# JOINT CNU-USTC-SUST SEMINAR ON $p$-ADIC DEFORMATION OF ALGEBRAIC CYCLE CLASSES AFTER BLOCH-ESNAULT-KERZ

## 1. Introduction

Our joint seminar program is to understand the paper [4] that study the *$p$-adic variation of Hodge conjecture* of Fontaine-Messing ([6], see also [4, Conjecture 1.2]) which states roughly as follows: For simplicity, let $X/W$ be a smooth projective scheme, with $W = W(k)$ the Witt ring of a perfect field $k$ of characteristic $p > 0$.[1] If the crystalline cycle class of an algebraic cycle $Z_1$ in the special fiber $X_1 = X \times k$ has the correct Hodge level under the canonical isomorphism of Berthelot-Ogus

$$H^*_{\mathrm{cris}}(X_1/W) \cong H^*_{\mathrm{dR}}(X/W), \tag{1.0.1}$$

then it can be lifted to an algebraic cycle $Z$ of $X$. This problem is known when the dimension of an cycle is zero dimensional (trivial by smoothness) and codimension one case (Betherlot-Ogus [2, Theorem 3.8]). The article [4] proved that, under some technical condition arising from integral $p$-adic Hodge theory, once the Hodge level is correct, the cycle can be *formally* lifted to the $p$-adic formal scheme $X_\cdot = $ "$\varinjlim_n$" $X \otimes W/p^n$ of $X$, and it remains to be an open problem that whether there exists some formal lifting which is algebraic.

The $p$-adic variation of Hodge conjecture is a $p$-adic analogue of the *infinitesimal variation of Hodge conjecture* ([4, Conjecture 1.1]): in this conjecture the base ring is the formal disk, i.e., the spectrum of $k[[t]]$ where $k$ is a characteristic zero field, and the Berthelot-Ogus isomorphism (1.0.1) is replaced by the trivialization of Gauß-Manin connection over the formal disk ([13, Proposition 8.9])

$$(H^*_{\mathrm{dR}}(X_1/k) \otimes k[[t]], id \otimes d) \cong (H^*_{\mathrm{dR}}(X/k[[t]]), \nabla_{\mathrm{GM}}). \tag{1.0.2}$$

It can be shown that the infinitesimal variation of Hodge conjecture is equivalent to the original version of Grothendieck on variation of Hodge conjecture ([11], page 103 footnote), which is indeed a direct consequence of Hodge conjecture after Deligne's global invariant cycle theorem. Moreover, it was shown by Maulik-Poonen ([19, Theorem 9.10]) that the $p$-adic variation of Hodge conjecture implies the variation of Hodge conjecture.

[1] The actual $p$-adic variation of Hodge conjecture is stated for any $p$-adic complete discrete valuation ring of perfect residue field.

## 2. Part I: Warm up

The aim of this part is to recall some results in crystalline theory used in [4], such as de Rham complex, de Rham-Witt complex and the crystalline cycle map. We would also like to take this chance to discuss the relations of the *Hodge conjecture*, and the various *variation of Hodge conjecture*.

**Talk 1: Variation of Hodge conjecture.**

The goal of this talk is to clarify the relations of variation of Hodge conjecture with Hodge conjecture, infinitesimal Hodge conjecture and $p$-adic variation of Hodge conjecture, and to state the main results in [4]. Possible topics include:

- Statement of Hodge conjecture, Grothendieck's variation of Hodge conjecture, Deligne's global invariant cycle theorem, Cattani-Deligne-Kaplan's algebraicity theorem and some renowned open problems (See [4, p. 674-675] for relevant references of these results).
- Equivalence of variation of Hodge conjecture and infinitesimal variation of Hodge conjecture.
- Explain the line bundle case of the infinitesimal variation of Hodge conjecture following the strategy of [4, p. 573].
- Explain that $p$-adic variation of Hodge conjecture implies variation of Hodge conjecture ([19, Theorem 9.10]).
- State the main results of the work [4] on $p$-adic variation of Hodge conjecture.

References: [4], [11], [6], [5].

**Talk 2: Continuous cohomology.**

The goal of this talk is to explain continuous etale cohomology theory of U. Jannsen, and to complete [4, Appendix B]. Later, we shall need this formalism for crystalline cohomology.

Prerequisites: some knowledge on unbounded complexes and model category for [4, Appendix B].

References: [22], [4, Appendix B].

**Talk 3: A reminder in crystalline cohomology.**

The goal is to collect some basics on crystalline cohomology and the theory of de Rham-Witt complex. Possible topics include:

- Review briefly the theory of crystalline cohomology: divided powers, divided power envelope, crystalline site, crystalline topos, crystalline cohomology over a nilpotent base, and calculation of crystalline cohomology using de Rham complex when the scheme is liftable ([1]).
- de Rham-Witt complex, and the computation of crystalline cohomology with de Rham-Witt complex.
- Crystalline cohomology over a complete base such as the Witt ring $W = W(k)$ (one may use continuous cohomology as recalled in [4, Appendix B7], see also [22]).

- Berthelot-Ogus's isomorphism relating crystalline cohomology with the de Rham cohomology ([2]).

**Talk 4: Crystalline cycle map.**

The aim of this talk is to explain the construction of cycle map

$$\mathrm{ch} : K_0(X_1) \longrightarrow \bigoplus_r H^{2r}_{\mathrm{cris}}(X_1/W)_{\mathbb{Q}}.$$

References:[10],[9]

**Talk 5: Line bundle case of the $p$-adic variation of Hodge conjecture.**

The goal of this talk is to prove the $p$-adic variation of Hodge conjecture in the line bundle case ([2, Theorem 3.8]).

References: [2, §3].

**Talk 6: Crystalline cohomology and de Rham cohomology.**

The aim of this talk is to introduce the main variants of de Rham complex and the de Rham-Witt complex following [4, §2]. Possible topics include:

- Recall the definition of Nisnevich site, and the categories $C_{\mathrm{pro}}(X_1)_{\text{ét/Nis}}$ and $D_{\mathrm{pro}}(X_1)_{\text{ét/Nis}}$. Define the pro-system of de Rham complex $\Omega^\bullet_{X.}$ and $W_\bullet\Omega^\bullet_{X_1}$ the pro-system of de Rham-Witt complexes for the étale and Nisnevich topology. Define the pro-system of étale/Nisnevich subsheaves $W_\bullet\Omega^r_{X_1,\log}$ of $n \mapsto W_n\Omega^r_{X_1}$.
- Compare $W_n\Omega^r_{X_1,\mathrm{Nis}}$ with its étale analogue $W_n\Omega^r_{X_1,\text{ét}}$ relative to the morphism of sites $\epsilon : X_{1,\text{ét}} \to X_{1,\mathrm{Nis}}$ ([4, Proposition 2.4]).
- For $r < p$, define $p(r)\Omega^\bullet_{X.}$ and $q(r)W_\bullet\Omega^\bullet_{X_1}$.
- Explain the canonical quasi-isomorphisms $\Omega^\bullet_{X.} \simeq W_\bullet\Omega^\bullet_{X_1}$, and $p(r)\Omega_{X.}^\bullet \simeq q(r)W_\bullet\Omega^\bullet_{X_1}$ for $r < p$ ([4, Proposition 2.8]).

References:[1], [4, §2]

## 3. Part II Obstructions of lifting algebraic cycles

We plan to cover §3-§8 of [4]. The crux of this part is Theorem 8.5 (1), which interprets the kernel of the Hodge-theoretical obstruction map

$$Ch^r(X_1) \stackrel{ob}{\to} H^{2r}_{cts}(X., p(r)\Omega^{<r}_{X.})$$

in terms of certain Chow group $\mathrm{CH}^r_{\mathrm{cont}}(X./W.)$ of the pro-scheme $X./W.$. Recall that $p(r)\Omega^{<r}_X$ is a subcomplex of $\Omega^{<r}_X$ and the obstruction map $ob$ comes from the composite of the following natural maps

$$\mathrm{CH}^r(X_1) \xrightarrow{cl_{cris}} H^{2r}_{cris}(X_1/W) \stackrel{\Phi^{-1}}{\cong} H^{2r}_{dR}(X/W) \to H^{2r}_{dR}(X/W)/Fil^r H^{2r}_{dR}(X/W).$$

Granted the fact that $\mathrm{CH}^r(X_1)$ is the cohomology of the so-called motivic complex $\mathbb{Z}_{X_1}(r)$ in Nisnevich topology (see (7.3) §7 and more into [20]), one would like to look for a natural morphism of complexes

$$\mathbb{Z}_{X_1}(r) \xrightarrow{?} p(r)\Omega^{<r}_{X.},$$

so that its cone shall yield the desired information by taking cohomologies. This is a natural method as suggested by the Deligne-Beilinson cohomology over $\mathbb{C}$. However, natural map exists only in the derived category (which can seen below). On the other hand, one notes that there is already a good $p$-adic analogue of the Deligne-Beilinson cohomology, the so-called syntomic cohomology as introduced by Fontaine-Messing in their $p$-adic integral comparison theorem and further explored in the work of K. Kato [12] (see e.g Remark (3.5) loc. cit.). The link between the motivic complex $\mathbb{Z}_{X_1}$ and the syntomic complex $\mathfrak{S}_{X./W.}$ is the Minlor $K$-sheaf $\mathcal{K}^M_{X_1}$ (which is mapped to the de Rham-Witt complex via $d\log[\ ]$ ($[\ ]$ denotes the Teichmüller lifting.) There are quasi-isomorphisms between the de Rham-Witt complex $W.\Omega^\bullet_{X_1}$ and de Rham complex $\Omega^\bullet_{X.}$, but non-canonical unless fixing a PD-envelope connecting both sides as the author did in (2.10) and Proposition 2.8 [4]. It is a marvelous work of the authors to manage the construction of the motivic complex $\mathbb{Z}_{X.}(r)$ (Definition 7.1 [4]) of the pro-scheme $X./W.$ which glues $\mathbb{Z}_{X_1}$, the part containing cycle information of the central fiber, with the syntomic complex $\mathfrak{S}_{X./W.}$ over something in common (which comes from $\mathcal{K}^M_{X_1}$) (the whole construction occupies §4, 5 and 7 [4]). The key property to the obstruction problem is the exact triangle in $D_{pro}(X_1)$:

$$p(r)\Omega^{<r}_{X.}[-1] \to \mathbb{Z}_{X.}(r) \to \mathbb{Z}_{X_1}(r) \to,$$

as given in Proposition 7.3 [4]. As verified in §6 [4], the connecting map

$$H^{2r}(\mathbb{Z}_{X_1}(r)) \to H^{2r}(p(r)\Omega^{<r}_{X.})$$

in the long exact sequence of cohomologies of the above exact triangle coincides with the obstruction map *ob* in the beginning.

Assembly of Notations:

- $k$ a perfect field of char $p$; $W = W(k)$ the Witt ring and $W_n$; $X$ smooth projective over $W(k)$; $X_n = X \otimes_W W_n$; $X_1$ the central fiber; $X./W.$ $p$-adic formal scheme.
- $\mathbb{Z}_{X_1}(r)$ Suslin-Voevodsky's cycle complex of $X_1$ (§7); $\mathbb{Z}_{X.}(r)$ pro-complex of $X.$ (Def. 7.1).
- $\mathrm{CH}^r_{\mathrm{cont}}(X.) := H^{2r}_{\mathrm{cont}}(X_1, \mathbb{Z}_{X.}(r))$ continous Chow group of $X.$ (Def. 8.1); $\xi_1 \in \mathrm{CH}^r(X_1)$; $c(\xi_1) \in H^{2r}_{\mathrm{cont}}(X_1, q(r)W.\Omega^{\cdot}_{X_1})$ refined crystalline class.

Thm. 8.5 asserts that the following sequence of abelian groups is exact:

$$(*)\ \mathrm{CH}^r_{\mathrm{cont}}(X.) \to \mathrm{CH}^r(X_1) \xrightarrow{Ob} H^{2r}_{\mathrm{cont}}(X_1, p(r)\Omega^{<r}_{X.}).$$

Note that the continuous Chow group $\mathrm{CH}^r_{\mathrm{cont}}(X.)$ maps to $\varinjlim_n K_0(X_n)$ surjectively (§9).

The obstruction $(*)$ is induced by taking cohomologies of the following exact triangle:

$$(**)\ p(r)\Omega^{<r}_{X.}[-1] \to \mathbb{Z}_{X.}(r) \to \mathbb{Z}_{X_1}(r) \to p(r)\Omega^{<r}_{X.}.$$

($**$) is related to the syntomic complex via (Prop. 7.3)

$$\begin{array}{ccccccc} p(r)\Omega^{<r}_{X.}[-1] & \longrightarrow & \mathbb{Z}_{X.}(r) & \longrightarrow & \mathbb{Z}_{X_1}(r) & \longrightarrow & \ldots \\ \| & & \downarrow & & \downarrow & & \\ p(r)\Omega^{<r}_{X.}[-1] & \longrightarrow & \mathfrak{S}_{X.}(r) & \longrightarrow & W.\Omega^r_{X_1,\log}[-r] & \longrightarrow & \ldots \end{array}$$

The syntomic complex $\mathfrak{S}_{X.}(r) := cone(J(r)\Omega^\bullet_{D.} \xrightarrow{1-f_r} \Omega^\bullet_{D.})$ was connected by Kato [12] to the Minor $K$-groups (and then to the $p$-adic vanishing cycles). The bottom exact triangle is the major result of the authors on the syntomic complex (Thm 5.4 [4]).

**Talk 7: Syntomic complex.**

**Goal**: Complete §4 [4].

Review the definitions of $\Omega^\bullet_{D.}$, $J(r)\Omega^\bullet_{D.}$, the de Rham-Witt complex $W.\Omega^\bullet_{X_1}$, $q(r)W.\Omega^\bullet_{X_1}$ and the statement Proposition 2.8 [4], especially the quasi-isomorphisms

$$\Phi(F) : I(r)\Omega^\bullet_{D.} \cong q(r)W.\Omega^\bullet_{X_1};\ J(r)\Omega^\bullet_{D.} \cong W^{\geq r}_{.}\Omega^\bullet_{X_1}.$$

Explain the diagram in the bottom of page 685 [4]. For this, one needs to collect corresponding facts on the Hasse maps $f_r$ and respectively $F_r$ in [12] and [18]. Explain the exact triangle in Corollary 4.6 [4], whose proof consists of Lemma 4.3 (middle), 4.4 (right hand side), 4.5 (left hand side) which will be also applied in §5. Note the usage of derived category is essential here (although the proofs of lemmas are standard calculations in complexes). We advice the speaker to leave the issue of Nisnevich topology in §4 to Talk 9.

Speak 2 hours.

References:[12], [18]

**Talk 8: Fundamental triangle.**

**Goal**: Complete §5,6 [4], particularly Theorem 5.4.

Fundamental triangle is the bridge connecting many things. These result decompose the syntomic complex of a formal scheme into a part from the special fiber and a part from the deformation of the special fiber.

Granted the results in §4, to prove Thm. 5.4 and Thm. 6.1, one only needs to operate several cone constructions by using standard homological algebra. Thm. 5.4 is the fundamental triangle

$$p(r)\Omega^{<r}_{X.}[-1] \to \mathfrak{S}_{X.}(r) \to W.\Omega^r_{X_1,\log}[-r]$$

and Thm. 6.1 describes the connecting map

$$\alpha : W.\Omega^r_{X_1,\log}[-r] \to p(r)\Omega^{<r}_{X.}$$

which implies the compatibility of $\alpha$ with the cycle map.

Speak about 2 hours.

References: Homological algebra, e.g [7].

**Talk 9: Nisnevich topology.**

**Goal**: This talk and Talk 10 are designed to be a mini-introduction to the motivic cohomology, especially the motivic complex $\mathbb{Z}_{X_1}(r)$ as used in the work [4].

One has natural morphism of sites

$$X_{et} \xrightarrow{\epsilon} X_{nis} \xrightarrow{\eta} X_{Zar},$$

and the Nisnevich topology enjoys good properties of Zariski topology and etale topology. An experienced speaker should be able to explain the meaning of the last sentence well. Explain all arguments used in [4] involving comparison of two sites (Proposition 2.4, Lemma 4.3 [4]).

Speak 1.5 hours.

References: [4], [20].

**Talk 10: Motivic cohomology and $K$-theory.**

**Goal**: Suslin-Voevodsky's cycle complex $\mathbb{Z}_X(r)$ for a smooth variety $X$ over a field $k$.

Introduce $\mathbb{Z}(r)$ as a sheaf with transfer over the big Nisnenich site $\mathrm{Sm}/k$. Sketch the idea of the proof of (7.2) [4], identifying the cohomology sheaves of $\mathbb{Z}(r)$ with the Minlor $K$-sheaves. For $X \in (\mathrm{Sm}/k)$, set $\mathbb{Z}_X(r)$ to be the restriction of $\mathbb{Z}(r)$ to the small Nisnevich site of $X$. Explain the properties (0)-(2) and (5) of Proposition 7.2 [4] for $\mathbb{Z}_X(r)$. Explain the idea of (7.3) [4]. This is a very valuable connection (but explaining (7.2) looks as a non-easy task). We expect the talk to be highly heuristic in some steps.

Speak 2.5 hours.

References: [20], [14]

**Talk 11: The motivic complex.**

**Goal**: Complete §7 [4].

Introduces the motivic complex $\mathbb{Z}_{X.}(r)$, and prove Proposition 7.2 and 7.3 (motivic fundamental triangle) [4].

Kato's work relates the syntomic complex $\mathfrak{S}_{X.}(r)$ with the Milnor $K$-sheaves, and the motivic complex $\mathbb{Z}_{X.}(r)$ of the pro-scheme $X.$ glues the motivic complex $\mathbb{Z}_{X_1}(r)$ of the central fiber and the syntomic complex along the de Rham realization of the Milnor $K$-sheaves. Proposition 7.2 comes up as a consequence of gluing corresponding results for $\mathbb{Z}_{X.}(r)$ and $\mathfrak{S}_{X.}(r)$.

Speak 2 hours.

References: [12].

**Talk 12: Crystalline-Hodge obstruction.**

**Goal**: Complete §8 [4]. the proof of Theorem 8.5 [4].

This is the main result of this session. Here the authors get the obstruction to lift a cycle class on the central fiber to continous Chow group $\mathrm{CH}^r_{\mathrm{cont}}(X.) := H^{2r}_{\mathrm{cont}}(X_1, \mathbb{Z}_{X.}(r))$.

If possible, the speaker comments on geometric meaning of the continuous Chow group.

Speak 2 hours.

References: no

**Talk 13: The motivic complex after Beilinson.**

**Goal**: A good retrospect on the construction of $\mathbb{Z}_{X.}(r)$ carried out in §4-5. Prove the quasi-isomorphism between the motivic complex $\mathbb{Z}_{X.}(r)$ and the motivic complex $\tilde{\mathfrak{S}}_{X.}(r)$ as proposed by Beilinson. Complete §3 and Appendix C [4].

Speak 2 hour.

References: Appendix C [4].

## 4. Part III: Chern character isomophism

Part III consists of sections 9-12. The aim is to connect the continuous Chow group, which is defined to be the cohomology of the motivic pro-complex over $X_1$, to a continuous $K_0$ of the attached pro-scheme $X.$. The main theorem of this part is [4, Theorem 11.1].

**Talk 14: Continuous $K$-Theory.**

**Goal**: The speaker should give a brief introduction to Algebraic $K$-Theory, particularly the basic set-up of the Quillen's higher $K$-theory. The standard reference on this topic for algebraic geometers is [24]. So briefly review Quillen's +-construction and $Q$-construction and etc. To the end of the talk, introduce the continuous $K$-groups as defined in Section 9.

Prerequisites:

References: [24], [4]

**Talk 15: Chern class.**

**Goal**: Report on the construction of Chern classes of Gillet [8] for higher $K$-theory. Apply his method to construct a continuous Chern class map from continuous $K$-theory to continuous motivic cohomology.

Prerequisites:

References: [8], [4]

**Talk 16: Topological cyclic homology.**

**Goal**: Explain the basics in topological cyclic homology theory [21]. Use deep results on topological cyclic homology theory to prove relevant $K$-theoretical results used in the sequel (Prop 10.5).

Prerequisites:

**Talk 17: Chern character isomorphism.**

**Goal**: Complette §11 [4], particularly Theorem 11.1.

Prerequisites:

References:

**Talk 18: Deformation of Milnor $K$-Groups.**

**Goal**: Complete §12 [4].

The result of this section Theorem 12.3 is used to show the formula

$$\mathcal{H}^r(\mathbb{Z}(r)) = \mathcal{K}_r^M,$$

generalizing the result for fields (Suslin-Voevodsky [23]).

Prerequisites:

References: [15], [16], [17]

## References


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