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Aim: introduce four complexes, syntomic $\mathfrak{S}_{X.}(r)$, and prove a triangle:

$$W.\Omega^r_{X_1,\log}[-r] \to q(r)W.\Omega^{\cdot}_{X_1} \xrightarrow{1-F_r} W.\Omega^{\cdot}_{X_1} \xrightarrow{[1]}$$

which is used to prove fundamental triangle.

$$p(r)\Omega^{<r}_{X.}[-1] \to \mathfrak{S}_{X.}(r) \to W.\Omega^r_{X_1,\log}[-r] \xrightarrow{[1]}.$$

§1. étale and Nisnevich top.. For a scheme X, associate 3 tops.

$$X_{et} \xrightarrow{\varepsilon} X_{Nis} \xrightarrow{\alpha} X_{Zar}$$

Refer to JinXing's talk for Nis topology. Note that $R^i\varepsilon_*\mathcal{F}$ does not necessarily vanish, but for coherent $\mathcal{F}$, $R^i\varepsilon_*\mathcal{F}=0$ for $i>0$.

§2: Setting.

- k char $=p$ field. $\cdot$ $W_n(k)$ witt ring.
- X_1 sm var$/k$. X_n sm$/W_n$.
- $$\begin{tikzcd} X_n \arrow[r, hook] \arrow[d] & Z_n \arrow[dl] \\ W_n & \end{tikzcd}$$ Z_n sm$/W_n$ with Frobenius $$\begin{tikzcd} Z_n \arrow[r, "F"] \arrow[d] & Z_n \arrow[d] \\ W_n \arrow[r, "F"] & W_n \end{tikzcd}$$ $F|_{Z_1} = \mathrm{Frob}_{Z_1}$
- D_n PD-envelope $D_{X_n}(Z_n)$. $\Omega^{\cdot}_{D_n/W_n} := \mathcal{O}_{D_n}\otimes_{\mathcal{O}_{Z_n}}\Omega^{\cdot}_{Z_n/W_n}$
- J_n ideal $X_n \hookrightarrow D_n$. $J_n^r := J_n^{[r]}$

  $I_n = (J_n, p)$, $\quad I_n^r = I_n^{[r]}$
- $W_n\Omega^r_{X_1,\log} = \left\langle \frac{d[x_1]}{[x_1]}\wedge\cdots\wedge\frac{d[x_r]}{[x_r]}\right\rangle$ $[x_i]$ Teichimu. lifting.
- $p^r: \mathcal{O}_{D_n} \hookrightarrow \mathcal{O}_{D_{n+r}}$ allow us $\underline{p^r\mathcal{O}_{D_{n+r}}} \xrightarrow{\frac{1}{p^r}} \mathcal{O}_{D_n}$

  $W_n \to W_{n+r}$ $\quad$ $p^rW. \xrightarrow{\frac{1}{p^r}} W.$

§3: 4 complexes. $(r < p)$

$J(r)\Omega_{D_\cdot}^{\cdot}: J_n^r \to J_n^{r-1}\otimes_{\mathcal{O}_{Z_n}}\Omega_{Z_n}^1 \to \cdots \to J\otimes\Omega_{Z_n}^{r-1} \to \mathcal{O}_{D_n}\otimes\Omega_{Z_n}^r \to \Omega_{D_n}^{r+1} \to \cdots$

$I(r)\Omega_{D}^{\cdot}: I_n^r \to \cdots$

$p(r)\Omega_{X_\cdot}^{\cdot}: p^r\mathcal{O}_{X_n} \to p^{r-1}\otimes\Omega_{X_n}^1 \to \cdots \to p\cdot\Omega_{X_n}^{r-1} \to \Omega_{X_n}^r \to \Omega_{X_n}^{r+1} \to \cdots$

$q(r)W\Omega_{X_1}^{\cdot}: p^{r-1}VW_{n-1}\mathcal{O}_{X_1} \to p^{r-2}VW_{n-1}\Omega_{X_1}^1 \to \cdots \to \cancel{p}VW_{n-1}\Omega_{X_1}^{r-1} \to W_n\Omega_{X_1}^r \to \cdots$ $(pdV = \cancel{V}d)$

Recall [BO] [Illusie]:

$$\begin{array}{ccc} \mathcal{O}_{D_n} & f & \leftrightarrow (f, F(f), F^2(f), \cdots F^{n-1}(f)) \\ \phi(F)\downarrow & \downarrow & \quad \| \quad\;\; \| \qquad\qquad\quad \| \\ W_n\mathcal{O}_{X_1} & ? & \leftrightarrow (w_1, w_2, \cdots\cdots\cdots w_n) \end{array}$$

$\rightsquigarrow \Omega_{D_n}^{\cdot} \overset{\phi(F)}{\underset{\sim}{\to}} W_n\Omega_{X_1}^{\cdot}$ $\quad J(r)\Omega_{D_n}^{\cdot} \xrightarrow{\sim} \Omega_{X_n}^{\geq r}$ and

$$\begin{array}{ccc} I(r)\Omega_{D_\cdot}^{\cdot} & \xrightarrow{\sim} & p(r)\Omega_{X_\cdot}^{\cdot} \\ \wr\downarrow\phi(F) & \overset{\sim}{\nearrow} * & \\ q(r)W_\cdot\Omega_{X_1}^{\cdot} & & \end{array}$$

$(*)$ is independent of Z_n.

de-Rham Witt Complex: $W_n\mathcal{O}_{X_1} \to W_n\Omega_{X_1}^1 \to W_n\Omega_{X_1}^2 \to \cdots$

$W_n\Omega_{X_1}^i = \Omega^i_{W_n\mathcal{O}_X}/\sim$ wt. $\underline{F}: W_{n+1} \to W_n$

$\underline{W_n = W/V^n + dV^n}$

$\underline{V}: W_m \to W_{n-1}$

$\underline{FV = p}: W_n \to W_{n+1}$

$\underline{FdV = d}$.

§4. Syntomic complex.

Construction: ~~Consider~~: $J(r)\cdot\Omega_{D_{n+r}}^{\cdot} \xrightarrow{F} \Omega_{D_{n+r}}^{\cdot}$, ~~we have~~

We can prove: $Im(F) \subseteq p^r\Omega_{D_{n+r}}^{\cdot}$, precisely:

for $J(r)\Omega_{D_{n+r}}^i$

$\downarrow$

$k \geq r,\ x \in J.$ $\quad x^{[k-i]}dx_1\wedge\cdots\wedge dx_i \overset{F}{\underset{\substack{x_1\mapsto x_1^p+py \\ \vdots \\ x_i\mapsto x_i^p+py_i}}{\longmapsto}} \boxed{\frac{(x^p+py)^{k-i}}{(k-i)!}}\cdot p^i dz_1\wedge\cdots\wedge dz_i$ $\quad = p^{k-i}\cdot ?$

§4. Continued: So we get factorizat:

$$\begin{tikzcd} J(r)\Omega^{\cdot}_{D_{n+r}} \arrow[r, "F"] \arrow[d, dotted] & p^r\cdot\Omega^{\cdot}_{D_{n+r}} \arrow[r, hook] & \Omega^{\cdot}_{D_{n+r}} \\ J(r)\Omega^{\cdot}_{D_n} \arrow[r, dashed, "\boxed{f_r}"] & \Omega^{\cdot}_{D_n} \arrow[u, "s\times p^r"'] & \end{tikzcd}$$

Syntomic complex: $\mathcal{S}_{X_\cdot}(r)_{et} = \mathrm{Cone}(J(r)\Omega^{\cdot}_{D_\cdot} \xrightarrow{1-f_r} \Omega^{\cdot}_{D_\cdot})[-1]$.

On Nis top: $\mathcal{S}_{X_\cdot}(r) = \tau_{\le r} \cdot R\varepsilon_* \mathcal{S}_{X_\cdot}(r)_{et}$

§5: Construct triangle:

$$W_\cdot\Omega^r_{X_1,\log}[-r] \to q(r)W_\cdot\Omega^{\cdot}_{X_1} \xrightarrow{1-F_r} W_\cdot\Omega^{\cdot}_{X_1} \xrightarrow{[1]}.$$

$$\begin{tikzcd} p^{r-1}VW_\cdot\mathcal{O}_{X_1} \arrow[r] \arrow[d] & p^{r-2}VW_\cdot\Omega^1_{X_1} \arrow[r] \arrow[d] & \cdots \arrow[r] & VW_n\Omega^{r-1}_{X_1} \arrow[r] \arrow[d] & W_n\Omega^r_{X_1} \arrow[r] \arrow[d, dotted] & W_\cdot\Omega^{r+1}_{X_1} \arrow[r] \arrow[d] & \cdots \\ W_\cdot\mathcal{O}_{X_1} \arrow[r] & W_\cdot\Omega^1_{X_1} \arrow[r] & \cdots \arrow[r] & W_\cdot\Omega^{r-1}_{X_1} \arrow[r] & W_\cdot\Omega^r_{X_1} \arrow[r] & W_\cdot\Omega^{r+1}_{X_1} \arrow[r] & \cdots \end{tikzcd}$$

strategy: Breaking into:

$$\begin{tikzcd} \tau_{<r}C^{\cdot} \arrow[r] \arrow[d, "②"] & C^{\cdot} \arrow[r] \arrow[d, dotted] & \tau_{\ge r}C^{\cdot} \arrow[r, "{[1]}"] \arrow[d, "①"] & {} \\ \tau_{<r}D^{\cdot} \arrow[r] & D^{\cdot} \arrow[r] & \tau_{\ge r}D^{\cdot} \arrow[r, "{[1]}"] & {} \end{tikzcd}$$

§5.1. ①:

$$\begin{tikzcd} 0 \arrow[r] & W_n\Omega^r_{X_1}/dVW_{n-1}\Omega^{r-1}_{X_1} \arrow[r] \arrow[d, "1-F_r"] & W_n\Omega^{r+1}_{X_1} \arrow[r] \arrow[d, "1-pF"] & W_n\Omega^{r+2}_{X_1} \arrow[r] \arrow[d, "1-p^2F"] & \cdots \\ 0 \arrow[r] & W_n\Omega^r_{X_1}/dW_n\Omega^{r-1}_{X_1} \arrow[r] & W_n\Omega^{r+1}_{X_1} \arrow[r] & W_n\Omega^{r+2}_{X_1} \arrow[r] & {} \end{tikzcd}$$

F_r:

$$\begin{tikzcd} V^n + dV^n \arrow[r, hook] \arrow[d] & W_{n+1}\Omega^r_{X_1} \arrow[r] \arrow[d, "F"] & W_n\Omega^r_{X_1} \arrow[r] \arrow[d, dotted, "F_r"] \arrow[dl, dotted] & W_n\Omega^r_{X_1}/dVW_{n-1}\Omega^{r-1}_{X_1} \arrow[d, dotted, "F_r"] \\ V^{n-1} + dV^{n-1} \arrow[r, hook] & W_n\Omega^r_{X_1} \arrow[r] & W_n\Omega^r_{X_1}/dV^{n-1}W_1\Omega^{r-1}_{X} \arrow[r] & W_n\Omega^r_{X_1}/dW_n\Omega^{r-1}_{X} \end{tikzcd}$$

§5.1. Continued.

Lemma 4.4: For $i>0$. $(1-p^iF): W_n\Omega^{r+i}_{X_1} \xrightarrow{\sim} W_n\Omega^{r+i}_{X}$.

Pf: $(1-p^iF)x=\beta$: $\quad x=(1+p^iF+p^{2i}F^2+\cdots+p^{2n}F^{2n})\beta$.

Lemma 4.3: On $X_{ét}$, we have exact sequence:

(*). $$0\to W_n\Omega^r_{X_1,\log}\to W_n\Omega^r_{X_1}/dVW_{n+1}\Omega^{r-1}_{X_1}\xrightarrow{1-F_r} W_n\Omega^r_{X_1}/dW_n\Omega^{r-1}_{X_1}\to 0$$

(**) On X_{Nis}: $0\to W_n\Omega^r_{X_1,\log}\to W_n\Omega^r_{X_1}/dV \xrightarrow{1-F_r} W_n\Omega^r_{X_1}/dW_n$ exact.

Pf: (*) $X_{ét}$

$$\begin{tikzcd}
 & 0 \arrow[r] & \boxed{} \arrow[r] & W_n\Omega^r_{X_1}/dVW_{n+1}\Omega^{r-1}_{X_1} \arrow[r, "1-F_r"] & W_n\Omega^r_{X_1}/dW_n \arrow[r] & \boxed{} \arrow[r] & 0 \\
\boxed{[\text{Illusie}]\Rightarrow} & 0 \arrow[r] & W_n\Omega^r_{X_1,\log} \arrow[r] & W_n\Omega^r_{X_1} \arrow[r, "1-F_r"] \arrow[u, two heads] & W_n\Omega^r_{X_1}/dV^{n-1} \arrow[r] \arrow[u, two heads] & 0 & \\
 & & & dVW_{n+1}\Omega^{r-1}_{X_1} \arrow[r, "\phi: 1-F_r"] \arrow[u, hook] & dW_n\Omega^{r-1}_{X_1}/dV^{n-1}\Omega^r_{X_1} \arrow[u, hook] & &
\end{tikzcd}$$

Applying snake lemma; enough to prove:

$$dVW_{n+1}\Omega^{r-1}_{X_1}\xrightarrow{\phi} dW_n\Omega^{r-1}_{X_1}/dV^{n-1}\Omega^r_{X_1} \text{ is isomorphic.}$$

Consider: V:
$$\begin{tikzcd}
dW_n\Omega^{r-1}_{X_1} \arrow[r] & pdVW_n\Omega^{r-1}_{X_1} \arrow[r, hook] & W_{n+1}\Omega^r_X \\
dW_n\Omega^{r-1}_{X_1}/dV^{n-1} \arrow[r, "\psi"] & dVW_{n+1}\Omega^{r-1}_{X_1} \arrow[u, "\times p"] &
\end{tikzcd}$$

$\psi: d\bar\alpha_n \mapsto V\bar\alpha_n \qquad \psi=\frac{V}{p},\ \psi^n=0$

Check: $\phi(\underbrace{\psi+\cdots+\psi^{n-1}}_{\text{is } -\phi^{-1}})=-1$. done!

§ 5.1. Continued.

Prove: (**): Applying $R\varepsilon_* : D(X_{et}) \to D(X_{Nis})$, since ε_* is left exact, we only need to verify:

$$\varepsilon_*\left(W_n\Omega^r_{X_1}/dV\right) = W_n\Omega^r_{X_1}/dV.$$

This follows since this is a quotient of coherent sheaves.

Lemma 4.3 + 4.4 $\Rightarrow$

(5.1) $W_n\Omega^r_{X_1,\log}[-r] \to \tau_{\geq r}(\mathcal{G}(r)W_n\Omega^\bullet_{X_1}) \to \tau_{\geq r}(W_n\Omega^\bullet_{X_1}) \xrightarrow{[1]} \cdots$

§ 5.2. ②: (5.2) $\tau_{<r}\, \mathcal{G}(r)W_n\Omega^\bullet_{X_1} \xrightarrow[\sim]{1-F_r} \tau_{<r}(W_n\Omega^\bullet_{X_1})$ isomorph-

$$\begin{array}{ccccccccccc}
0 \to & p^{r-1}VW_\bullet\mathcal{O}_{X_1} & \to & p^{r-2}VW_\bullet\Omega^1_{X_1} & \to \cdots \to & VW_\bullet\Omega^{r-1}_{X_1} & \to & dVW_\bullet\Omega^{r-1}_{X_1} & \to 0 \\
& \downarrow 1-\frac{F}{p^r} & & \downarrow 1-\frac{F}{p^{r-1}} & & \downarrow 1-\frac{F}{p} & & \downarrow & \\
0 \to & W_\bullet\mathcal{O}_{X_1} & \to & W_\bullet\Omega^1_{X_1} & \to \cdots \to & W_\bullet\Omega^{r-1}_{X_1} & \to & dW_\bullet\Omega^{r-1}_{X_1} & \to 0
\end{array}$$

$$\begin{array}{ccc}
VW_\bullet\Omega^{r-1}_{X_1} & \longrightarrow & W_\bullet\Omega^{r-1}_{X_1}, \\
\downarrow & & \downarrow \\
(V\alpha_{n-1})_n & \longmapsto & (V\alpha_{n-1} - \alpha_n)_n \\
\| & & \| \\
(0, V\alpha_1, \dots) & & (\beta_n)_n = (-\alpha_1, V\alpha_1 - \alpha_2, \dots)
\end{array}$$

$(V\beta - \quad)$ inverse map

$= -(0, V\beta_1, V\beta_2 + V^2\beta_1, V\beta_3 + V^2\beta_2 + V^3\beta_1, \dots)$

$F(V\alpha_{n-1}) = p\alpha_{n-1}$

$$\begin{array}{ccc}
F : VW_{n-1}\Omega^{r-1}_X & \to & W_{n-1}\Omega^{r-1}_X \\
& \searrow & \downarrow \\
& & W_n\Omega^{r-1}_X
\end{array}$$

By (5.1) + (5.2). we prove.

$$W_\bullet\Omega^r_{X_1,\log}[-r] \to q(r)W_\bullet\Omega^\bullet_{X_1} \xrightarrow{1-F_r} W_\bullet\Omega^\bullet_{X_1} \xrightarrow{[1]}$$

§ 5.3: By construction we have

$$\begin{tikzcd}
J(r)\Omega^r_{D_n} \arrow[r, "f_r = \frac{F}{p^r}"] \arrow[d, "\phi(F)"] & \Omega^r_{D_n} \arrow[d, "\phi(F)"] \arrow[r, "f_r"] & \Omega^r_{W_n\mathcal{O}_X} \ \cancel{\text{and}} \arrow[dl] \\
W_n\Omega^r_{X_1} \arrow[r, "F_r"] & W_n\Omega^r_X / dV^{n-1}\Omega^r_{X_1} &
\end{tikzcd}$$

$$\begin{tikzcd}
J(r)\Omega^\bullet_{D_\bullet} \arrow[r, "f_r"] \arrow[d, "\phi(F)"] & \Omega^\bullet_{D_\bullet} \arrow[d, "\phi(F)"] \\
q(r)W_\bullet\Omega^\bullet_{X_1} \arrow[r, "F_r"] & W_\bullet\Omega^\bullet_X
\end{tikzcd}$$