Universal Quantized Spin-Hall Conductance Fluctuation in Graphene

Zhenhua Qiao,[1] Jian Wang,[1,*] Yadong Wei,[2] and Hong Guo[3]

[1]Department of Physics and the Center of Theoretical and Computational Physics, The University of Hong Kong, Hong Kong, China
[2]Department of Physics, School of Physics, Shenzhen University, Shenzhen, China
[3]Department of Physics, McGill University, Montreal, PQ, Canada H3A 2T8
(Received 1 December 2007; published 1 July 2008)

We report theoretical investigations of the quantized spin-Hall conductance fluctuation of graphene in the presence of disorder. Two graphene models that exhibit the quantized spin-Hall effect (QSHE) are analyzed. Model I is with unitary symmetry under an external magnetic field $B \neq 0$ but with a zero spin-orbit interaction, $t_{SO} = 0$. Model II is with symplectic symmetry where $B = 0$ but $t_{SO} \neq 0$. The two models give exactly the same universal QSHE conductance fluctuation value $0.285 \pm 0.005e/4\pi$ regardless of symmetry. We also examined a third model that exhibits QSHE but with quadratic dispersion and obtained the same results. Finally, all three models of QSHE have a one-sided log-normal distribution for spin-Hall conductance. Our results strongly suggest that the quantized spin-Hall conductance fluctuation belongs to a new universality class.

DOI: 10.1103/PhysRevLett.101.016804 PACS numbers: 73.43.Cd, 71.70.Ej, 72.15.Rn, 81.05.Uw

One of the most important transport features of mesoscopic conductors is the *universal* conductance fluctuation (UCF) in the diffusive regime caused by disorder scattering and quantum coherence [1]. The universality characterized by the value of UCF only depends on the dimensionality and symmetry of the system. According to random matrix theory (RMT) [2], there are three ensembles or universalities due to symmetry: (1) When time-reversal and spin-rotation symmetries are present, i.e., when the magnetic field $B = 0$ and spin-orbit interaction (SOI) $t_{SO} = 0$, the Hamiltonian H of the system is an orthogonal matrix and one has circular orthogonal ensemble (COE). COE is characterized by a symmetry index $\beta = 1$. (2) If time-reversal symmetry is broken by $B \neq 0$, H is unitary and one has the circular unitary ensemble (CUE) characterized by $\beta = 2$. (3) If spin-rotation symmetry is broken by $t_{SO} \neq 0$ while time-reversal symmetry is maintained, one has the circular symplectic ensemble (CSE) for which $\beta = 4$. While different ensembles have different values of UCF, it is amazing that the multitude of possibilities of electron dynamics in nature can be classified by only a few ensembles [3]. For instance, in one dimension (1D) the UCF value is given by [2] $[\mathrm{rms}(G)]^2 = 2/(15\beta)$.

Recently, *universal* fluctuation was also found to occur in 2D mesoscopic spin-Hall effect (SHE) [4]. SHE can be induced by spin-orbit interaction, for instance Rashba SOI in 2D, such that chemical potentials of the spin-up or -down channels become different at the two boundaries of a mesoscopic sample [5,6]. With disorder, numerical calculations showed [4] that the spin-Hall conductance G_{SH} of a 2D mesoscopic system fluctuates from sample to sample with a value $\mathrm{rms}(G_{SH}) \approx 0.18e/4\pi$: this is independent of system details thus universal, and the phenomenon is termed universal spin-Hall conductance fluctuation (USCF). The numerical value of USCF has been quantitatively confirmed by RMT [7]. For most situations, G_{SH} itself may have any value in units of $e/4\pi$ depending on system details. On the other hand, several authors have advanced the notion of *quantized* SHE (QSHE) for situations where electronic edge states exist: in QSHE G_{SH} takes integer multiples of $e/4\pi$. In particular, QSHE is shown to occur in 2D graphene due to SOI plus the peculiarity of graphene electronic structure [8]. QSHE is also predicted to occur in graphene without SOI but with an external magnetic field [9]. Therefore, using the language of RMT [2], QSHE occurs in graphene with CUE where $B \neq 0$ but $t_{SO} = 0$, and with CSE where $B = 0$ but $t_{SO} \neq 0$.

Several important and interesting questions therefore arise concerning the universality of QSHE: is it still classifiable by the RMT ensembles? As the disorder is increased, is there a USCF for QSHE and if there is, is the value different from the USCF for SHE that is $0.18e/4\pi$? What is the distribution of G_{SH} in QSHE? Indeed, all these questions are related to the curiosity, i.e., whether or not the Dirac dispersion relation of graphene brings new physics to the spin-Hall conductance fluctuation in the quantized SHE. It is the purpose of this work to investigate these issues.

To be more specific, we investigate the two graphene models that exhibit QSHE [8,9] as mentioned above. In the first model, model I [9], SOI is neglected in the graphene but a magnetic field is applied causing a Zeeman splitting. Model I has unitary symmetry and importantly is in the quantum Hall regime where edge states are present. Because of the Zeeman splitting and graphene energy spectrum both electronlike and holelike edge states exist near the Fermi level forming countercirculating edge states in graphene that has been confirmed experimentally [10]. It is these countercirculating edge states that lead to QSHE [9]. The second model, model II, is the one proposed by Kane and Mele [8] where intrinsic SOI gives rise to "spin filtered" edge states that cause QSHE based on an idea discussed by Haldane [11]. Clearly, model II has symplectic symmetry. Although the value of SOI parameter t_{SO} for graphene is small [12], model II is nevertheless very useful for our purpose, namely to investigate the universality class of QSHE. As we show later, the value of t_{SO}—as long as it is nonzero—turns out to be irrelevant as far as universality is concerned. From the symmetry point of view, one would expect these two models to belong to different universality classes. To our surprise, extensive numerical results indicate that in the presence of edge states, the QSHE dominates the physics and these two models give exactly the same universal value for USCF $= 0.285e/4\pi$ regardless of symmetry. The distribution of G_{SH} in the QSHE regime is found to obey a one-sided log-normal distribution: this is qualitatively different from the conventional UCF for charge and USCF for SHE where it is a Gaussian distribution. To find out whether the universal feature is solely due to graphene structures or not, we have studied the spin-Hall fluctuations in a third model that has symplectic symmetry on a square lattice with a quadratic dispersion relation [13]. The quantized spin-Hall effect in this model can be realized in Te based materials [13]. Our results show that the same universal value for USCF is obtained with the one-sided log-normal distribution.

In a tight-binding representation, the Hamiltonian for 2D honeycomb lattice of graphene can be written as

$$H_1 = H_0 - \sum_{\langle ij\rangle} te^{i2\pi\phi_{ij}} c^\dagger_{i\sigma} c_{j\sigma} + g_s \sum_{i\sigma} c^\dagger_{i\sigma}(\boldsymbol{\sigma}\cdot\mathbf{B})c_{i\sigma}$$

for model I, and

$$H_2 = H_0 - \sum_{\langle ij\rangle} tc^\dagger_{i\sigma} c_{j\sigma} + \frac{2i}{\sqrt{3}} t_{SO} \sum_{\langle\langle ij\rangle\rangle} c^\dagger_i \boldsymbol{\sigma}\cdot(\mathbf{d}_{kj}\times\mathbf{d}_{ik})c_j$$

for model II, where $H_0 = \sum_{i\sigma}\epsilon_i c^\dagger_{i\sigma}c_{i\sigma}$ and $c^\dagger_{i\sigma}$ ($c_{i\sigma}$) is the creation (annihilation) operator for an electron with spin σ on site i. H_0 is the on-site single particle energy where diagonal disorder is introduced by drawing ϵ_i randomly from a uniform distribution in the interval $[-W/2, W/2]$. Here W measures strength of disorder. The second term in H_1 is due to nearest neighbor hopping and the presence of a magnetic field, the last term in H_1 is due to Zeeman energy. Here $g_s = (1/2)g\mu_B$ (with $g = 4$) is the Lande g factor, phase $\phi_{ij} = \int \mathbf{A}\cdot dl/\phi_0$, $\phi_0 = h/e$ is the quantum of flux, and the spin-Hall conductance and its fluctuation are in units of $e/4\pi$. In H_2 the last term is the SOI that involves next nearest sites of indices i, j with k the common nearest neighbor of i and j, and $\mathbf{d}_{ik}$ describes a vector pointing from k to i.

We use the four-probe device schematically shown in the inset of Fig. 1 to investigate USCF in QSHE. The four probes are exact extensions from the central scattering region; i.e., the probes are graphene nanoribbons. The number of sites in the scattering region is denoted as $N = n_x \times n_y$, where there are $n_x = 8\times n + 1$ sites on $n_y = 4\times n$ chains. We apply external bias voltages V_i with $i = 1, 2, 3, 4$ at the four different probes as $V_i = (v/2, 0, -v/2, 0)$. The spin-Hall conductance G_{SH} can be calculated from the multiprobe Landauer-Buttiker formula [4] $G_{SH} = (e/8\pi)[(T_{2\uparrow,1} - T_{2\downarrow,1}) - (T_{2\uparrow,3} - T_{2\downarrow,3})]$, where the transmission coefficient is given by $T_{2\sigma,1} = \mathrm{Tr}(\Gamma_{2\sigma}G^r\Gamma_1 G^a)$ with $G^{r,a}$ being the retarded and the advanced Green functions of the central disordered region which can be evaluated numerically. The quantities $\Gamma_{i\sigma}$ are the linewidth functions describing coupling of the probes and the scattering region and are obtained by calculating self-energies Σ^r of the semi-infinite leads using a transfer matrices method [14]. The spin-Hall conductance fluctuation is defined as $\mathrm{rms}(G_{SH}) \equiv \sqrt{\langle G^2_{SH}\rangle - \langle G_{SH}\rangle^2}$, where $\langle\cdots\rangle$ denotes averaging over an ensemble of samples with different configurations of the same disorder strength W. In the following, our numerical data are mainly collected on a system with $n = 8$, i.e., with 32×65 sites in the graphene. In the rest of the Letter, we fix units by setting energy E, disorder strength W, SOI coupling t_{SO} in terms of the hopping parameter t, and the magnetic field in terms of magnetic flux ϕ.

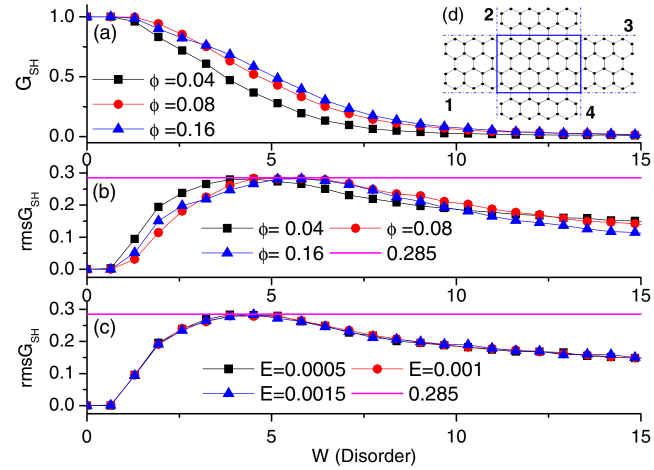

FIG. 1 (color online). (a)–(c) Spin-Hall conductance and its fluctuation versus disorder strength at different energies and magnetic fluxes for the first model. (d) Schematic plot of the four-terminal graphene device.

We first examine model I which has unitary symmetry. In the absence of disorder, countercirculating edge states are formed [9,10]. A transverse flow of spin current between probes 2 and 4 is generated when the bias voltage is applied across probes 1 to 3 leading to the QSHE. In the regime of QSHE, we now increase disorder strength W. This causes a breakdown of the integer value of G_{SH} and induces sample-to-sample fluctuations of G_{SH}. Figure 1(a) plots the average G_{SH} by calculating 5000 samples for each point on the figure; Fig. 1(b) plots the corresponding fluctuation $\mathrm{rms}(G_{SH})$, as a function of W. When W is increased, G_{SH} decreases from its quantized value $G_{SH} = 1$ and $\mathrm{rms}(G_{SH})$ increases. The breakdown of quantized

G_{SH} is due to W that causes a direct transmission from probe 1 to 3; this is shown in Fig. 3(c) where the direct transmission T_{31} is plotted against W. From T_{31} we conclude that the graphene device is in an insulating regime at small W, i.e., zero or very small T_{31}; it is in a diffusive regime for intermediate W and finally reentrant to the insulating regime for large W. We note that for a different set of system parameters, the diffusive regime corresponds to a different range of W. For a given E or ϕ, rms(G_{SH}) develops a "plateau" region, e.g., in the range $W = [3, 7]$ in Fig. 1(b). This plateau is at rms(G_{SH}) $= 0.285$ in unit of $e/4\pi$. Although the plateau range of W depends on specific values of E or ϕ, our results show that it is always inside the diffusive regime. From Fig. 3 we found that rms(G_{SH}) $= 0.285$ is always true if there is a plateau, i.e., if the diffusive transport regime is established. We therefore identify rms(G_{SH}) $= 0.285$ as a "universal" value. This USCF value is different from that of the conventional SHE situation [4,7] where the universal value is 0.18. Therefore QSHE and SHE belong to different universality classes due to these different statistical properties.

Next, we investigate model II that has a symplectic symmetry. For such a graphene device there is an energy gap between $-1 < E < 1$, within which edge states exist [8]. Figure 2 plots averaged G_{SH} and rms(G_{SH}) versus W for a given set of E, t_{SO} parameter values. Five thousand samples were calculated for the disorder averaging. Similar behavior is found as that of model I. For different values of t_{SO}, rms(G_{SH}) reaches a plateau at a different range of W [see Fig. 2(b)]. Amazingly, all plateaus have the same value and this value is precisely rms(G_{SH}) $= 0.285$. To further confirm this finding, Fig. 2(c) plots rms(G_{SH}) versus W for a fixed $t_{SO} = 0.7$ but several different values of energy E. Again, the same conclusion is obtained. This indicates that there exists a transport regime where the QSHE conductance fluctuation has a universal behavior independent of disorder (albeit in a region roughly $2t \sim 5$ eV), energy, and SOI.

Now we examine the third model that has a symplectic symmetry but on the square lattice with four probes. The Hamiltonian is given in Ref. [13]. Note that the quadratic dispersion relation of this model is different from the Dirac dispersion relation in graphene. Figure 3 shows the averaged G_{SH} and rms(G_{SH}) versus W for different energies E. The plateau values of rms(G_{SH}) are, again, ~ 0.285. Results of Figs. 1–3 strongly suggest that there is a universal spin-Hall conductance fluctuation in the quantized spin-Hall regime with USCF $= 0.285 \pm 0.005$ in units of $e/4\pi$. This is different from the conventional SOI induced SHE where USCF $= 0.18$ [4].

Very importantly, it appears that symmetry does not play a role in the QSHE regime at least for the CUE and CSE cases we have examined: both give rms(G_{SH}) $= 0.285$. To further support this finding, we calculated the distribution function of G_{SH}, $P(G_{SH})$, in the QSHE regime. Such a distribution is a Gaussian for conventional SHE in the diffusive regime [4]. For QSHE, Figs. 4(a)–4(d) plot $P(G_{SH})$ for four different values of W in the universal regime for model II, which has CSE symmetry. Data were collected by calculating 84 000 samples for each W. The distributions are completely different from a Gaussian. We found that by using $\ln(G_{SH})$ as a variable and plotting $P(\ln(G_{SH}))$, all the distributions become one-sided log-normal [see Figs. 4(e)–4(h)]. For model I, which has CUE symmetry, and the third model, our results give the same conclusion; i.e., the distribution of quantum spin-Hall conductance is a one-sided log-normal. Therefore, for the three models we investigated, not only is USCF rms(G_{SH}) $= 0.285$ the same, but also the distribution function is the same. This strongly indicates that in the presence

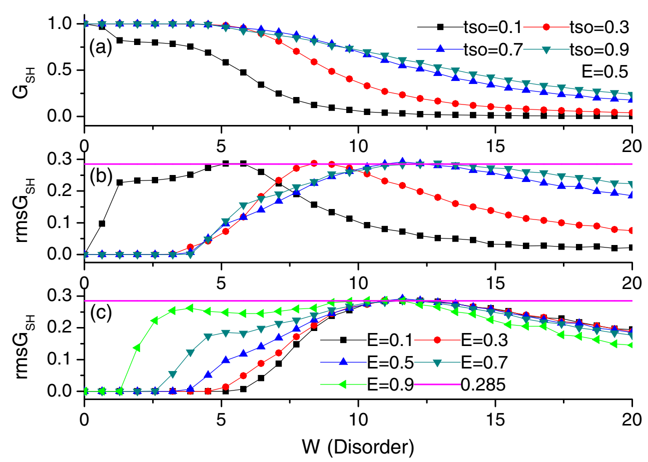

FIG. 2 (color online). (a)–(c) Spin-Hall conductance and its fluctuation versus disorder strength at different energies and magnetic fluxes for the second model. The parameters and symbols used in (a) and (b) are the same.

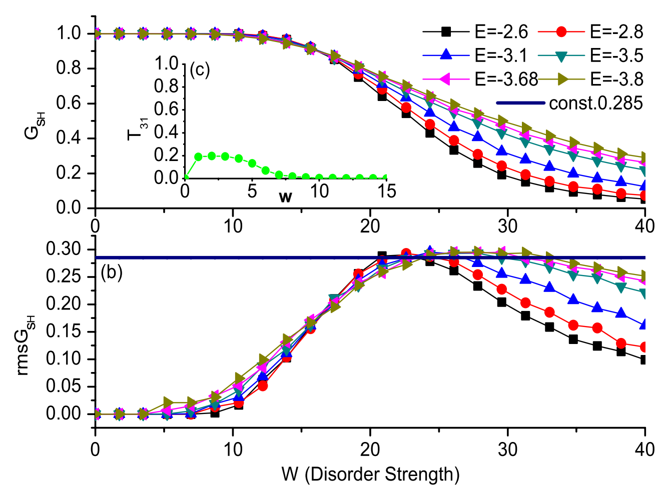

FIG. 3 (color online). (a),(b) Spin-Hall conductance and its fluctuation versus disorder strength at different energies for the third model for a 40×40 square lattice with 5000 configurations for each data. (c) The transmission coefficient T_{31} versus disorders for the first model [Fig. 1(c)].

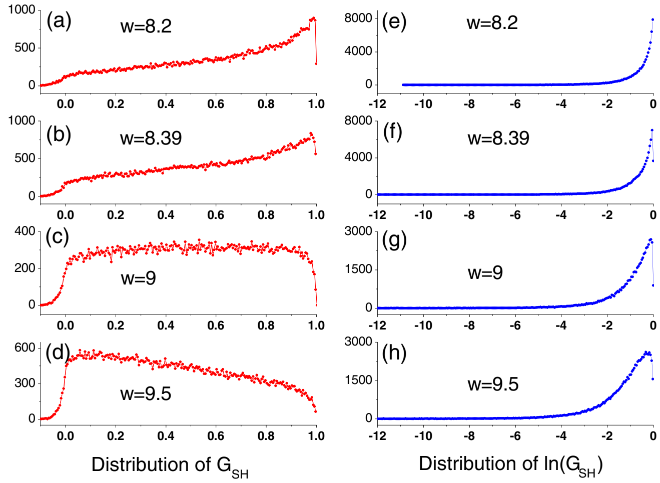

FIG. 4 (color online). (a)–(d) The distribution of spin-Hall conductance at different disorder strengths for the second model. (e)–(h) The distribution of $\ln(G_{\rm SH})$.

of edge states (i.e., QSHE), systems with unitary symmetry and symplectic symmetry belong to the same universality class that is different from the conventional SHE.

Finally, as a further confirmation of the QSHE universality class, we have carried out extensive calculation on spin-Hall conductance fluctuation for the same four-probe graphene device with additional Rashba SOI t_R [15]. For *nonzero* t_R, three cases are of interest. (1) $B = 0$ and $t_{\rm SO} = 0$. For this situation it is obvious that there is no edge state and therefore the spin-Hall effect caused by t_R is not quantized. Indeed, here we did not obtain the USCF for QSHE but obtained a value of 0.18 for all energies, i.e., the same as the conventional USCF found before [4,7]. As expected, for this case the distribution of $G_{\rm SH}$ was found to be a Gaussian. (2) When $|E| < 1$, for both model I and model II our numerical results show that USCF $= 0.285$ remains the same as long as t_R does not destroy the edge states. (3) When $|E| > 1$, there is no edge state in model II [8]; our results show that USCF $= 0.18$ for any $t_{\rm SO}$. Therefore, edge states dominate the quantized spin-Hall physics and t_R is an irrelevant parameter (for both model I and model II). On the other hand, if edge states are absent $t_{\rm SO}$ becomes an irrelevant parameter (for model II). This clearly shows the landscape of the universality class and it is the edge state that drives the system from the universality of USCF $= 0.18$ to the new universality we have discussed.

In summary, we have investigated quantized spin-Hall conductance fluctuation for three models with unitary and symplectic symmetry, respectively. Our numerical results show that three models exhibit the same universal quantized spin-Hall conductance fluctuation with the value USCF $= 0.285 \pm 0.005 e/4\pi$. The fact that both Dirac dispersion relation and quadratic dispersion relation give rise to the same USCF indicates that the edge states dominate the physics in the QSHE regime. Because of the presence of edge states, the distribution of quantum spin-Hall conductance obeys one-sided log-normal distribution for three models. This strongly suggests that the quantized spin-Hall conductance fluctuation for systems with both unitary symmetry and symplectic symmetry belongs to the same universality class that is different from the usual spin-Hall conductance fluctuation in the absence of edge states.

This work was financially supported by a RGC grant (HKU 704607P) from the government of HKSAR and LuXin Energy Group (J. W.), NSFC under Grant No. 10574093 (Y. D. W.), and NSERC of Canada, FQRNT of Québec and Canadian Institute of Advanced Research (H. G.). We thank Dr. W. Ren for useful discussions on the analysis of the log-normal distributions.

*jianwang@hkusua.hku.hk

[1] L. B. Altshuler, JETP Lett. **41**, 648 (1985); P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985); P. A. Lee, A. D. Stone, and H. Fukuyama, Phys. Rev. B **35**, 1039 (1987).
[2] C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
[3] An exceptional case to the COE, CUE, and CSE universalities is in the integer quantum Hall regime where it is the edge states that carry charge current. As disorder is increased and quantized Hall plateaus destroyed, the Hall conductance fluctuates from sample to sample. The universality class of Hall conductance fluctuation in the quantum Hall regime has not been well understood.
[4] W. Ren *et al.*, Phys. Rev. Lett. **97**, 066603 (2006).
[5] S. Murakami *et al.*, Science **301**, 1348 (2003).
[6] J. Sinova *et al.*, Phys. Rev. Lett. **92**, 126603 (2004).
[7] J. H. Bardarson *et al.*, Phys. Rev. Lett. **98**, 196601 (2007).
[8] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[9] D. A. Abanin, P. A. Lee, and L. S. Levitov, Phys. Rev. Lett. **96**, 176803 (2006).
[10] D. A. Abanin *et al.*, Phys. Rev. Lett. **98**, 196806 (2007).
[11] F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
[12] H. Min *et al.*, Phys. Rev. B **74**, 165310 (2006).
[13] B. A. Bernevig *et al.*, Science **314**, 1757 (2006).
[14] López-Sancho *et al.*, J. Phys. F **14**, 1205 (1984).
[15] L. Sheng *et al.*, Phys. Rev. Lett. **95**, 136602 (2005).