mathcal{J}_{X/T} \arrow[u, hook] & & & \end{tikzcd}$$

with kernel K

P.D ideal ~~structure~~.

locally $(U, \bar{U}, \delta)$

$$\mathcal{O}_{\bar{U}}\langle T_1, \cdots T_n \rangle \longrightarrow \mathcal{O}_U$$

P.D ring hom.

Note:

$$1 + K \xrightarrow{\log} K$$

$$1 + x \longmapsto \sum (-1)^{n-1} (n-1)! \, \delta_n(x)$$

P.D structure on K

$$\Rightarrow \quad H^1(X_0, \mathcal{O}_{X_0}^*) \to H^2_{cris}(X_0/T, 1+\mathcal{J}_{X/T}) \xrightarrow{\log} H^2_{cris}(X_0/T, \mathcal{J}_{X/T}) \to H^2_{cris}(X_0/T)$$

$$H^1(X_0, \mathcal{O}_{X_0}^*) \to H^2_{cris}(X_0/T, \ker^\times) \xrightarrow{\log} H^2_{cris}(X_0/T, \ker) \to H^2_{dR}(X_0/T)$$

$$H^2(X, \mathcal{I}^\times_{X/T}) \to H^2(X, \mathcal{I}_{X/T})$$

$u_{X/T*} i_* = id.$

$Ru_{X/T*} \ker^\times$

$Ru_{X/T*} \ker$

$I + K \xrightarrow{d\log} L\Omega^1 \cdots$

$K \xrightarrow{d} L\Omega^1_{X/T} \to \cdots$

$K \cong L\mathcal{O}_X$

$I + I\mathcal{O}_X \xrightarrow{d\log} \Omega \cdots$

$I\mathcal{O}_X \xrightarrow{d} \Omega^1 \cdots$

$Ru_{X/T*}(L\mathcal{O}_X)^* \cong \mathcal{O}_X^*$

$Ru_{X/T*} i_* \mathcal{O}_{X_0} \cong \mathcal{O}_{X_0}^*$

$\Rightarrow Ru_{X/T} K = I\mathcal{O}_X$

$\Rightarrow \square$.

- deRham to Obstruction

$I = (p)$

take $T = W_n \; W_{n+1} \cdots$ take $\boxed{\text{the limit}}$

so we can pretend $T = W(k)$ $I = (p) \subseteq W$

$$H^1(X_0, \mathcal{O}_{X_0}^*) \to H^2(X, \mathcal{I}^\times_{X/T}) \xrightarrow{\log} H^2(X, \mathcal{I}_{X/T}) \to H^2_{dR}(X)$$

$$H^1(X, \mathcal{O}_X^*) \to H^1(X_0, \mathcal{O}_{X_0}^*) \to H^2(X, I + I\mathcal{O}_X) \xrightarrow{\log} H^2(X, I\mathcal{O}_X)$$

proj.

$0 \to I + I\mathcal{O}_X \to \mathcal{O}_X^* \to i_* \mathcal{O}_{X_0}^* \to 0$

$p\mathcal{O}_X \cong \mathcal{O}_X.$

$H^2(X, I\mathcal{O}_X) \cong H^2(X, \mathcal{O}_X)$

for $T = W_n$, $\log : I + I\mathcal{O}_X \xrightarrow{\cong} I\mathcal{O}_X$ if $\underline{\text{char} > 2}$

since inverse map $\exp$ $I\mathcal{O}_X \to I + I\mathcal{O}_X$

$x \mapsto \sum \delta_n(x)$ is well-defined

($(p) \subseteq W$ $\delta_n(p)$ is p-adically nilpotent).

$$H^2_{dR}(X)$$

$$\mathrm{Ob}: H^1(X_0, \mathcal{O}_{X_0}^*) \xrightarrow{c^1_{cris} = c^2_{dR}} H^2(X, \mathcal{O}_X)$$

We show the ~~thm~~. p-adic V.H.C. for Line bundle.

Namely: $L \in \mathrm{Pic}(X_0)_{\mathbb{Q}}$ lifts to $\mathrm{Pic}(X)_{\mathbb{Q}}$

iff $c^1_{cris}(\mathcal{L}) \in F^1 H^2_{dR}(X)_{\mathbb{Q}}$.

$p = 2$?? $\quad (W(k), (p^2))$.

$\exists k.\ \mathcal{L}^{\otimes k} \dashrightarrow \mathcal{L}' \quad \boxed{\overset{\sim}{\pi}}$ — log maps has inverse map exp.

liftable to X' by obstruction Theory.

$$\begin{array}{ccccc} X_0 & \dashrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ k & \longrightarrow & W(k)/\pi & \longrightarrow & W(k) \end{array}$$

$\mathrm{ob} \in H^2(X_0, \mathcal{O}_{X_0}) \otimes (p)$

$\mathrm{ob}(L \otimes L')$

$= \mathrm{ob}(L) + \mathrm{ob}(L')$. $\qquad 1 + p\mathcal{O}_{X'} \overset{Ab}{\simeq} p\mathcal{O}_{X'} = \mathcal{O}_{X_0} \otimes_k (p)$