EE356 Elementary Plasma Physics H. O. #4

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LECTURE 2: SINGLE PARTICLE MOTION

2.1 INTRODUCTION

Depending on the density of charged particles, plasmas behave either as a fluid with collective effects being dominant or as a collection of individual particles. In dense plasmas, the electrical forces between particles couples them to each other and to the electromagnetic fields which affects their motions. In rarefied plasmas, the charged particles do not interact with one another and their motions do not constitute a large enough current to significantly affect the electromagnetic fields. Under these conditions, the motion of each particle can be treated independently from any other, by solving the Lorentz force equation for prescribed electric and magnetic fields, a procedure known as the single particle approach. In magnetized plasmas, under the influence of an external static or slowly varying magnetic field, the single particle approach is only applicable if the external magnetic field is quite strong, compared to the magnetic field produced by the electric current due to the particle motions. Although the single particle approach may only be valid in special circumstances, understanding the individual particle motions is also an important first step in understanding the collective behavior of plasmas. Accordingly, we shall study single particle motions in this and the following two lectures.

In all of the following, the fundamental equation of motion for the particles is under the influence of the Lorentz force is given by

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad [2.1]$$

where m is the particle mass and $\mathbf{v}$ is its velocity. While we only consider nonrelativistic[1] motion ($|\mathbf{v}| \ll c$), the above equation is valid for the relativistic case, if we simply replace the mass m with $m = m_0\left(1 - v^2/c^2\right)^{-1/2}$, where m_0 is the rest mass and $v = |\mathbf{v}|$. More commonly, the relativistic version of equation [2.1] is written simply in terms of the particle momentum $\mathbf{p} = m\mathbf{v}$, rather than velocity $\mathbf{v}$.

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[1] By the same token, we neglect any radiation produced by the acceleration of charged particles. At non-relativistic velocities, such radiation is quite negligible; the radiated electric field at a distance R from the particle is proportional to $q^2a^2/(c^2R^2)$, where q is charge of the particle, c is the speed of light in free space, and a is the acceleration. For a discussion at an appropriate level, see Chapter 7 of J. B. Marion, *Classical Electromagnetic Radiation*, Academic Press, New York, 1965.

2.2 MOTION IN A UNIFORM B FIELD: GYRATION

We start by considering the simplest cases of motion in uniform fields. When a particle is under the influence of a static electric field that is uniform in space, the particle simply moves with a constant acceleration along the direction of the field, so that this case does not warrant further study. On the other hand, the motion of a charged particle under the influence of static and uniform magnetic field is of fundamental interest, and is studied in this section.

With only a static and uniform magnetic field present, equation [2.1] becomes

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \qquad [2.2]$$

Taking the dot product of [2.2] with $\mathbf{v}$ we have

$$\begin{aligned}
\mathbf{v} \cdot m\frac{d\mathbf{v}}{dt} &= \mathbf{v} \cdot q(\mathbf{v} \times \mathbf{B}) \\
m\frac{1}{2}\frac{d(\mathbf{v} \cdot \mathbf{v})}{dt} &= q[\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B})] \\
\frac{d}{dt}\left(\frac{mv^2}{2}\right) &= 0
\end{aligned}$$

where $v = |\mathbf{v}|$ is the particle speed and where we have noted that $(\mathbf{v} \times \mathbf{B})$ is perpendicular to $\mathbf{v}$ so that the right hand side is zero. It is clear that a static magnetic field cannot change the kinetic energy of the particle, since the force is always perpendicular to the direction of motion. Note that this is true even for a spatially nonuniform field, since our derivation above did not use the fact that the field is uniform in space.

We first consider the case of a magnetic field configuration consisting of field lines that are straight and parallel, with the magnetic field intensity constant in time and space. Later on, we will allow the magnetic field intensity to vary in the plane perpendicular to the field, while continuing to assume that the field lines are straight and parallel. We can decompose the particle velocity into its components parallel and perpendicular to the magnetic field, i.e.,

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$$

in which case we can rewrite [2.2] as

$$\frac{d\mathbf{v}_{\parallel}}{dt} + \frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}(\mathbf{v}_{\perp} \times \mathbf{B})$$

since $\mathbf{v}_{\parallel} \times \mathbf{B} = 0$. This equation can be split into two equations in terms of $\mathbf{v}_{\parallel}$ and $\mathbf{v}_{\perp}$

respectively:

$$\frac{d\mathbf{v}_\parallel}{dt} = 0 \qquad \longrightarrow \qquad \mathbf{v}_\parallel = \text{const.}$$
$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B})$$

It is clear from the above that the magnetic field has no effect on the motion of the particle in the direction along it, and that it only affects the particle velocity in the direction perpendicular to it. To examine the character of the perpendicular motion, consider a static magnetic field oriented along the z axis, namely $\mathbf{B} = \hat{\mathbf{z}}B$. We can write [2.2] in component form as

$$m\frac{dv_x}{dt} = qBv_y \qquad [2.3a]$$
$$m\frac{dv_y}{dt} = -qBv_x \qquad [2.3b]$$
$$m\frac{dv_z}{dt} = 0 \qquad [2.3c]$$

The component of the velocity parallel to the magnetic field is often denoted as $v_\parallel = v_z$ and is constant since the Lorentz force $q(\mathbf{v} \times \mathbf{B})$ is perpendicular to $\hat{\mathbf{z}}$.

To determine the time variations of v_x and v_y, we can take the second derivatives of [2.3a] and [2.3b] and substitute to find

$$\frac{d^2v_x}{dt^2} + \omega_c^2 v_x = 0 \qquad [2.4a]$$
$$\frac{d^2v_y}{dt^2} + \omega_c^2 v_y = 0 \qquad [2.4b]$$

where $\omega_c = -qB/m$ is the *gyrofrequency* or *cyclotron frequency*. Note that ω_c is an angular frequency in units of rad-m^{-1} and can be positive or negative, depending on the sign of q. A positive value of ω_c in a right-handed coordinate system indicates that the sense of rotation is along the direction of positive ϕ, where ϕ is the cylindrical coordinate azimuthal angle, measured *from* the x axis as shown in Figure 2.1.

The solution of [2.4] is in the form of a harmonic motion given by

$$v_x = v_\perp \cos(\omega_c t + \psi) \qquad [2.5a]$$
$$v_y = -v_\perp \sin(\omega_c t + \psi) \qquad [2.5b]$$
$$v_z = v_\parallel \qquad [2.5c]$$

where ψ is some arbitrary phase angle which defines the orientation of the particle velocity at $t = 0$ and $v_\perp = \sqrt{v_x^2 + v_y^2}$ is the constant speed in the plane perpendicular to $\mathbf{B}$.

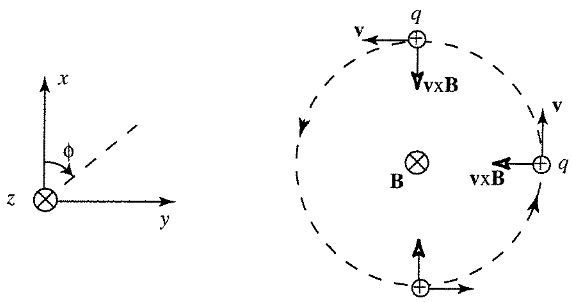

Fig. 2.1. Motion of a particle in a magnetic field. A particle with positive charge q and with a velocity $\mathbf{v}$ experiences a force of $q\mathbf{v} \times \mathbf{B}$ in the presence of a magnetic field $\mathbf{B}$. [gyration]

To appreciate the above result physically, consider the coordinate system and the forces on the particle (assumed to be a positive charge q) as shown in Figure 2.1 at different points along its orbit. It is clear that the particle experiences a $\mathbf{v} \times \mathbf{B}$ force directed inward at all times, which balances the centrifugal force, resulting in a circular motion. For a z-directed magnetic field, electrons rotate in the right hand sense, i.e., have a positive value of ω_c; in other words, if the thumb points in the direction of the magnetic field, the fingers rotate in the direction of the electron motion. The radius of the circular trajectory can be determined by considering the fact that the $\mathbf{v} \times \mathbf{B}$ force is balanced by the centrifugal force so that we have

$$\frac{mv_\perp^2}{r} = |q|\mathbf{v} \times \mathbf{B} = qv_\perp B \qquad \longrightarrow \qquad r_c = \frac{mv_\perp}{|q|B} = \frac{v_\perp}{\omega_c}$$

where r_c is known as the *gyroradius* or *Larmor radius*. Note that the magnitude of the particle velocity remains constant, since the magnetic field force is at all times perpendicular to the motion. The magnetic field cannot change the kinetic energy of the particle; however, it does change the direction of its momentum. It is important to note that the gyrofrequency ω_c of the charged particle does not depend on its velocity (or kinetic energy) and is only a function of the intensity of the magnetic field. Particles with higher velocities (and thus higher energies) orbit in circles with larger radii but complete one revolution in the same time as particles with lower velocities which orbit over smaller circles. Particle with larger masses also orbit in circles with larger radii, however, they complete one revolution in a longer time compared with smaller masses. A convenient expression for the gyrofrequency f_{ce} for electrons is

$$f_{ce} = \frac{\omega_c}{2\pi} \simeq 2.8 \times 10^6 B \text{ Hz}$$

where B is in units of Gauss (note that 10^4 Gauss=1 Tesla or wb-m^{-2}). As an example, the Earth's magnetic field at the surface is of oredr $\sim$0.5 Gauss, corresponding to a gyrofrequency of $f_{ce} \simeq 1.4$ MHz.

The particle position as a function of time can be found by integrating [2.5]

$$x = r_c \sin(\omega_c t + \psi) + (x_0 - r_c \sin\psi) \qquad [2.6a]$$
$$y = r_c \cos(\omega_c t + \psi) + (y_0 - r_c \cos\psi) \qquad [2.6b]$$
$$z = z_0 + v_\| t \qquad [2.6c]$$

where x_0, y_0, and z_0 are the coordinates of the location of the particle at $t = 0$ and ψ is simply the phase with respect to a particular time of origin. Equations [2.6] show that the particle moves in a circular orbit perpendicular to **B** with an angular frequency ω_c and radius r_c about a *guiding center* $\mathbf{r}_g = \hat{\mathbf{x}}x_0 + \hat{\mathbf{y}}y_0 + \hat{\mathbf{z}}(z_0 + v_\| t)$. The concept of a guiding center is useful in considering particle motion in inhomogeneous fields, since the gyration is often much more rapid than the motion of the guiding center. Note from [2.6] that in the present case, the guiding center simply moves linearly along z at a uniform speed $v_\|$, although the particle motion itself is helical, as shown in Figure 2.2. The *pitch angle* of the helix is defined as

$$\boxed{\alpha = \tan^{-1}\left(\frac{v_\perp}{v_\|}\right)} \qquad [2.7]$$

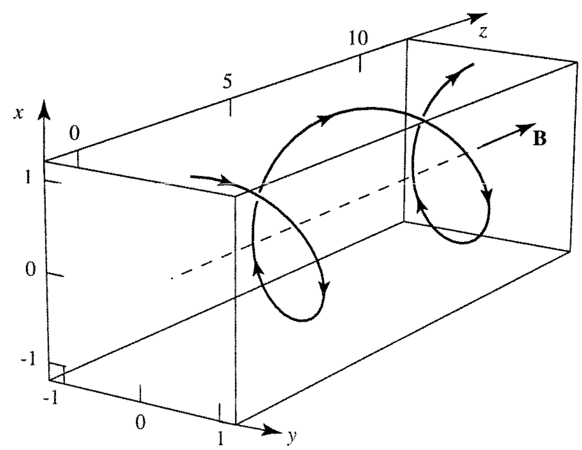

Fig. 2.2. Electron guiding center motion in a magnetic field $\mathbf{B} = \hat{\mathbf{z}}B$. Also see Chapter 4, Fig. 5 on p. 44 of Bittencourt. [helical]

It is interesting to note that for both positive and negative charges, the particle gyration constitutes an electric current in the $-\phi$ direction (i.e., opposite to the direction of the fingers of the right hand when the thumb points in the direction of the $+z$ axis). The magnetic moment associated with such a current loop is given by μ = current $\times$ area or

$$\mu = \underbrace{\left(\frac{q\omega_c}{2\pi}\right)}_{\text{Current}} \underbrace{(\pi r_c^2)}_{\text{Area}} = \frac{mv_\perp^2}{2B}$$

Note that the direction of the magnetic field generated by the gyration is opposite to that of the external field. Thus, freely mobile particles in a plasma respond to an external magnetic field with a tendency to *reduce* the total magnetic field. In other words, a plasma is a *diamagnetic* medium and has a tendency to exclude magnetic fields, as we shall further study later in the context of our discussions of magnetohydrodynamics and magnetic pressure.

2.3 E × B DRIFT

When both electric and magnetic fields are present, the particle motion is found to be a superposition of gyrating motion in the plane perpendicular to the magnetic field and a drift of the guiding center in the direction parallel to **B**. Assuming once again that the magnetic field is in the z direction, i.e., $\mathbf{B} = \hat{\mathbf{z}}B$, we decompose the electric field **E** into its components parallel and perpendicular to **B**:

$$\mathbf{E} = \mathbf{E}_\perp + \hat{\mathbf{z}}E_\parallel = \hat{\mathbf{x}}E_\perp + \hat{\mathbf{z}}E_\parallel$$

where we have taken the electric field to be in the x direction, with no loss of generality. Noting that we can also decompose the particle velocity into its two components, i.e., $\mathbf{v}(t) = \mathbf{v}_\perp(t) + \hat{\mathbf{z}}v_z(t)$, the equation of motion can be written as

$$m\frac{d\mathbf{v}_\perp}{dt} = q(\hat{\mathbf{x}}E_\perp + \mathbf{v}_\perp \times \hat{\mathbf{z}}B) \qquad [2.8a]$$

$$m\frac{dv_\parallel}{dt} = qE_\parallel \qquad [2.8b]$$

Equation [2.8b] simply indicates constant acceleration along **B**. For the transverse component, we seek a solution of the form

$$\mathbf{v}_\perp(t) = \mathbf{v}_{\mathrm{E}} + \mathbf{v}_{\mathrm{ac}}(t) \qquad [2.9]$$

where $\mathbf{v}_{\mathrm{E}}$ is a constant velocity and $\mathbf{v}_{\mathrm{ac}}$ is the alternating component. Using [2.9] in [2.8a] we have

$$m\frac{d\mathbf{v}_{\mathrm{ac}}}{dt} = q(\hat{\mathbf{x}}E_\perp + \mathbf{v}_{\mathrm{E}} \times \hat{\mathbf{z}}B + \mathbf{v}_{\mathrm{ac}} \times \hat{\mathbf{z}}B) \qquad [2.10]$$

We know from our discussions in the previous section that the left hand side term and the last term on the right hand side term in [2.10] simply describe circular motion (gyration) at

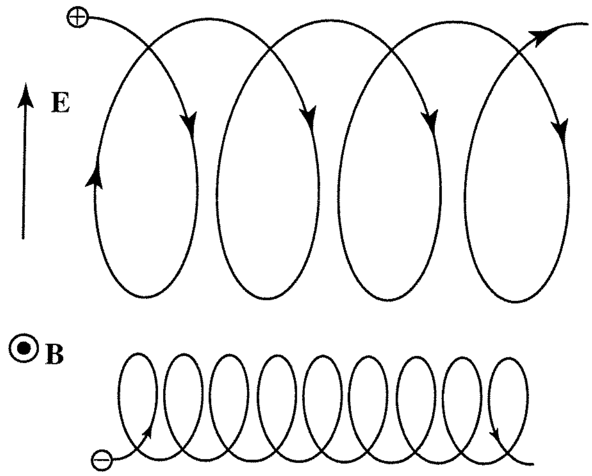

Fig. 2.3. Particle drifts in crossed E and B fields. The negatively charged particle is assumed to have the same velocity ($v_\perp$) as the positively charged one but a smaller mass and therefore a smaller gyroradius. The $\mathbf{E} \times \mathbf{B}$ drift speed $\mathbf{v}_{\mathrm{E}}$ for both particles is the same. [cycloid]

a rate $\omega_c = -qB/m$. Thus, if we choose $\mathbf{v}_{\mathrm{E}}$ such that the first two terms on the right hand side of [2.10] cancel, i.e.,

$$\hat{\mathbf{x}}E_\perp + \mathbf{v}_{\mathrm{E}} \times \hat{\mathbf{z}}B = 0 \qquad \rightarrow \qquad \boxed{\mathbf{v}_{\mathrm{E}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}} \tag{2.11}$$

then [2.10] reduces to the form

$$m\frac{d\mathbf{v}_{\mathrm{ac}}}{dt} = q\mathbf{v}_{\mathrm{ac}} \times \hat{\mathbf{z}}B \tag{2.12}$$

which, as mentioned, simply describes rotation at a frequency $\omega_c = -qB/m$. Note that we can use $\mathbf{E}$ rather than $\mathbf{E}_\perp$ in [2.11] since $\hat{\mathbf{z}}E_\| \times \mathbf{B} \equiv 0$. Thus, we see that the particle motion in the presence of is given by

$$\mathbf{v}(t) = \hat{\mathbf{z}}v_\|(t) + \mathbf{v}_{\mathrm{E}} + \mathbf{v}_{\mathrm{ac}}(t) \tag{2.13}$$

consisting of motion steady acceleration along $\mathbf{B}$, uniform drift velocity $\mathbf{v}_{\mathrm{E}}$ perpendicular to $\mathbf{B}$ and $\mathbf{E}$, and the gyration. Taking the time average of $\mathbf{v}(t)$ over one gyroperiod ($T_c = 2\pi/\omega_c$), we have

$$< \mathbf{v} >= \frac{1}{T_c}\int_0^{T_c} \mathbf{v}(t)dt = \hat{\mathbf{z}}v_\| + \mathbf{v}_{\mathrm{E}}$$

showing that $\mathbf{v}_{\mathrm{E}} = (\mathbf{E} \times \mathbf{B})/B^2$ is the average perpendicular velocity.

It is interesting to note that the drift velocity $\mathbf{v}_E$ is independent of q, m, and $v_\perp = |\mathbf{v}_\perp|$. The reason can be seen from physical picture of the drift as shown in Figure 2.3. As the positively charged particle moves downward (against the electric field) during the first half of its cycle, it loses energy and its r_c decreases. In the second half of its cycle, it regains this energy back as it now moves in the direction of the electric field. This difference in the radius of curvature of its orbit near the top versus bottom of its orbit is the reason for the drift $\mathbf{v}_E$. A negatively charged particle gyrates in the opposite direction but also gains/loses energy in the opposite directions compared to the positively charged particle. Since we have assumed the negatively charged particle to be lighter it has a smaller gyroradius r_c. However, at the same time its gyrofrequency is larger and the two effects cancel each other out, resulting in the same drift velocity. Two particles of the same mass but different energy (i.e., different $\frac{1}{2}mv_\perp^2$ or $v_\perp$) have the same gyrofrequency ω_c, and although the one with the higher velocity has a higher r_c and hence gains more energy from $\mathbf{E}$ in a half cycle, the fractional change in r_c for a given change in energy is smaller, so that the two effects cancel out and $\mathbf{v}_E$ is independent of $v_\perp$.

The basic source of the $\mathbf{E} \times \mathbf{B}$ drift derived above was the component of electric field perpendicular to $\mathbf{B}$. It is clear from the above procedure that any other constant transverse force $\mathbf{F}_\perp$ acting on a particle gyrating in a constant magnetic field would produce a drift perpendicular to both $\mathbf{F}_\perp$ and $\mathbf{B}$, with the drift velocity given by

$$\boxed{\mathbf{v}_F = \frac{(\mathbf{F}_\perp/q) \times \mathbf{B}}{B^2}} \qquad [2.14]$$

We shall use [2.14] later in studying the motion of charged particles in nonuniform magnetic fields.

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LECTURE 3: PARTICLE MOTION IN NONUNIFORM B FIELDS

3.1 INTRODUCTION

Both naturally occurring plasmas and those that one encounters in many applications often exist in the presence of magnetic fields that do not vary appreciably in time but which vary with one or more coordinates of space, i.e., which are *nonuniform*. An important example of a nonuniform magnetic field is the so-called magnetic mirror configuration which is commonly used to confine plasmas, and which is also the mechanism by which energetic particles are trapped in the earth's radiation belts.

Assuming the absence of an electric field, and no temporal variations of the magnetic field[1], the kinetic energy of the particle must remain zero, since the magnetic force is at all times perpendicular to the motion of the particles as discussed in Lecture 2. In general, exact anaytical solutions for charged particle motions in a nonuniform magnetic field cannot be found. However, one very important configuration that can be studied analytically is the case in which the gyroradius r_c is much smaller than the spatial scales over which the magnetic field varies. In such cases, the motion of the particle can be decomposed into the fast gyromotion plus some type of relatively slow drift motion. The slow drift is associated with the motion of the guiding center, and the separation of its motion from the rapid gyration is similar to the simplest case analyzed in Lecture 2, where we saw that the guiding center simply moved linearly along the magnetic field as the particle executed its complicated gyromotion.

We now examine particle motion in different types of nonuniform magnetic fields, assuming the presence of only one type of inhomogeneity in each case.

[1] Note that any time variations of the magnetic field would lead to an electric field via Faraday's law, i.e., $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, which can in turn accelerate the particles.

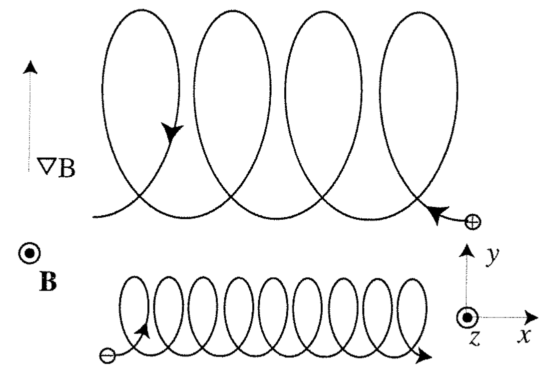

Fig. 3.1. Particle drifts due to a magnetic field gradient. [gradient]

3.2 GRADIENT DRIFT

We first consider a magnetic field the intensity of which varies in a direction perpendicular to the magnetic field vector. Without loss of generality, let $\mathbf{B}(y) = \hat{\mathbf{z}}B_z(y)$, as depicted in Figure 3.1. Since the field strength has a nonzero gradient ∇B_z in the y direction, we note that the local gyroradius r_c (i.e., the radius of curvature of the particle orbit) is large in regions where B is small and vice versa. Thus, on physical grounds alone, we expect a positive charge to drift to the left and a negative charge to drift to the right.

To find an expression for the particle drift velocity, we take advantage of expression [2.14] which gives drift velocity for any force perpendicular to the **B** field. For the field geometry depicted in Figure 3.1, this means a force in either x or y directions. Since we started with the premise that the gyration was much more rapid than the relatively slow drift, it is appropriate to determine the net resultant force averaged over one gyroperiod. The force perpendicular to **B** is the Lorentz force given by

$$\begin{aligned}\mathbf{F} &= q(\mathbf{v}\times\mathbf{B}) = \hat{\mathbf{x}}qv_yB_z - \hat{\mathbf{y}}qv_xB_z \\ &\simeq \hat{\mathbf{x}}\,q\,v_y\left(B_0 + y\frac{\partial B_z}{\partial y}\right) - \hat{\mathbf{y}}qv_x\left(B_0 + y\frac{\partial B_z}{\partial y}\right)\end{aligned}$$

where we have expanded $B_z(y)$ into a Taylor series[2] around the guiding center of the particle (taken with no loss of generality to be at $x_g = 0$ and $y_g = 0$), with B_0 being the magnetic field intensity at the guiding center and y is the distance from the guiding center. In other words, we have written

$$B_z(z) = B_0 + y\frac{\partial B_z}{\partial y} + \cdots$$

and neglected higher order terms.

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[2] In the more general case this expansion can be written as

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r}\cdot\nabla)\mathbf{B}_0 + \cdots$$

where $\mathbf{B}_0$ is the field at the guiding center and $\mathbf{r}$ is the position vector with the origin chosen as the guiding center.

We thus have the two transverse components of the Lorentz force as

$$F_x = qv_y\left(B_0 + y\frac{\partial B_z}{\partial y}\right) \tag{3.1a}$$

$$F_y = -qv_x\left(B_0 + y\frac{\partial B_z}{\partial y}\right) \tag{3.1b}$$

We wish to determine $< F_x >$ and $< F_y >$, where the brackets denote averaging over one gyroperiod. To do this, we can assume that the particles by and large follow the orbits for a uniform field, as determined in Lecture 2 (see [2.5 and [2.6]), namely:

$$x_c = r_c \sin(\omega_c t + \psi) \tag{3.2a}$$

$$y_c = r_c \cos(\omega_c t + \psi) \tag{3.2b}$$

$$v_x = v_\perp \cos(\omega_c t + \psi) \tag{3.2c}$$

$$v_y = -v_\perp \sin(\omega_c t + \psi) \tag{3.2d}$$

where ω_c has the same sign as q (i.e., is negative for electrons). Substituting in [3.1a] and [3.1b] we have

$$F_x = -qv_\perp \sin(\omega_c t + \psi)\left[B_0 + r_c\cos(\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right] \tag{3.3a}$$

$$F_y = -qv_\perp \cos(\omega_c t + \psi)\left[B_0 + r_c\cos(\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right] \tag{3.3b}$$

The average over one gyroperiod ($2\pi/\omega_c$) of F_x is zero, since it contains the product of sine and cosine terms. The averaging of F_y has the product of a cosine with a cosine which results in a factor of $\frac{1}{2}$. We thus have,

$$\langle F_y \rangle = -\frac{qv_\perp r_c}{2}\frac{\partial B_z}{\partial y} \tag{3.4}$$

The drift velocity is then given from [2.14] as

$$\mathbf{v}_\nabla = \frac{(\mathbf{F}_\perp/q)\times\mathbf{B}}{B^2} = \frac{\langle F_y\rangle\hat{\mathbf{y}}\times\hat{\mathbf{z}}B_z}{qB_z^2} = -\frac{v_\perp r_c}{2B_z}\frac{\partial B_z}{\partial y}\,\hat{\mathbf{x}} \tag{3.5}$$

where the subscript '∇' indicates that the drift velocity is due to the gradient drift. Since the magnetic field direction was chosen arbitrarily, we can write [3.5] more generally as

$$\mathbf{v}_\nabla = \frac{v_\perp r_c}{2}\frac{\mathbf{B}\times\nabla B}{B^2} = \frac{mv_\perp^2}{2q}\frac{\mathbf{B}\times\nabla B}{B^3} \tag{3.6}$$

for any magnetic field $\mathbf{B} = \hat{\mathbf{B}}B$, and noting that $r_c = mv_\perp^2/(qB)$. The corresponding more general expression for the perpendicular gradient force $\mathbf{F}_\nabla$ is

$$\mathbf{F}_\nabla = -\frac{\frac{1}{2}mv_\perp^2}{B}\nabla B = -\frac{W_\perp}{B}\nabla B \qquad [3.7]$$

where $W_\perp$ is the perpendicular kinetic energy of the particle.

Equation [3.6] exhibits the dependencies that we expect on physical bases. Electrons and ions drift in opposite directions and the drift velocity is proportional to the perpendicular energy of the particle, namely $W_\perp = \frac{1}{2}mv_\perp^2$. Faster particles drift faster, since they have a larger gyroradiius and their orbits span a larger range of the inhomogeneity of the field.

3.3 CURVATURE DRIFT

When particles rapidly gyrate while moving along a magnetic field line which is curved, as depicted in Figure 3.2a, they experience a centrifugal force perpendicular to the magnetic field, which produces a drift as defined by [2.14]. Assuming once again that the spatial scale of the curvature is much larger than the gyroradius, we can focus our attention to the motion of the guiding center. The outward centrifugal force in the frame of reference moving with the guiding center at a velocity $v_\parallel$ is given by

$$\mathbf{F}_{\text{cf}} = mv_\parallel^2\frac{\mathbf{R}_c}{R_c^2} \qquad [3.8]$$

where $\mathbf{R}_c$ is the vector pointing radially outward from the center of the circle described by the local curvature of the field and R_c has a magnitude equal to the radius of curvature.

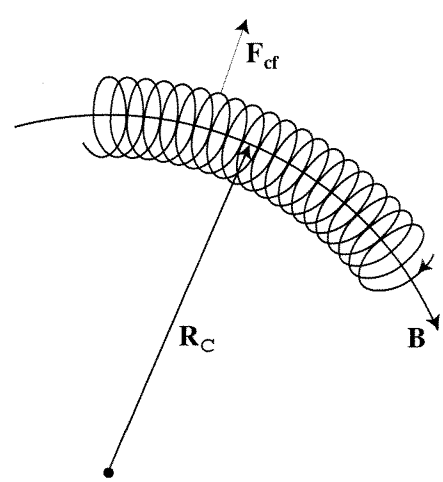

Fig. 3.2. Curvature drift. Particle drift in a curved magnetic field. ([curvature]

Using the force given in [3.8] in [2.14] we find the curvature drift velocity as

$$\mathbf{v}_\mathrm{R} = \frac{(\mathbf{F}_\mathrm{cf}/q)\times\mathbf{B}}{B^2} = \frac{mv_\parallel^2}{q}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2B^2} \qquad [3.9]$$

In vacuum, curvature drift cannot by itself be the only drift since the curl of the magnetic field must be zero, namely $\nabla\times\mathbf{B}=0$. Considering cylindrical coordinates with $\mathbf{B}=B_\phi(r)\hat{\phi}$, we must then have

$$(\nabla\times\mathbf{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = 0 \qquad \rightarrow \qquad B_\phi = \frac{A}{r}$$

where A is a constant. The gradient of $\mathbf{B}$ is then $\partial B_\phi/\partial r = -B_\phi/r = -A/r$. More generally, we can write the resulting gradient as $\nabla = -(B/R_c^2)\mathbf{R}_c$. Thus, the total drift due to both gradient and curvature effects can be written as

$$\mathbf{v}_\mathrm{total} = \mathbf{v}_\mathrm{R} + \mathbf{v}_\nabla = (v_\parallel^2 + \frac{1}{2}v_\perp)\frac{\mathbf{B}\times\nabla B}{w_cB^2} \qquad [3.10]$$

An example of gradient plus curvature drift is the *longitudinal drift* of radiation belt electrons around the Earth (see Figure 3.3). Note that the direction of the earth's magnetic field is from south-to-north.

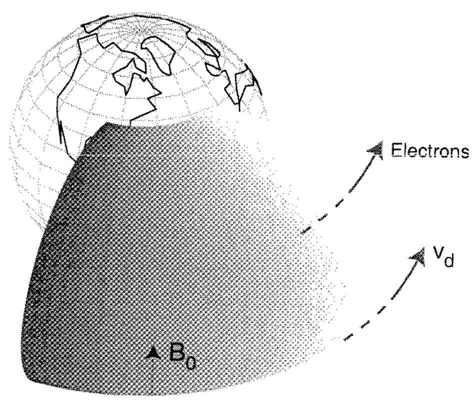

Fig. 3.3. Longitudinal drift of radiation belt electrons. Note that the direction of the earth's magnetic field $\mathbf{B}_0$ is from south to north. [longitudinal]

It is interesting to note that both the gradient and curvature drift velocities are inversely proportional to the charge q so that electrons and ions drift in opposite directions. The oppositely directed drifts of electrons and ions leads to a transverse current. The gradient drift current is given by

$$\begin{aligned}\mathbf{J}_\nabla &= N|q_e|[(\mathbf{v}_\nabla)_i - (\mathbf{v}_\nabla)_e] \\ &= \frac{N}{B^3}[(W_\perp)_i + (W_\perp)_e](\mathbf{B}\times\nabla B)\end{aligned}$$

where $N = N_i = N_e$ is the plasma density and $W_\perp = \frac{1}{2}mv_\perp^2$ is the perpendicular particle energy. Note that the gradient drift current $\mathbf{J}_\nabla$ flows in a direction perpendicular to both the magnetic field and its gradient.

Similarly, the different directional curvature drifts of the electrons and ions leads to a curvature drift current given by

$$\begin{aligned}\mathbf{J}_\mathrm{R} &= N|q_e|[(\mathbf{v}_\mathrm{R})_i - (\mathbf{v}_\mathrm{R})_e] \\ &= \frac{2N}{R_c^2 B^2}[(W_\parallel)_i + (W_\parallel)_e)](\mathbf{R}_c\times\mathbf{B})\end{aligned}$$

where $W_\parallel = \frac{1}{2}mv_\parallel^2$ is the parallel particle energy. The curvature drift current $\mathbf{J}_\mathrm{R}$ flows in a direction perpendicular to both the magnetic field and its curvature.

In the earth's magnetosphere, the gradient and curvature drift currents described above create a large scale current called the *ring current*, the magnitude of which can exceed several million amperes during moderate size magnetic storms, when the number of particles in the ring current region increases. The ring current produces a magnetic field that decreases the earth's field within the drift orbits of the particles. This effect is observed as a major decrease in the geomagnetic field during magnetic storms.

3.4 ADIABATIC INVARIANCE OF THE MAGNETIC MOMENT

In Lecture 2, we recognized that a gyrating particle constitutes an electric current loop with a magnetic dipole moment given by $\mu = mv_\perp/(2B)$. In this section, we demonstrate that this quantity has a remarkable tendency to be conserved (i.e., to be invariant), in spite of spatial or temporal changes in the magnetic field intensity, as long as the changes in B are small over a gyroradius or gyroperiod. This kind of constancy of a variable is termed *adiabatic invariance*, to distinguish them from quantities that may be absolute invariants, such as total charge, energy or momentum in a physical system.

Consider a particle gyrating in magnetic field oriented primarily in the z direction but varying in intensity as a function of z, as depicted in Figure. Assume the field to be azimuthally symmteric, so that there is no ϕ-component, i.e., $B_\phi = 0$, and no variations of any of the quantities in ϕ, i.e., $\partial(\cdot)/\partial\phi = 0$. As the particle gyrates around $\mathbf{B}$ with a perpendicular velocity $v_\perp$ while moving along it at $v_\parallel$, we are primarily concerned with the motion of its guiding center, which moves along the z axis. The force acting on the particle

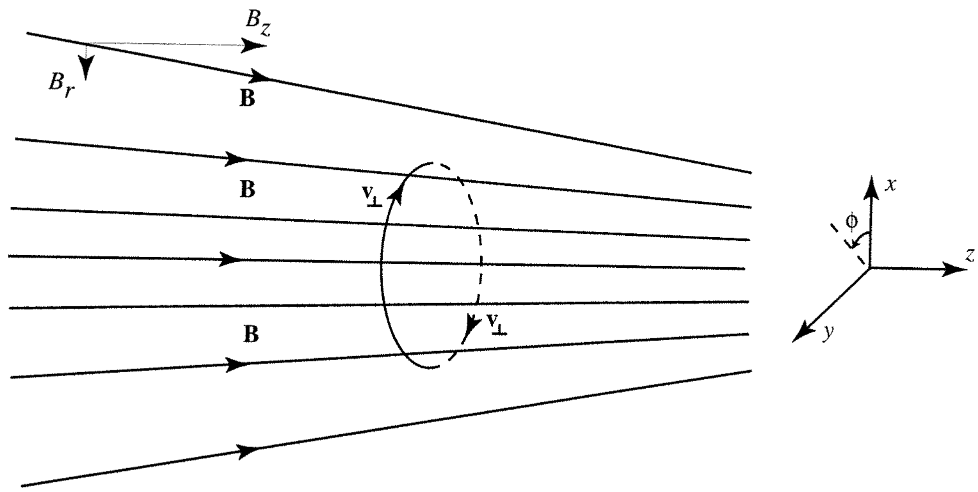

Fig. 3.4. Drift of a particle in a magnetic mirror configuration. [mirrorlecturethree]

during this motion can be found from [3.7], by noting that the magnetic field has a nonzero gradient in the z direction. We have

$$F_z = -\frac{\frac{1}{2}mv_\perp^2}{B_z}\frac{\partial B_z}{\partial z} = -\frac{W_\perp}{B_z}\frac{\partial B}{\partial z} = -\mu\frac{\partial B}{\partial z} \qquad [3.11]$$

Alternatively, the force F_z can be found directly from the Lorentz force equation. The z component of the Lorentz force results from $q\mathbf{v}_\perp \times \mathbf{B}$ or

$$F_z = q\mathbf{v}_\perp \times \mathbf{B} = qv_\perp B_r \qquad [3.12]$$

where B_r can be found using the fact that we must have $\nabla \cdot \mathbf{B} = 0$, so that

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0 \quad \longrightarrow \quad B_r \simeq -\frac{r}{2}\frac{\partial B_z}{\partial z} \qquad [3.13]$$

assuming that $\partial B_z/\partial z$ does not vary significantly with r. In other words, the total magnetic field in the case of a converging field line geometry of Figuremust be given by

$$\mathbf{B} = B_r\hat{\mathbf{r}} + B_z\hat{\mathbf{z}}$$

Evaluating B_r as given in [3.13] at $r = r_c$, and substituting in [3.12] we find the same expression for F_z as in [3.11].

With F_z determined, we can examine the variations of the parallel and perpendicular energies of the particle as its guiding center moves along z. Consider the total energy of the particle

$$W = W_\perp + W_\parallel$$

which must remain constant in the absence of electric fields, so that we have

$$\frac{dW_\perp}{dt} + \frac{dW_\parallel}{dt} = 0 \qquad [3.14]$$

Noting that $W_\perp = \mu B$, the time rate of change of the transverse energy can be written as

$$\frac{dW_\perp}{dt} = \frac{d(\mu B)}{dt} = \mu\frac{dB}{dt} + B\frac{d\mu}{dt} = \mu v_\parallel \frac{dB}{dz} + B\frac{d\mu}{dt} \qquad [3.15]$$

where we have noted that the dB/dt term is simply the variation of the magnetic field as seen by the particle as its guiding center moves to new locations, so that this term can be written as $dB/dt = (dz/dt)(\partial B/\partial z) = v_\parallel(\partial B/\partial z)$. The rate of change of the parallel energy $W_\parallel$ is determined by the force F_z via the equation of motion, namely

$$\begin{aligned} m\frac{dv_\parallel}{dt} &= F_z \\ m\frac{dv_\parallel}{dt} &= -\mu\frac{dB}{dz} \\ mv_\parallel\frac{dv_\parallel}{dt} &= -v_\parallel\mu\frac{dB}{dz} \\ \frac{d\left(\frac{1}{2}mv_\parallel\right)}{dt} &= -v_\parallel\mu\frac{dB}{dz} \\ \frac{dW_\parallel}{dt} &= -v_\parallel\mu\frac{dB}{dz} \end{aligned} \qquad [3.16]$$

Substituting [3.16] and [3.15] in [3.14] we find

$$\mu v_\parallel\frac{dB}{dz} + B\frac{d\mu}{dt} - v_\parallel\mu\frac{dB}{dz} = 0 \quad \longrightarrow \quad \frac{d\mu}{dt} = 0$$

which indicates that the magnetic moment μ is an invariant of the particle motion.

Note from [3.15] and [3.16] that the perpendicular energy of the particle increases while the parallel energy decreases as it moves toward regions of higher **B**-field, so that $dB/dz > 0$. As the particle moves into regions of higher and higher B, its parallel velocity $v_\parallel$ eventually reduces to zero, and it 'reflects' back, moving in the other direction. In a symmetric magnetic field geometry, such as that in a dipole magnetic field similar to that of the earth and other magnetized planets, the particle would encounter a similar convergence of magnetic field lines as it travels to the other end of the system, from which it also reflects, thereby being forever trapped in a 'magnetic bottle'.

Until now we have assumed that the magnetic field exhibited no temporal changes, so that $\partial B/\partial t = 0$ and there are no induced electric fields. However, the magnetic moment

μ is still conserved even when there are time variations, as long as those variations occur slowly in comparison with the gyroperiod of the particles. Noting that temporal variations of the magnetic field would create a spatially varying electric field via Faraday's law, i.e., $-\partial\mathbf{B}/\partial t = \nabla \times \mathbf{E}$, let us consider the change in perpendicular energy $W_\perp$ due to an electric field.

Consider the equation of motion under the influence of the Lorentz force, namely

$$m\frac{d(\mathbf{v}_\perp + \mathbf{v}_\parallel)}{dt} = q[\mathbf{E} + (\mathbf{v}_\perp + \mathbf{v}_\parallel) \times \mathbf{B}]$$

and take the dot product of this equation with $\mathbf{v}_\perp$ to find

$$\frac{dW_\perp}{dt} = q(\mathbf{E} \cdot \mathbf{v}_\perp)$$

where we have used the fact that $W_\perp = \frac{1}{2}mv_\perp^2$. The increase in particle energy over one gyration can be found by averaging over a gyroperiod

$$\Delta W_\perp = q\int_0^{T_c} (\mathbf{E} \cdot \mathbf{v}_\perp)dt$$

where $T_c = 2\pi/\omega_c$. Assuming that the field changes slowly, the particle orbit is not perturbed significantly, and we can replace the integration in time with a line integral over the unperturbed circular orbit. In other words,

$$\Delta W_\perp = q\oint_C \mathbf{E} \cdot d\mathbf{l} = q\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

where $d\mathbf{l}$ is a line element along the closed gyroorbit C while $d\mathbf{s}$ is a surface element over the surface S enclosed by the gyroorbit. For changes much slower than the gyroperiod, we can replace $\partial\mathbf{B}/\partial t$ with $\omega_c\Delta B/(2\pi)$, with ΔB being the average change during one gyroperiod. We thus have

$$\Delta W_\perp = \frac{1}{2}q\omega_c r_c^2 \Delta B = \mu\Delta B \qquad [3.17]$$

using previously derived expressions for ω_c, r_c and μ. However, we know from [3.15] that

$$\Delta W_\perp = \mu\Delta B + B\Delta\mu \qquad [3.18]$$

Comparing [3.17] and [3.18] we find that $\Delta\mu = 0$, indicating that the magnetic moment is invariant even when particles are accelerated in electric field induced by slow temporal variations in the magnetic field.

3.5 OTHER GRADIENTS OF B

We have studied particle motion in nonuniform magnetic fields with particular types of inhomogeneities. The various spatial gradients of teh magnetic field can be summarized in tensor or dyadic notation as $\nabla \mathbf{B}$:

$$\nabla \mathbf{B} = \begin{bmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial Bx}{\partial y} & \frac{\partial B_x}{\partial z} \\ \frac{\partial B_y}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_y}{\partial z} \\ \frac{\partial B_z}{\partial x} & \frac{\partial B_z}{\partial y} & \frac{\partial B_z}{\partial z} \end{bmatrix}$$

Note that only eight of the nine components of $\nabla \mathbf{B}$ are independent, since the condition $\nabla \cdot \mathbf{B} = 0$ allows us to determine one of the diagonal terms in terms of the other two. In regions where there are no currents ($\mathbf{J} = 0$), we must also have $\nabla \times \mathbf{B} = 0$, imposing additional restrictions on the various components of $\nabla \mathbf{B}$.

The diagonal terms terms are sometimes referred to as the divergence terms and represent gradients along the **B** direction, i.e., $\nabla_{\parallel} B$, one of which $(\partial B_z/\partial z)$ was responsible for the mirror effect discussed in Section 3.4.

The terms $\partial B_z/\partial x$ and $\partial B_z/\partial y$ are known as the gradient terms and represent transverse gradients $(\nabla_{\perp} B)$ responsible for the gradient drift studied in Section 3.2.

The terms $\partial B_x/\partial z$ and $\partial B_y/\partial z$ are known as the curvature terms and represent change of direction of **B**, i.e., curvature, and were studied in Section 3.3.

The remaining terms (i.e., $\partial B_x/\partial y$ and $\partial B_y/\partial x$) are known as the shear terms and represent twisting of the magnetic field lines and are not important in particle motion.

EE356 Elementary Plasma Physics H. O. #6

Inan 10 April 2006

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LECTURE 3: PARTICLE MOTION IN NONUNIFORM B FIELDS

3.1 INTRODUCTION

Both naturally occurring plasmas and those that one encounters in many applications often exist in the presence of magnetic fields that do not vary appreciably in time but which vary with one or more coordinates of space, i.e., which are *nonuniform*. An important example of a nonuniform magnetic field is the so-called magnetic mirror configuration which is commonly used to confine plasmas, and which is also the mechanism by which energetic particles are trapped in the earth's radiation belts.

Assuming the absence of an electric field, and no temporal variations of the magnetic field[1], the kinetic energy of the particle must remain zero, since the magnetic force is at all times perpendicular to the motion of the particles as discussed in Lecture 2. In general, exact anaytical solutions for charged particle motions in a nonuniform magnetic field cannot be found. However, one very important configuration that can be studied analytically is the case in which the gyroradius r_c is much smaller than the spatial scales over which the magnetic field varies. In such cases, the motion of the particle can be decomposed into the fast gyromotion plus some type of relatively slow drift motion. The slow drift is associated with the motion of the guiding center, and the separation of its motion from the rapid gyration is similar to the simplest case analyzed in Lecture 2, where we saw that the guiding center simply moved linearly along the magnetic field as the particle executed its complicated gyromotion.

We now examine particle motion in different types of nonuniform magnetic fields, assuming the presence of only one type of inhomogeneity in each case.

[1] Note that any time variations of the magnetic field would lead to an electric field via Faraday's law, i.e., $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, which can in turn accelerate the particles.

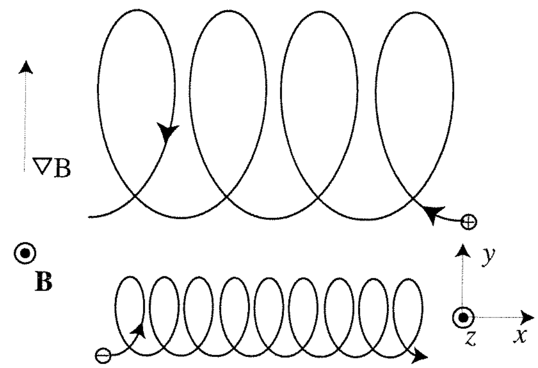

Fig. 3.1. Particle drifts due to a magnetic field gradient. [gradient]

3.2 GRADIENT DRIFT

We first consider a magnetic field the intensity of which varies in a direction perpendicular to the magnetic field vector. Without loss of generality, let $\mathbf{B}(y) = \hat{\mathbf{z}}B_z(y)$, as depicted in Figure 3.1. Since the field strength has a nonzero gradient ∇B_z in the y direction, we note that the local gyroradius r_c (i.e., the radius of curvature of the particle orbit) is large in regions where B is small and vice versa. Thus, on physical grounds alone, we expect a positive charge to drift to the left and a negative charge to drift to the right.

To find an expression for the particle drift velocity, we take advantage of expression [2.14] which gives drift velocity for any force perpendicular to the $\mathbf{B}$ field. For the field geometry depicted in Figure 3.1, this means a force in either x or y directions. Since we started with the premise that the gyration was much more rapid than the relatively slow drift, it is appropriate to determine the net resultant force averaged over one gyroperiod. The force perpendicular to $\mathbf{B}$ is the Lorentz force given by

$$\begin{aligned}\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) &= \hat{\mathbf{x}}qv_yB_z - \hat{\mathbf{y}}qv_xB_z \\ &\simeq \hat{\mathbf{x}}\,q\,v_y\left(B_0 + y\frac{\partial B_z}{\partial y}\right) - \hat{\mathbf{y}}qv_x\left(B_0 + y\frac{\partial B_z}{\partial y}\right)\end{aligned}$$

where we have expanded $B_z(y)$ into a Taylor series[2] around the guiding center of the particle (taken with no loss of generality to be at $x_g = 0$ and $y_g = 0$), with B_0 being the magnetic field intensity at the guiding center and y is the distance from the guiding center. In other words, we have written

$$B_z(z) = B_0 + y\frac{\partial B_z}{\partial y} + \cdots$$

and neglected higher order terms.

---

[2] In the more general case this expansion can be written as

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r}\cdot\nabla)\mathbf{B}_0 + \cdots$$

where $\mathbf{B}_0$ is the field at the guiding center and $\mathbf{r}$ is the position vector with the origin chosen as the guiding center.

We thus have the two transverse components of the Lorentz force as

$$F_x = qv_y\left(B_0 + y\frac{\partial B_z}{\partial y}\right) \tag{3.1a}$$

$$F_y = -qv_x\left(B_0 + y\frac{\partial B_z}{\partial y}\right) \tag{3.1b}$$

We wish to determine $< F_x >$ and $< F_y >$, where the brackets denote averaging over one gyroperiod. To do this, we can assume that the particles by and large follow the orbits for a uniform field, as determined in Lecture 2 (see [2.5 and [2.6]), namely:

$$x_c = r_c \sin(\omega_c t + \psi) \tag{3.2a}$$

$$y_c = r_c \cos(\omega_c t + \psi) \tag{3.2b}$$

$$v_x = v_\perp \cos(\omega_c t + \psi) \tag{3.2c}$$

$$v_y = -v_\perp \sin(\omega_c t + \psi) \tag{3.2d}$$

where ω_c has the same sign as q (i.e., is negative for electrons). Substituting in [3.1a] and [3.1b] we have

$$F_x = -qv_\perp \sin(\omega_c t + \psi)\left[B_0 + r_c \cos(\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right] \tag{3.3a}$$

$$F_y = -qv_\perp \cos(\omega_c t + \psi)\left[B_0 + r_c \cos(\omega_c t + \psi)\frac{\partial B_z}{\partial y}\right] \tag{3.3b}$$

The average over one gyroperiod ($2\pi/\omega_c$) of F_x is zero, since it contains the product of sine and cosine terms. The averaging of F_y has the product of a cosine with a cosine which results in a factor of $\frac{1}{2}$. We thus have,

$$\langle F_y \rangle = -\frac{qv_\perp r_c}{2}\frac{\partial B_z}{\partial y} \tag{3.4}$$

The drift velocity is then given from [2.14] as

$$\mathbf{v}_\nabla = \frac{(\mathbf{F}_\perp/q) \times \mathbf{B}}{B^2} = \frac{\langle F_y \rangle \hat{\mathbf{y}} \times \hat{\mathbf{z}} B_z}{qB_z^2} = -\frac{v_\perp r_c}{2B_z}\frac{\partial B_z}{\partial y}\,\hat{\mathbf{x}} \tag{3.5}$$

where the subscript '∇' indicates that the drift velocity is due to the gradient drift. Since the magnetic field direction was chosen arbitrarily, we can write [3.5] more generally as

$$\mathbf{v}_\nabla = \frac{v_\perp r_c}{2}\frac{\mathbf{B} \times \nabla B}{B^2} = \frac{mv_\perp^2}{2q}\frac{\mathbf{B} \times \nabla B}{B^3} \tag{3.6}$$

for any magnetic field $\mathbf{B} = \hat{\mathbf{B}}B$, and noting that $r_c = mv_\perp^2/(qB)$. The corresponding more general expression for the perpendicular gradient force $\mathbf{F}_\nabla$ is

$$\mathbf{F}_\nabla = -\frac{\frac{1}{2}mv_\perp^2}{B}\nabla B = -\frac{W_\perp}{B}\nabla B \qquad [3.7]$$

where $W_\perp$ is the perpendicular kinetic energy of the particle.

Equation [3.6] exhibits the dependencies that we expect on physical bases. Electrons and ions drift in opposite directions and the drift velocity is proportional to the perpendicular energy of the particle, namely $W_\perp = \frac{1}{2}mv_\perp^2$. Faster particles drift faster, since they have a larger gyroradiius and their orbits span a larger range of the inhomogeneity of the field.

3.3 CURVATURE DRIFT

When particles rapidly gyrate while moving along a magnetic field line which is curved, as depicted in Figure 3.2a, they experience a centrifugal force perpendicular to the magnetic field, which produces a drift as defined by [2.14]. Assuming once again that the spatial scale of the curvature is much larger than the gyroradius, we can focus our attention to the motion of the guiding center. The outward centrifugal force in the frame of reference moving with the guiding center at a velocity $v_\parallel$ is given by

$$\mathbf{F}_{\rm cf} = mv_\parallel^2\frac{\mathbf{R}_c}{R_c^2} \qquad [3.8]$$

where $\mathbf{R}_c$ is the vector pointing radially outward from the center of the circle described by the local curvature of the field and R_c has a magnitude equal to the radius of curvature.

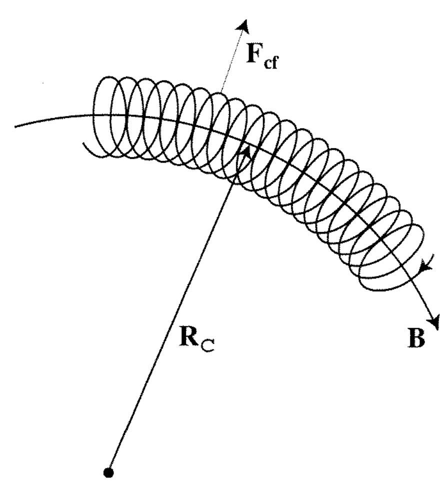

Fig. 3.2. Curvature drift. Particle drift in a curved magnetic field. ([curvature]

Using the force given in [3.8] in [2.14] we find the curvature drift velocity as

$$\mathbf{v}_\mathrm{R} = \frac{(\mathbf{F}_\mathrm{cf}/q) \times \mathbf{B}}{B^2} = \frac{m v_\parallel^2}{q} \frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \qquad [3.9]$$

In vacuum, curvature drift cannot by itself be the only drift since the curl of the magnetic field must be zero, namely $\nabla \times \mathbf{B} = 0$. Considering cylindrical coordinates with $\mathbf{B} = B_\phi(r)\hat{\phi}$, we must then have

$$(\nabla \times \mathbf{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(r B_\phi) = 0 \qquad \rightarrow \qquad B_\phi = \frac{A}{r}$$

where A is a constant. The gradient of $\mathbf{B}$ is then $\partial B_\phi/\partial r = -B_\phi/r = -A/r$. More generally, we can write the resulting gradient as $\nabla = -(B/R_c^2)\mathbf{R}_c$. Thus, the total drift due to both gradient and curvature effects can be written as

$$\mathbf{v}_\mathrm{total} = \mathbf{v}_\mathrm{R} + \mathbf{v}_\nabla = (v_\parallel^2 + \frac{1}{2} v_\perp) \frac{\mathbf{B} \times \nabla B}{w_c B^2} \qquad [3.10]$$

An example of gradient plus curvature drift is the *longitudinal drift* of radiation belt electrons around the Earth (see Figure 3.3). Note that the direction of the earth's magnetic field is from south-to-north.

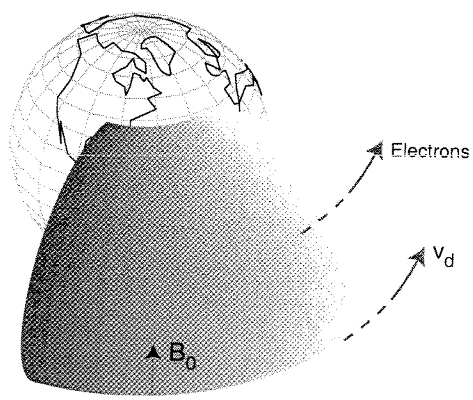

Fig. 3.3. Longitudinal drift of radiation belt electrons. Note that the direction of the earth's magnetic field $\mathbf{B}_0$ is from south to north. [longitudinal]

It is interesting to note that both the gradient and curvature drift velocities are inversely proportional to the charge q so that electrons and ions drift in opposite directions. The oppositely directed drifts of electrons and ions leads to a transverse current. The gradient drift current is given by

$$\begin{aligned}\mathbf{J}_\nabla &= N|q_e|[(\mathbf{v}_\nabla)_i - (\mathbf{v}_\nabla)_e] \\ &= \frac{N}{B^3}[(W_\perp)_i + (W_\perp)_e](\mathbf{B} \times \nabla B)\end{aligned}$$

where $N = N_i = N_e$ is the plasma density and $W_\perp = \frac{1}{2}mv_\perp^2$ is the perpendicular particle energy. Note that the gradient drift current $\mathbf{J}_\nabla$ flows in a direction perpendicular to both the magnetic field and its gradient.

Similarly, the different directional curvature drifts of the electrons and ions leads to a curvature drift current given by

$$\begin{aligned}\mathbf{J}_\mathrm{R} &= N|q_e|[(\mathbf{v}_\mathrm{R})_i - (\mathbf{v}_\mathrm{R})_e] \\ &= \frac{2N}{R_c^2 B^2}[(W_\parallel)_i + (W_\parallel)_e)](\mathbf{R}_c \times \mathbf{B})\end{aligned}$$

where $W_\parallel = \frac{1}{2}mv_\parallel^2$ is the parallel particle energy. The curvature drift current $\mathbf{J}_\mathrm{R}$ flows in a direction perpendicular to both the magnetic field and its curvature.

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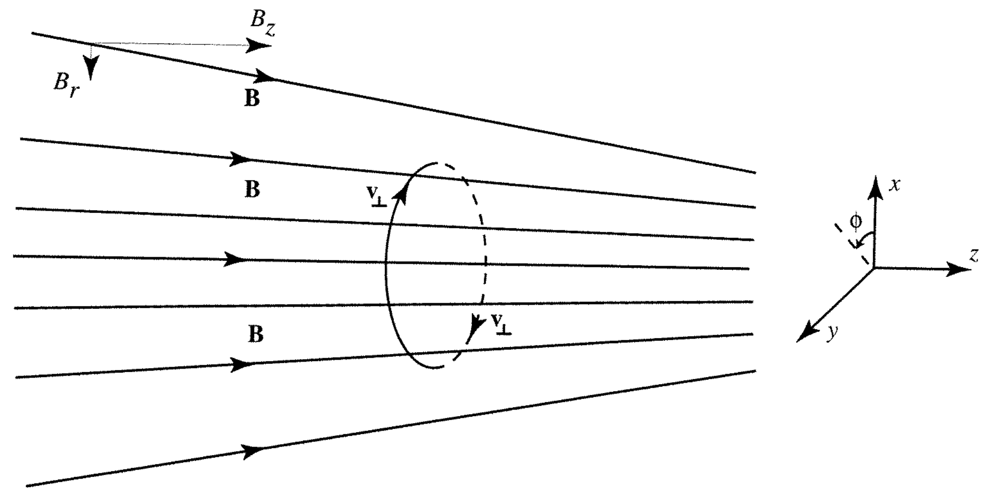

Fig. 3.4. Drift of a particle in a magnetic mirror configuration. [mirrorlecturethree]

during this motion can be found from [3.7], by noting that the magnetic field has a nonzero gradient in the z direction. We have

$$F_z = -\frac{\frac{1}{2}mv_\perp^2}{B_z}\frac{\partial B_z}{\partial z} = -\frac{W_\perp}{B_z}\frac{\partial B}{\partial z} = -\mu\frac{\partial B}{\partial z} \qquad [3.11]$$

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where B_r can be found using the fact that we must have $\nabla \cdot \mathbf{B} = 0$, so that

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0 \quad \longrightarrow \quad B_r \simeq -\frac{r}{2}\frac{\partial B_z}{\partial z} \qquad [3.13]$$

assuming that $\partial B_z/\partial z$ does not vary significantly with r. In other words, the total magnetic field in the case of a converging field line geometry of Figuremust be given by

$$\mathbf{B} = B_r\hat{\mathbf{r}} + B_z\hat{\mathbf{z}}$$

Evaluating B_r as given in [3.13] at $r = r_c$, and substituting in [3.12] we find the same expression for F_z as in [3.11].

With F_z determined, we can examine the variations of the parallel and perpendicular energies of the particle as its guiding center moves along z. Consider the total energy of the particle

$$W = W_\perp + W_\parallel$$

which must remain constant in the absence of electric fields, so that we have

$$\frac{dW_\perp}{dt} + \frac{dW_\parallel}{dt} = 0 \qquad [3.14]$$

Noting that $W_\perp = \mu B$, the time rate of change of the transverse energy can be written as

$$\frac{dW_\perp}{dt} = \frac{d(\mu B)}{dt} = \mu\frac{dB}{dt} + B\frac{d\mu}{dt} = \mu v_\parallel \frac{dB}{dz} + B\frac{d\mu}{dt} \qquad [3.15]$$

where we have noted that the dB/dt term is simply the variation of the magnetic field as seen by the particle as its guiding center moves to new locations, so that this term can be written as $dB/dt = (dz/dt)(\partial B/\partial z) = v_\parallel(\partial B/\partial z)$. The rate of change of the parallel energy $W_\parallel$ is determined by the force F_z via the equation of motion, namely

$$\begin{aligned} m\frac{dv_\parallel}{dt} &= F_z \\ m\frac{dv_\parallel}{dt} &= -\mu\frac{dB}{dz} \\ mv_\parallel\frac{dv_\parallel}{dt} &= -v_\parallel\mu\frac{dB}{dz} \\ \frac{d\left(\frac{1}{2}mv_\parallel\right)}{dt} &= -v_\parallel\mu\frac{dB}{dz} \\ \frac{dW_\parallel}{dt} &= -v_\parallel\mu\frac{dB}{dz} \end{aligned} \qquad [3.16]$$

Substituting [3.16] and [3.15] in [3.14] we find

$$\mu v_\parallel\frac{dB}{dz} + B\frac{d\mu}{dt} - v_\parallel\mu\frac{dB}{dz} = 0 \quad \longrightarrow \quad \frac{d\mu}{dt} = 0$$

which indicates that the magnetic moment μ is an invariant of the particle motion.

Note from [3.15] and [3.16] that the perpendicular energy of the particle increases while the parallel energy decreases as it moves toward regions of higher **B**-field, so that $dB/dz > 0$. As the particle moves into regions of higher and higher B, its parallel velocity $v_\parallel$ eventually reduces to zero, and it 'reflects' back, moving in the other direction. In a symmetric magnetic field geometry, such as that in a dipole magnetic field similar to that of the earth and other magnetized planets, the particle would encounter a similar convergence of magnetic field lines as it travels to the other end of the system, from which it also reflects, thereby being forever trapped in a 'magnetic bottle'.

Until now we have assumed that the magnetic field exhibited no temporal changes, so that $\partial B/\partial t = 0$ and there are no induced electric fields. However, the magnetic moment

μ is still conserved even when there are time variations, as long as those variations occur slowly in comparison with the gyroperiod of the particles. Noting that temporal variations of the magnetic field would create a spatially varying electric field via Faraday's law, i.e., $-\partial\mathbf{B}/\partial t = \nabla \times \mathbf{E}$, let us consider the change in perpendicular energy $W_\perp$ due to an electric field.

Consider the equation of motion under the influence of the Lorentz force, namely

$$m\frac{d(\mathbf{v}_\perp + \mathbf{v}_\parallel)}{dt} = q[\mathbf{E} + (\mathbf{v}_\perp + \mathbf{v}_\parallel) \times \mathbf{B}]$$

and take the dot product of this equation with $\mathbf{v}_\perp$ to find

$$\frac{dW_\perp}{dt} = q(\mathbf{E} \cdot \mathbf{v}_\perp)$$

where we have used the fact that $W_\perp = \frac{1}{2}mv_\perp^2$. The increase in particle energy over one gyration can be found by averaging over a gyroperiod

$$\Delta W_\perp = q\int_0^{T_c} (\mathbf{E} \cdot \mathbf{v}_\perp)dt$$

where $T_c = 2\pi/\omega_c$. Assuming that the field changes slowly, the particle orbit is not perturbed significantly, and we can replace the integration in time with a line integral over the unperturbed circular orbit. In other words,

$$\Delta W_\perp = q\oint_C \mathbf{E} \cdot d\mathbf{l} = q\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

where $d\mathbf{l}$ is a line element along the closed gyroorbit C while $d\mathbf{s}$ is a surface element over the surface S enclosed by the gyroorbit. For changes much slower than the gyroperiod, we can replace $\partial\mathbf{B}/\partial t$ with $\omega_c \Delta B/(2\pi)$, with ΔB being the average change during one gyroperiod. We thus have

$$\Delta W_\perp = \frac{1}{2}q\omega_c r_c^2 \Delta B = \mu\Delta B \qquad [3.17]$$

using previously derived expressions for ω_c, r_c and μ. However, we know from [3.15] that

$$\Delta W_\perp = \mu\Delta B + B\Delta\mu \qquad [3.18]$$

Comparing [3.17] and [3.18] we find that $\Delta\mu = 0$, indicating that the magnetic moment is invariant even when particles are accelerated in electric field induced by slow temporal variations in the magnetic field.

3.5 OTHER GRADIENTS OF B

We have studied particle motion in nonuniform magnetic fields with particular types of inhomogeneities. The various spatial gradients of teh magnetic field can be summarized in tensor or dyadic notation as $\nabla\mathbf{B}$:

$$\nabla\mathbf{B} = \begin{bmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial Bx}{\partial y} & \frac{\partial B_x}{\partial z} \\ \frac{\partial B_y}{\partial x} & \frac{\partial B_y}{\partial y} & \frac{\partial B_y}{\partial z} \\ \frac{\partial B_z}{\partial x} & \frac{\partial B_z}{\partial y} & \frac{\partial B_z}{\partial z} \end{bmatrix}$$

Note that only eight of the nine components of $\nabla\mathbf{B}$ are independent, since the condition $\nabla \cdot \mathbf{B} = 0$ allows us to determine one of the diagonal terms in terms of the other two. In regions where there are no currents ($\mathbf{J} = 0$), we must also have $\nabla \times \mathbf{B} = 0$, imposing additional restrictions on the various components of $\nabla\mathbf{B}$.

The diagonal terms terms are sometimes referred to as the divergence terms and represent gradients along the **B** direction, i.e., $\nabla_{\parallel} B$, one of which $(\partial B_z/\partial z)$ was responsible for the mirror effect discussed in Section 3.4.

The terms $\partial B_z/\partial x$ and $\partial B_z/\partial y$ are known as the gradient terms and represent transverse gradients ($\nabla_{\perp} B$) responsible for the gradient drift studied in Section 3.2.

The terms $\partial B_x/\partial z$ and $\partial B_y/\partial z$ are known as the curvature terms and represent change of direction of **B**, i.e., curvature, and were studied in Section 3.3.

The remaining terms (i.e., $\partial B_x/\partial y$ and $\partial B_y/\partial x$) are known as the shear terms and represent twisting of the magnetic field lines and are not important in particle motion.