Nisnevich Topology

Definition: Suppose X is a scheme. A family of morphisms between schemes $\{f_\alpha : Y_\alpha \to X\}_{\alpha \in I}$ is called a Nisnevich cover, if each f_α is an étale morphism and $\forall x \in X$, $\exists$ $\alpha \in I$, $\exists y \in Y_\alpha$, s.t. $k(x) \overset{f^\#}{\underset{\sim}{\to}} k(y)$, and $X = \bigcup_{\alpha \in I} f(Y_\alpha)$.

Remark: By definition, any Zariski cover is a Nisnevich cover, and any Nisnevich cover is an ~~étale~~ étale cover.

Definition: Suppose X is a scheme. The Nisnevich topology X_{Nis} is defined to be the Grothendieck topology whose underlying category is $\{U \xrightarrow{f} X \mid f \text{ étale}\}$, and whose coverings are Nisnevich covers.

If we denote the Zariski and étale topology on X by X_{Zar}, $X_{ét}$ respectively, then we have morphisms between Grothendieck topologies:

$$X_{ét} \to X_{Nis} \to X_{Zar}.$$

Intuitively, the Nisnevich ~~topology~~ is finer than Zariski topology and coarser than étale topology.

Example: Suppose k is a field. $X = \operatorname{Spec} k$.
Then a Nisnevich presheaf on X is also a Nisnevich sheaf, since Nisnevich covers of X are trivial.
~~Moreover, we have the following equivalence of categories:~~

Moreover, it is easy to see that, a Nisnevich sheaf on $X = \mathrm{Spec}\, k$ is equivalent to a compatible $\mathrm{Gal}(L/k)$ modules M_L, when L goes over finite Galois extensions of k in a fixed seperable closure k^s.

Here by compatible $\mathrm{Gal}(L/k)$-module we mean the following diagram is commutative, for any $k \subset L \subset K$.

$$
\begin{array}{ccccc}
\mathrm{Gal}(K/k) & \times & M_K & \longrightarrow & M_K \\
\downarrow & & \uparrow & & \uparrow \\
\mathrm{Gal}(L/k) & \times & M_L & \longrightarrow & M_L
\end{array}
$$

On the other hand, an étale sheaf on $X = \mathrm{Spec}\, k$ is equivalent to a $\mathrm{Gal}(k^s/k)$-module M, and also equivalent to a compatible $\mathrm{Gal}(L/k)$-modules M_L, s.t. $k \subset L \subset k^s$ and L/k finite Galois, and for any $k \subset L \subset K \subset k^s$, we have $(M_K)^{\mathrm{Gal}(K/L)} = M_L$.

By the discussion above, we have

Proposition 1: If k is a field, $\mathcal{F}$ is a Nisnevich sheaf on $\mathrm{Spec}\, k$, then $H^i(\mathrm{Spec}\, k, \mathcal{F}) = 0$, $\forall\, i > 0$.

§ sheafification and stalks.

If $\mathcal{F}$ is a presheaf on X_{Nis}, we can get a Nisnevich sheaf $\tilde{\mathcal{F}}$ on X_{Nis} satisfies universal property as that the similar well-know in Zariski and étale topology.

the construction of $\tilde{\mathcal{F}}$ is also similar. Let us recall the construction briefly.

Fistly, for any Nisnevich cover $\mathcal{U}$ of a scheme, we define $\check{H}^0(\mathcal{U}, \mathcal{F}) = \ker (C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}))$ as the Čech 0-th cohomology group of $\mathcal{F}$ ~~respectively~~ with respect to $\mathcal{U}$.

Let $\mathcal{F}^+$ be the Nisnevich presheaf such that for any étale $U \to X$, $\mathcal{F}^+(U) = \check{H}^0(U, \mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^0(\mathcal{U}, \mathcal{F}|_U)$. ~~where~~

$\mathcal{U}$ Nisnevich cover of U.

Then $\tilde{\mathcal{F}} := (\mathcal{F}^+)^+$ is ~~the shea~~ a Nisnevich sheaf on X_{Nis} and it is just the sheafification of $\mathcal{F}$.

As usual, for an injective morphism between Nisnevich sheaves $\mathcal{F} \hookrightarrow \mathcal{G}$, we define $\mathcal{G}/\mathcal{F}$ to be the sheafification of the presheaf $U \to \mathcal{G}(U)/\mathcal{F}(U)$.

Then it is easy to see sheaves on X_{Nis} form an abelian category, in particular, we can say short exact sequence: $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$.

We can characterize short exact sequence by something like stalks.

If $\mathcal{F}$ is a (pre) sheaf on X_{Nis}, for $x \in X$ and a finite seperable field extension $k(x) \hookrightarrow k$, we define $\mathcal{F}_{x,k} = \varinjlim_{U} \mathcal{F}(U)$, where in the direct limit, U goes over the diagram

$$\begin{tikzcd} & U \arrow[dr, "\text{ét}"] & \\ \operatorname{Spec} k \arrow[ur] \arrow[rr] & & X \end{tikzcd}$$

such that $U \to X$ étale and $\operatorname{Spec} k \to X$ corresponds to $k(x) \hookrightarrow k$.

Prop 2: A Nisnevich sheaf $\mathcal{F} = 0 \iff \forall x \in X$, $\forall$ finite seperable $k(x) \hookrightarrow k$, $\mathcal{F}_{x,k} = 0$.

proof: By definition. #

Prop 3: A complex of Nisnevich sheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence if and only if: $\forall x \in X$, $\forall$ finite seperable $k(x) \hookrightarrow k$, $0 \to \mathcal{F}_{1,x,k} \to \mathcal{F}_{2,x,k} \to \mathcal{F}_{3,x,k} \to 0$ is exact.

proof: This follows from the definitions of sheafification and stalks. #

Prop 4: If $f: Y \to X$ is a finite morphism, $\mathcal{F}$ is a sheaf on Y_{Nis}, $x \in X$. Then

$$(f_* \mathcal{F})_{x, k(x)} \simeq \bigoplus_{y \in f^{-1}(x)} \mathcal{F}_{y, k(y)}.$$

proof: If $A \to B$ is a finite extension of rings, and (A, m) is a local ring, then

$$B \otimes_A A^h \simeq \bigoplus_{n_i} B^h_{n_i}.$$

where A^h is the henselization of A. and n_i goes over the prime (maximal) ideals of B over m. For a proof of this purely algebraic lemma, see [Fu Lei, Etale Cohomology Theory, proposition 2.8.12].

Then one can argue similarly as that in §3 in [Freitag, Kiehl, Etale Cohomology and the Weil Conjecture] to conclude. #

Cor 5: Suppose $f: Y \to X$ is a finite morphism. Then f_* is an exact functor from Nisnevich sheaves on Y to Nisnevich sheaves on X.

proof: For any étale $U \to X$, apply prop 4 to $U \times_X Y \to U$. #

~~Now we consider some~~

§ Vanishing theorems.

We want to show an analogy of Grothendieck Vanishing Theorm for Nisnevich sheaves. Let us firstly consider some special Nisnevich covers.

Def: A commutative square of the form

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ A & \xhookrightarrow{i} & X \end{array}$$

is called upper distinguished if $B = A \times_X Y$, f is étale, $i: A \to X$ is an open embedding and $Y - B \to X - A$ is an isomorphism. Here we put the reduced closed subscheme structure on $X-A$ and $Y-B$.

Prop 6: A presheaf $\mathcal{F}$ is a Nisnevich sheaf if and only if
$$\begin{array}{ccc} F(B) & \leftarrow & F(Y) \\ \uparrow & & \uparrow \\ F(A) & \leftarrow & F(X) \end{array}$$
is Cartesian

for every upper distinguished square

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

proof: See [Mazza, Voevodsky, Weibel, Lecture Notes on Motivic Cohomology], Lemma 12.7. #

Cor 7: If $\mathcal{F}$ is a Nisnevich sheaf on X, and

$$\begin{array}{ccc} B & \longrightarrow & Y \\ \downarrow & & \downarrow f \\ A & \xrightarrow{i} & X \end{array}$$

is an upper distinguished square, then we have the following "generalized" Mayer-Vietoris long exact sequence:

$$\cdots \to H^i_{Nis}(X,\mathcal{F}) \to H^i_{Nis}(A,\mathcal{F}) \oplus H^i_{Nis}(Y,\mathcal{F}) \to H^i_{Nis}(A\times_X Y,\mathcal{F})$$
$$\to H^{i+1}_{Nis}(X,\mathcal{F}) \to \cdots$$

proof: See [F. Morel, V. Voevodsky, A^1- Homotopy theroy of schemes, Publications mathématiques de l'I.H.É.S., tome 90 (1999), p. 45-143.] Remark 3.1.7. #

Now we can state a vanishing theorem about Nisnevich sheaves:

Theorem 8: Let X be a Noetherian scheme of dimension $\leq d$, then for any sheaf of Abelian groups $\mathcal{F}$ on X_{Nis}, one has $H^i_{Nis}(X, \mathcal{F}) = 0$ for $i > d$.

proof: Consider the natural morphism $\varphi: X_{Nis} \to X_{Zar}$. The Grothendieck vanishing theorem and Leray spectral sequence implies that it is sufficient to prove $\operatorname{codim} \operatorname{supp} R^j\varphi_* \mathcal{F} \geq j$, for any $j \geq 1$.

By taking direct limits, it is not difficult to see that we only need to prove the following ~~lemma~~.

~~Lemma~~ Claim: If $S = \operatorname{Spec} A$, (A, m) is a Noetherian local ring of dimension d, Suppose $\mathcal{F}$ is a sheaf of Abelian groups on S_{Nis}, then

$$H^i_{Nis}(S, \mathcal{F}) = 0 \quad \text{for } i > d.$$

proof of claim: Let $U = S \setminus \{m\}$ be the open subset of S removing the closed point m.

Let $\mathcal{F} \to I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} \cdots \to I^i \xrightarrow{d^i} I^{i+1} \to \cdots$

be the injective resolution of $\mathcal{F}$, and $t \in H^0_{Nis}(S, I^i)$ with $d^i t = 0$.

then there is an étale $V \longrightarrow S$ such that

$$\begin{tikzcd} U \times_S V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r, hook] & S \end{tikzcd}$$

is an upper distinguished square,

and $t|_V = d^{i-1}(u)$. for some $u \in H^0_{Nis}(V, I^{i-1})$.

Then by the Mayer-Vietoris sequence in Cor 7, we can see $t = 0 \in H^i_{Nis}(X, \mathcal{F})$. Here we used the inductive hypothesis and the fact that $\dim U \leq d-1$

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Remark: The above proof is essentially from [Morel - Voevodsky, $\mathbb{A}^1$-Homotopy theory of schemes, publ. I.H.E.S. 99 (1990)] Prop. 3.1.8.

§ Pushforward and Pull-back between $X_{\acute{e}t}$ and X_{Nis}.

We have a natural morphism $\varepsilon : X_{\acute{e}t} \to X_{Nis}$.

Obviously we have:

Prop: If $\mathcal{F}$ is a sheaf on $X_{\acute{e}t}$. then for any $U \to X$ étale, $\varepsilon_* \mathcal{F}(U) = \mathcal{F}(U)$, moreove, we have $\varepsilon^* \varepsilon_* \mathcal{F} = \mathcal{F}$.

Prop: ε^* is an exact functor from sheaves on X_{Nis} to sheaves on $X_{\acute{e}t}$.

proof: we can check this on stalks as usual. #

For a complex of sheaves $\mathcal{F}^{\cdot}$, we define the canonical truncation $\tau_{\le r}\mathcal{F}^{\cdot}$ to be

$$\to \mathcal{F}^{r-2} \to \mathcal{F}^{r-1} \to \ker d_r \to 0 \cdots$$

where $d_r : \mathcal{F}^r \to \mathcal{F}^{r+1}$ is the homomorphism in $\mathcal{F}^{\cdot}$.

Then $\mathcal{H}^i(\tau_{\le r}\mathcal{F}^{\cdot}) = \begin{cases} \mathcal{H}^i(\mathcal{F}^{\cdot}), & \forall\, i \le r \\ 0, & \forall\, i > r \end{cases}$.

and we have a natural inclusion $\tau_{\le r}\mathcal{F}^{\cdot} \to \mathcal{F}^{\cdot}$.

Prop: If $\mathcal{F}^{\cdot} \in D_+(X_{\acute{e}t})$, the derived category of complex of sheaves on $X_{\acute{e}t}$ bounded from below, and suppose $\mathcal{H}^i(\mathcal{F}^{\cdot}) = 0$, $\forall\, i > r$, then

$$\mathcal{F}^{\cdot} \simeq \varepsilon^* \tau_{\le r} R\varepsilon_* \mathcal{F}^{\cdot}$$

proof: Choose an injective resolution $\mathcal{F}^{\cdot} \to I^{\cdot}$, then $\tilde{\mathcal{F}}^{\cdot} := \varepsilon^* \tau_{\le r} R\varepsilon_* \mathcal{F}^{\cdot} = \varepsilon^* \tau_{\le r} \varepsilon_* I^{\cdot}$, so $\tilde{\mathcal{F}}^i = I^i$, $\forall\, i < r$.
Since ε_* is left exact, we see $\tilde{\mathcal{F}}^{\cdot} = \tau_{\le r} I^{\cdot}$. so $\mathcal{F}^{\cdot} \to I^{\cdot}$ and $\tilde{\mathcal{F}}^{\cdot} \to I^{\cdot}$ are quasi-isomorphisms. #

Now we consider the natural morphism $\varphi : X_{Nis} \to X_{zar}$.

We say a sheaf of $\mathcal{O}_X$-module $\mathcal{F}$ on X_{Nis} is quasi-coherent, if there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}'$ on X_{zar}, such that $\mathcal{F} \simeq \varphi^* \mathcal{F}' = \varphi^{-1}\mathcal{F}' \otimes_{\mathcal{O}_{X,zar}} \mathcal{O}_{X,Nis}$.

Similarly as étale site, we have the following vanishing theorem about quasi-coherent sheaves on X_{Nis}.

Theorem: Suppose $X = \operatorname{Spec} A$ is affine, and $\mathcal{F}$ is a quasi-coherent sheaf on X_{Nis}. Then

$$H^i_{Nis}(X, \mathcal{F}) = 0, \quad \text{for } i > 0.$$

proof: Similarly as in the étale case, we only need to show: If $Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ is a Nisnevich cover, then $\check{H}^i(\mathcal{U}, \mathcal{F}) = 0$, for $i > 0$. Here $\mathcal{U} = \{Y \to X\}$ is the Nisnevich cover of X.

Let $M = \Gamma(X, \mathcal{F})$, then it is easy to see $\Gamma(Y, \mathcal{F}) = M \otimes_A B$, and $\Gamma(\underbrace{Y \times_X Y \times_X \cdots \times_X Y}_{n}, \mathcal{F}) = M \otimes_A \underbrace{B \otimes_A \cdots \otimes_A B}_{n}$.

So the Čech complex

$$\mathcal{F}(X) \to C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

is isomorphic to the A-modules complex

$$M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to \cdots$$

which is exact since $A \to B$ is a faithfully flat extension.

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