Crystalline cohomology and de Rham cohomology

k, perfect field, char $k = p > 0$

$S = \mathrm{Spec}\, W(k)$

$X \to S$ Smooth (projective)

Form a pro-system of schemes $(X_n)_{n \in \mathbb{N}}$, $X_n = X \times_W W_n$

$(X_1)_{et} \xrightarrow{\sim} (X_n)_{et}$. $\forall n \in \mathbb{N}$.

Use étale topology throughout the lecture.

Introduce three players of the game:

(1) de-Rham

$$(\Omega^\bullet_X, d) : \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^d_X.$$

Considered as a ~~pro~~ pro-system of complex of sheaves on $(X_1)_{et}$.

i.e.

$$\begin{tikzcd}
\mathcal{O}_{X_1} \arrow[r, "d"] & \Omega^1_{X_1} \arrow[r, "d"] & \cdots \arrow[r] & \Omega^d_{X_1} \\
\mathcal{O}_{X_2} \arrow[r, "d"] \arrow[u] & \Omega^1_{X_2} \arrow[r, "d"] \arrow[u] & \cdots \arrow[r] & \Omega^d_{X_2} \arrow[u] \\
\cdots \arrow[u] & \cdots \arrow[u] & & \cdots \arrow[u]
\end{tikzcd}$$

with each $\Omega^i_{X_n}$ as an étale sheaf of abelian grps on X_1.

$\forall r,\ (r<p)$

define subcomplexes:

$$\Omega^{\geq r}_{X_\cdot} \subset p(r)\,\Omega_{X_\cdot} \quad (\subset \Omega_{X_\cdot})$$

$\Omega^{\geq r}_{X_\cdot}$: stupid truncation, i.e.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \Omega^r_{X_\cdot} \arrow[r, "d"] \arrow[d, equal] & \Omega^{r+1}_{X_\cdot} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & \Omega^d_{X_\cdot} \arrow[d, equal] \\
\mathcal{O}_{X_\cdot} \arrow[r, "d"] & \cdots \arrow[r] & \Omega^{r-1}_{X_\cdot} \arrow[r, "d"] & \Omega^r_{X_\cdot} \arrow[r, "d"] & \Omega^{r+1}_{X} \arrow[r] & \cdots \arrow[r] & \Omega^d_{X_\cdot}
\end{tikzcd}$$

$p(r)\Omega_{X}^{\cdot}$:

$$p^r\mathcal{O}_X \xrightarrow{d} p^{r-1}\Omega^1_X \xrightarrow{d} \cdots \xrightarrow{d} p\,\Omega^{r-1}_X \longrightarrow \Omega^r_X \to \cdots \to \Omega^d_X$$

(More generally, for any sequence

$$\nu_*: \nu_0 \geq \nu_1 \geq \cdots \geq \nu_d \geq 0$$

one can define

$$\begin{array}{ll} P(\nu_*)\Omega^{\cdot}_X : & p^{\nu_0}\mathcal{O}_X \longrightarrow p^{\nu_1}\Omega^1_X \to \cdots \to p^{\nu_d}\Omega^d_X \quad) \\ \cap & \\ \Omega^{\cdot}_X & \end{array}$$

(2) Crystalline.

Fix a closed immersion,

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] & \mathbb{P}^n_W \overset{\Delta}{=} Z \arrow[dl] \\ S & \end{tikzcd}$$

(important for crystalline is $\exists$ Frob lifting F_Z: on Z).

$\mathcal{D}$ = PD envelope (hull) of X in $\mathbb{P}^n_W$

$= $ PD envelope of X_1 in $\mathbb{P}^n_W$

(note, we fix the PD-base $(W, (p))$ as always).

They have different PD-ideals:

$$\begin{tikzcd} X_n \arrow[r, hook] \arrow[d] & D_n \arrow[r] \arrow[dl] & Z_n \arrow[dll] \\ W_n & & \end{tikzcd}$$

$J_n = $ (PD-)ideal of X_n in $D_n \quad \subset \mathcal{O}_{D_n}$

$I_n = $ (PD-)ideal of X_1 in $D_n \quad \subset$

$\quad = J_n + (p) \quad \subset \mathcal{O}_{D_n}$

Note: $(D_n)_{ét} \xleftarrow{\sim} (X_n)_{ét} \xleftarrow{\sim} (X_1)_{ét}$

the sheaves I_n, J_n are all nilpotent, using

$$(x^{[n]})^m = \frac{(mn)!}{(n!)^m}\, x^{[nm]}$$

$$= (m!) \underbrace{\left[\frac{(mn)!}{m!(n!)^m}\right]}_{= C_{m,n} \in \mathbb{Z}}$$

$$\Omega^{\cdot}_{D_n} = \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega^{\cdot}_{Z_n}$$

$$\mathcal{O}_{D_n} = \mathcal{O}_{Z_n}\left(\gamma^i(f),\ i \geq 0,\ f \in J_{X_n}(\cancel{Z_n}\mathcal{O}_{Z_n}) \right)$$

$$d(\gamma^i(f)) = \gamma^{i-1}(f)\, dx$$

Thus

$$\begin{tikzcd} \mathcal{O}_{Z_n} \arrow[r, "d"] \arrow[d] & \Omega_{Z_n} \arrow[d] \\ \mathcal{O}_{D_n} \arrow[r, "d"] & \mathcal{O}_{D_n} \otimes_{\mathcal{O}_{Z_n}} \Omega_{Z_n} \end{tikzcd}$$

extends to .

define subcomplexes: $\left(\cancel{} \ r < p \ \cancel{} \Rightarrow \begin{array}{l} J_{\cdot}^{[i]} = J_{\cdot}^{i} \\ I_{\cdot}^{[i]} = I_{\cdot}^{i} \end{array} \quad \forall i \leq r \right)$

$$J(r)\Omega^{\cdot}_{D_\cdot} \subset I(r)\Omega^{\cdot}_{D_\cdot} \ (\subset \Omega^{\cdot}_{D_\cdot})$$

$$\begin{tikzcd} J_{\cdot}^{[r]}\mathcal{O}_{D_\cdot} \arrow[r] & J_{\cdot}^{[r-1]}\Omega^{1}_{D_\cdot} \arrow[r] & \cdots \arrow[r] & J_{\cdot}\Omega^{r-1}_{D_\cdot} \arrow[r] & \Omega^{r}_{D_\cdot} \arrow[r] & \cdots \arrow[r] & \Omega^{d}_{D_\cdot} \\ I_{\cdot}^{[r]}\mathcal{O}_{D_\cdot} \arrow[r] \arrow[u, phantom, "\cup"] & I_{\cdot}^{[r-1]}\Omega^{1}_{D_\cdot} \arrow[r] \arrow[u, phantom, "\cup"] & \cdots \arrow[r] & I_{\cdot}\Omega^{r-1}_{D_\cdot} \arrow[r] \arrow[u, phantom, "\cup"] & \Omega^{r}_{D_\cdot} \arrow[r] \arrow[u, phantom, "\parallel"] & \cdots \arrow[r] & \Omega^{d}_{D_\cdot} \arrow[u, phantom, "\parallel"] \end{tikzcd}$$

(3) de Rham-Witt

$W_\cdot \Omega^\cdot_{X_1}$: de Rham-Witt complex

(operators: F, V.)

$$
\begin{tikzcd}
\mathcal{O}_{X_1} \arrow[r, "d"] & \Omega^1_{X_1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^d_{X_1} \\
W_2(\mathcal{O}_{X_1}) \arrow[u] \arrow[r, "d"] & W_2(\Omega^1_{X_1}) \arrow[u] \arrow[r, "d"] & \cdots \arrow[r, "d"] & W_2(\Omega^d_{X_1}) \arrow[u] \\
\cdots \arrow[u] & \cdots \arrow[u] & \cdots & \cdots \arrow[u]
\end{tikzcd}
$$

define subcomplexes $(r < p)$.

$$W_\cdot \Omega^{\leq r}_{X_1} \subset q(r) W_\cdot \Omega^\cdot_{X_1} \subset W_\cdot \Omega^\cdot_{X_1}.$$

$q(r) W_\cdot \Omega^\cdot_{X_1}$:

$$p^{r-1} V W_\cdot \mathcal{O}_{X_1} \xrightarrow{d} p^{r-2} V W_\cdot \Omega^1_{X_1} \xrightarrow{d} \cdots \to p V W_\cdot \Omega^{r-2}_{X_1} \to V W_\cdot \Omega^{r-1}_{X_1}$$
$$\xrightarrow{d} W_\cdot \Omega^r_{X_1} \xrightarrow{d} \cdots \to W_\cdot \Omega^{d-1}_{X_1} \to W_\cdot \Omega^d_{X_1}$$

Here use the formula: $p\, dV = V d$

to conclude this is a subcomplex.

The main result of §2 [BEK] is the following

Theorem (prop 2.8 [BEK]). $(r<p)$

There are natural quasi-isomorphisms

$$\begin{tikzcd}
& \Omega^\cdot_D \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & & \\
\Omega^\cdot_X & \cup & W_\cdot\Omega^\cdot_{X_1} & \\
\cup & I(r)\Omega^\cdot_D \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \cup & \text{BEK} \arrow[l] \\
p(r)\Omega^\cdot_X & \cup & q(r)W_\cdot\Omega^\cdot_{X_1} & \\
\cup & J(r)\Omega^\cdot_D \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \cup & \\
\Omega^{\geq r}_X & & W_\cdot\Omega^{\geq r}_{X_1} &
\end{tikzcd}$$

Berthelot Illusie

Sketch of proof:

Step 1 (~~Bey...~~ PD - poincaré Lemma)

Problem is local: and assume $X \hookrightarrow Z$ given by hypersurf defn

$$\begin{tikzcd}
\mathcal{O}_Z \arrow[r] \arrow[d, equal] & \mathcal{O}_X \arrow[d, equal] \\
A[X] \arrow[r, "\pi"] & A \\
& W \arrow[u] \arrow[ul, bend left]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_Z \arrow[r] \arrow[dr] & \mathcal{O}_X \\
& \mathcal{O}_D \arrow[u, "\pi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}_Z & \mathcal{O}_X \arrow[l] \arrow[dl, "i"] \\
\mathcal{O}_D &
\end{tikzcd}$$

Then $\cancel{\mathcal{I}} = (x)$ $\qquad x^{[n]} = \frac{x^n}{n!}$

$$\mathcal{O}_D = \mathcal{O}_Z\left(\frac{x^n}{n!},\ n \geq 1\right) \quad (\subset \mathcal{O}_Z \otimes k)$$

$$J = \left(\frac{x^n}{n!} \,\middle|\, n \geq 1\right), \subset I = \left(p, \frac{x^n}{n!} \,\middle|\, n \geq 1\right)$$

Then $\cancel{\mathcal{O}_D}$ $i: A \longrightarrow A[X]$ induces.

$$\Omega^\cdot_D \underset{i^*}{\overset{\pi^*}{\rightleftarrows}} \Omega^\cdot_X$$

Claim: $\pi^* \circ i^* \overset{\text{homotopy}}{\simeq} \mathrm{id}$.

i.e. $\exists$ homotopy operator:

$$h^\cdot : \Omega^\cdot_D \to \Omega^{\cdot-1}_D \quad \text{s.t}$$

$$\mathrm{Id} - \pi^* \cdot i^* = dh + hd$$

$$\begin{tikzcd}
\Omega_D^{n-1} \arrow[r, "d"] \arrow[d, "1-\pi^* i^*"'] & \Omega_D^{n} \arrow[r, "d"] \arrow[d, "1-\pi^* i^*"] \arrow[dl, "h"'] & \Omega_D^{n+1} \arrow[d, "1-\pi^* i^*"] \arrow[dl, "h"'] \\
\Omega_D^{n-1} \arrow[r, "d"] & \Omega_D^{n} \arrow[r, "d"] & \Omega_D^{n+1}
\end{tikzcd}$$

Write:

$$\begin{tikzcd}
\Omega_D^{n} \arrow[d, "h"'] & = & \Omega_X^n \otimes \mathcal{O}_D \arrow[dl, "0"'] & \oplus & \Omega_X^{n-1} \otimes \mathcal{O}_D\{dx\} \arrow[dll, "\int"] \\
\Omega_D^{n-1} & = & \Omega_X^{n-1} \otimes \mathcal{O}_D & \oplus & \Omega_X^{n-2} \otimes \mathcal{O}_D\{dx\}.
\end{tikzcd}$$

where $\int$ is induced by

$$\int\left(\frac{x^k}{k!}\,dx\right) = \frac{x^{k-1}}{(k-1)!}.$$

Easy to check:

$$\mathrm{id} - \pi^* i^* = hd \pm dh.$$

$$(i: \Omega_D \longrightarrow \Omega_X)$$

The reason that $J(r)\,\Omega_D \longrightarrow \Omega_X^{\geq r}$ (the same for $I(r)\,\Omega_D^i \longrightarrow P(r)\,\Omega_X^i$)

is that, for $i \leq r < p$, $\frac{f^{i}}{i!} = \underbrace{\left(\frac{1}{i!}\right)}_{\text{invertible}} \cdot f^i \longmapsto 0$, as $f = 0$ in $\mathcal{O}_X$.

Step 2 (Witt calculus)

assume $X_1 = \mathrm{Spec}\, \cancel{A}\, \overset{\overset{A}{\|\wr}}{B}/p \hookrightarrow \mathrm{Spec}\, \mathcal{B} = X = D = Z$

Let $F : \mathcal{B} \to \mathcal{B}$ be an absolute Frob. lifting

(most time. $\mathcal{B} = W[T]$, $F(T) = T^p$, $F|_W = \sigma$)

After Illusie ch II, §1, the quasi-iso

$\Omega^{\cdot}_{D_\cdot} \to W_\cdot\Omega^{\cdot}_{X_1}$ is induced by the following

$$\begin{tikzcd}
\mathcal{O}_{D_n} \arrow[r, "t"] \arrow[dr, "s"] \arrow[ddr] & W_n(\mathcal{O}_{X_1}) \\
& W_n(\mathcal{O}_{D_n}) \arrow[u] \arrow[d, "w"] \\
& \prod_{i=0}^{n-1} \mathcal{O}_{D_n}
\end{tikzcd}$$

w = ghost map

$x \mapsto (x, F(x), \ldots, F^{n-1}(x))$

$$\left(\Rightarrow \begin{tikzcd} \Omega^{\cdot}_{D_n} \arrow[r] \arrow[dr] & \Omega^{\cdot}_{W_n(\mathcal{O}_{X_1})} \arrow[d] \\ & W_n\Omega^{\cdot}_{X_1} \end{tikzcd} \right)$$

Solve s inductively

$$\begin{cases} s_0 = x \\ s_0^p + p s_1 = F(x) \\ \vdots \\ s_0^{p^{n-1}} + p s_1^{p^{n-2}} + \cdots + p^{n-1} s_{n-1} = F^{n-1}(x) \end{cases}$$

étale locally, $X_1 = \mathrm{Spec}(k[T_1, \dots, T_d])$

In this case, the above data is given as follows:

(i) $A = k[T]$

$B = W(k)[T]$

$$C = \bigcup_{r \geq 0} k[T^{\frac{1}{p^r}}] = k[T_1^{\frac{1}{p^r}}, \dots, T_d^{\frac{1}{p^r}}]$$

A form $\omega \in \Omega^m_{C/k}$ is integral, if

$$\parallel$$

$$\sum_{\text{finite}} a_{i_1 \dots i_m} \frac{dT_{i_1}}{T_{i_1}} \wedge \dots \wedge \frac{dT_{i_m}}{T_{i_m}}$$

$$a_I \in \bigcup_{r \geq 0} \cancel{W} [T^{\frac{1}{p^r}}], \quad \forall I.$$

$$F(T_i) = T_i^p, \qquad V(T_i) = p T_i^{\frac{1}{p}}.$$

$$E^{\cdot} \subset \Omega^{\cdot}_{C/k}, \qquad F, V \curvearrowright E^{\cdot},$$

$$E^m = \{ \omega \in \Omega^m_{C/k} \mid \omega, d\omega \text{ are integral} \}$$

Set $E^m_n = E^m / V^n E^m + dV^n E^{m-1}$

Then

$$\begin{array}{ccccccc}
E_1^0 & \to & E_1^1 & \to & \cdots \to & E_1^d & : \; E_1^\cdot \\
\uparrow & & \uparrow & & & \uparrow & \quad \uparrow \\
E_2^0 & \to & E_2^1 & \to & \cdots \to & E_2^d & : \; E_2^\cdot \\
\uparrow & & \uparrow & & & \uparrow & \quad \uparrow \\
 & & & & & & \quad \vdots
\end{array}$$

is can. isomorphic to $W.\Omega^\cdot_{X_1}$.

e.g. $E_n^0 \simeq W_n A$

$$\begin{array}{ccc} E_n^0 & \simeq & W_n A \\ \Psi & & \Psi \\ T_i & \longmapsto & [T_i] \end{array}$$

(2). $\Omega^\cdot_{D_n} \xrightarrow[\sim]{\text{quasi-iso}} W.\Omega^\cdot_{X_1}$ is the following calculation

~~$\Omega^\cdot_{D_K}$~~ $\Omega^\cdot_B \hookrightarrow E^\cdot$

$$E^\cdot \cong \Omega^\cdot_B \oplus E^\cdot_{\text{frac}}.$$

and $E^\cdot_{\text{frac}} \overset{\text{homotopy}}{\simeq} 0$

(by Deligne).

look at $d=1$ case carefully:

Clearly $f \in K[T]$ is integral

and $df \in K[T]\,dT$ is integral

$$\Longleftrightarrow \quad f \in W[T]$$

Consider $f \in K[T^{\frac{1}{p}}]$.

write $W[T^{\frac{1}{p}}] = W[T]\{1, T^{\frac{1}{p}}, \dots T^{\frac{p-1}{p}}\}$

Thus $f = \sum_{i=0}^{p-1} a_i T^{\frac{i}{p}}$, $a_i \in W[T]$; ~~and~~

$$df = \sum_i da_i \cdot T^{\frac{i}{p}} + \sum \left(a_i \cdot \frac{i}{p}\right) T^{\frac{i}{p}} \frac{dT}{T}$$

is integral $\Longleftrightarrow \sum_{i=0}^{p-1} \left(T^{\frac{i}{p}} a_i \cdot \frac{i}{p}\right) \frac{dT}{T}$ integral

$$\Longleftrightarrow \sum_{i=0}^{p-1} a_i \cdot \frac{i}{p} T^{\frac{i}{p}} \in W[T^{\frac{1}{p}}]$$

$$\Longleftrightarrow \quad a_i \in pW[T],\ \forall i>0$$

Then, it is easy to see that

$$E^0 = W[T]\{1, pT^{\frac{1}{p}}, \dots pT^{\frac{p-1}{p}}, p^2T^{\frac{1}{p^2}}, \dots p^2T^{\frac{p-1}{p^2}}, p^3T^{\frac{1}{p^3}}, \dots\}.$$

$$= W[T] \oplus W[T]\{pT^{\frac{1}{p}}, \dots\}$$

$$E^1 = W[T]\,dT \oplus W[T]\{T^{\frac{1}{p}}, \dots, T^{\frac{p-1}{p}}, T^{\frac{1}{p^2}}, \dots\}\frac{dT}{T}$$

Then $E^\bullet = \Omega^\bullet_B \oplus E^\bullet_{Frac}$

define an obvious homotopy

$$\begin{tikzcd}
E^0_{Frac} \arrow[r] \arrow[d, "id"'] & E^1_{Frac} \arrow[d] \arrow[dl, "h=\int"'] \\
E^0_{Frac} \arrow[r] & E^1_{Frac}
\end{tikzcd}$$

note $h(T^s \cdot T^{\frac{1}{p^r}} \frac{dT}{T})$, $\forall s \geq 0,\ r \geq 1$

$$= p^r \underbrace{(1+p^r(s))^{-1}}_{\text{unit.}} T^s \cdot T^{\frac{1}{p^r}} \in E^0_{Frac}.$$

step 3 (BEK)

The step-up is as step 2. i.e $X = D = Z$. local.

Set. $\nu_\bullet = \nu_0 \geq \nu_1 \geq \dots \geq \nu_d \geq \nu_{d+1} \geq 0$

$$\nu_{i+1} \geq \nu_i - 1, \quad \nu_i < p, \quad \nu_{d+1} = \max\{0, \nu_d - 1\}.$$

define $q(\nu_\cdot)W_\cdot\Omega^{\cdot}_{X_1}$ by

$$q(\nu_i)^{\cdot}W_\cdot\Omega^{\cdot}_{X_1} = \begin{cases} p^{\nu_i}W_\cdot\Omega^{\cdot}_{X_1}, & \nu_i = \nu_{i+1} \\ p^{\nu_{i+1}}VW_\cdot\Omega^{\cdot}_{X_1}, & \nu_i = \nu_{i+1}+1 \end{cases}$$

going to show: $\left(\begin{array}{l}\text{Theorem is for} \\ \nu_i = \max\{0, r-i\}, \quad r<p.\end{array}\right)$

$$p^{\nu_\cdot}\Omega^{\cdot}_{X_\cdot} \xrightarrow{\sim} q(\nu_\cdot)W_\cdot\Omega^{\cdot}_{X_1}.$$

DO induction on $N = \Sigma \nu_i$

$N=0$ is step 2.

~~Let~~ i be the ~~place~~ such that

$$\nu_0 = \nu_1 = \cdots = \nu_i > \nu_{i+1}$$

Define $\mu_\cdot$: $\mu_j = \begin{cases} \nu_j, & j \geq i+1 \\ \nu_j - 1, & j \leq i \end{cases}$

Thus, by induction,

$$p^{\mu_\cdot}\Omega_{X_\cdot} \xrightarrow{\sim} q(\mu_\cdot)W_\cdot\Omega^{\cdot}_{X_1}$$

However, $$p^{\mu_\cdot}\Omega_{X_\cdot}/p^{\nu_\cdot}\Omega_{X_\cdot} \cong \mathcal{O}_{X_1} \to \cdots \to \Omega^{i}_{X_1}$$

$$q(\mu.)W\Omega^{\cdot}_{X_1} \big/ q(\nu.)W\Omega^{\cdot}_{X_1} \cong$$

$$W(\mathcal{O}_{X_1})/pW(X_1) \longrightarrow \cdots \longrightarrow W\Omega^{i-1}_{X_1}/pW\Omega^{i-1}_{X_1} \longrightarrow W\Omega^{i}_{X_1}/VW\Omega^{i}_{X_1}$$

(*). (reverse direction).

By Illusie, §1. Cor 3.20, the induced map (from something as

$$\begin{tikzcd}
W(\mathcal{O}_{X})/p \arrow[r] \arrow[d] & \cdots \arrow[r] & W\Omega^{i-1}_{X_1}/p \arrow[r] \arrow[d] & W\Omega^{i}_{X_1}/VW\Omega^{i}_{X_1} \arrow[d] \\
\mathcal{O}_{X_1} \arrow[r] & \cdots \arrow[r] & \Omega^{i-1}_{X_1} \arrow[r] & \Omega^{i}_{X_1}
\end{tikzcd}$$

is an quasi-isomorphism

Thus

$$\begin{tikzcd}
0 \arrow[r] & p^{\nu.}\Omega_{X} \arrow[r] \arrow[d] & p^{\mu.}\Omega_{X} \arrow[r] \arrow[d, "\wr"] & p^{\mu.}\Omega_{X}/p^{\nu.}\Omega_{X} \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & q(\nu.)W\Omega \arrow[r] & q(\mu.)W\Omega \arrow[r] & q(\mu.)W\Omega/q(\nu.)W\Omega_{X} \arrow[r] & 0
\end{tikzcd}$$

$\Rightarrow$ $p^{\nu.}\Omega_X \simeq q(\nu.)W\Omega.$

#