Universal Quantized Spin-Hall Conductance Fluctuation in Graphene

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We report theoretical investigations of the quantized spin-Hall conductance fluctuation of graphene in the presence of disorder. Two graphene models that exhibit the quantized spin-Hall effect (QSHE) are analyzed. Model I is with unitary symmetry under an external magnetic field $B \neq 0$ but with a zero spin-orbit interaction, $t_{SO} = 0$. Model II is with symplectic symmetry where $B = 0$ but $t_{SO} \neq 0$. The two models give exactly the same universal QSHE conductance fluctuation value $0.285 \pm 0.005/e/4\pi$ regardless of symmetry. We also examined a third model that exhibits QSHE but with quadratic dispersion and obtained the same results. Finally, all three models of QSHE have a one-sided log-normal distribution for spin-Hall conductance. Our results strongly suggest that the quantized spin-Hall conductance fluctuation belongs to a new universality class.

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One of the most important transport features of mesoscopic conductors is the universal conductance fluctuation (UCF) in the diffusive regime caused by disorder scattering and quantum coherence [1]. The universality characterized by the value of UCF only depends on the dimensionality and symmetry of the system. According to random matrix theory (RMT) [2], there are three ensembles or universalities due to symmetry: (1) When time-reversal and spin-rotation symmetries are present, i.e., when the magnetic field $B = 0$ and spin-orbit interaction (SOI) $t_{SO} = 0$, the Hamiltonian $H$ of the system is an orthogonal matrix and one has circular orthogonal ensemble (COE). COE is characterized by a symmetry index $\beta = 1$. (2) If time-reversal symmetry is broken by $B \neq 0$, $H$ is unitary and one has the circular unitary ensemble (CUE) characterized by $\beta = 2$. (3) If spin-rotation symmetry is broken by $t_{SO} \neq 0$ while time-reversal symmetry is maintained, one has the circular symplectic ensemble (CSE) for which $\beta = 4$. While different ensembles have different values of UCF, it is amazing that the multitude of possibilities of electron dynamics in nature can be classified by only a few ensembles [3]. For instance, in one dimension (1D) the UCF value is given by [2] $[\text{rms}(G)]^2 = 2/(15\beta)$.

Recently, universal fluctuation was also found to occur in 2D mesoscopic spin-Hall effect (SHE) [4]. SHE can be induced by spin-orbit interaction, for instance Rashba SOI in 2D, such that chemical potentials of the spin-up or -down channels become different at the two boundaries of a mesoscopic sample [5,6]. With disorder, numerical calculations showed [4] that the spin-Hall conductance $G_{SH}$ of a 2D mesoscopic system fluctuates from sample to sample with a value $\text{rms}(G_{SH}) = 0.18e/4\pi$: this is independent of system details thus universal, and the phenomenon is termed universal spin-Hall conductance fluctuation (USCF). The numerical value of USCF has been quantitatively confirmed by RMT [7]. For most situations, $G_{SH}$ itself may have any value in units of $e/4\pi$ depending on system details. On the other hand, several authors have advanced the notion of quantized SHE (QSHE) for situations where electronic edge states exist: in QSHE $G_{SH}$ takes integer multiples of $e/4\pi$. In particular, QSHE is shown to occur in 2D graphene due to SOI plus the peculiarity of graphene electronic structure [8]. QSHE is also predicted to occur in graphene without SOI but with an external magnetic field [9]. Therefore, using the language of RMT [2], QSHE occurs in graphene with CUE where $B = 0$ but $t_{SO} = 0$, and with CSE where $B = 0$ but $t_{SO} \neq 0$.

Several important and interesting questions therefore arise concerning the universality of QSHE: is it still classifiable by the RMT ensembles? As the disorder is increased, is there a USCF for QSHE and if there is, is the value different from the USCF for SHE that is $0.18e/4\pi$? What is the distribution of $G_{SH}$ in QSHE? Indeed, all these questions are related to the curiosity, i.e., whether or not the Dirac dispersion relation of graphene brings new physics to the spin-Hall conductance fluctuation in the quantized SHE. It is the purpose of this work to investigate these issues.

To be more specific, we investigate the two graphene models that exhibit QSHE [8,9] as mentioned above. In the first model, model I [9], SOI is neglected in the graphene but a magnetic field is applied causing a Zeeman splitting. Model I has unitary symmetry and importantly is in the quantum Hall regime where edge states are present. Because of the Zeeman splitting and graphene energy spectrum both electronlike and holelike edge states exist near the Fermi level forming countercirculating edge states in graphene that has been confirmed experimentally [10]. It is these countercirculating edge states that lead to QSHE [9]. The second model, model II, is the one proposed by Kane and Mele [8] where intrinsic SOI gives rise to “spin filtered” edge states that cause QSHE based on an idea...
discussed by Haldane [11]. Clearly, model II has symplectic symmetry. Although the value of SOI parameter $t_{SO}$ for graphene is small [12], model II is nevertheless very useful for our purpose, namely to investigate the universality class of QSHE. As we show later, the value of $t_{SO}$—as long as it is non-zero—turns out to be irrelevant as far as universality is concerned. From the symmetry point of view, one would expect these two models to belong to different universality classes. To our surprise, extensive numerical results indicate that in the presence of edge states, the QSHE dominates the physics and these two models give exactly the same universal value for USCF = $0.285e/4\pi$ regardless of symmetry. The distribution of $G_{SH}$ in the QSHE regime is found to obey a one-sided log-normal distribution: this is qualitatively different from the conventional UCF for charge and USCF for SHE where it is a Gaussian distribution. To find out whether the universal feature is solely due to graphene structures or not, we have studied the spin-Hall fluctuations in a third model that has symplectic symmetry on a square lattice with a quadratic dispersion relation [13].

The quantized spin-Hall effect in this model can be realized in Te based materials [13]. Our results show that the same universal value for USCF is obtained with the one-sided log-normal distribution.

In a tight-binding representation, the Hamiltonian for 2D honeycomb lattice of graphene can be written as

$$H_1 = H_0 - \sum_{\langle ij \rangle} t e^{2i\pi \phi_{ij}} c_{i\sigma}^\dagger c_{j\sigma} + g_z \sum_{i\sigma} c_{i\sigma}^\dagger (\sigma \cdot \mathbf{B}) c_{i\sigma}$$

for model I, and

$$H_2 = H_0 - \sum_{\langle ij \rangle} t c_{i\sigma}^\dagger c_{j\sigma} + \frac{2i}{\sqrt{3}} t_{SO} \sum_{\langle ij \rangle} c_{i\sigma}^\dagger (\mathbf{d}_{ij} \times \mathbf{d}_{ik}) c_{j\sigma}$$

for model II, where $H_0 = \sum_{i\sigma} \epsilon_i c_{i\sigma}^\dagger c_{i\sigma}$ and $c_{i\sigma}^\dagger$ ($c_{i\sigma}$) is the creation (annihilation) operator for an electron with spin $\sigma$ on site $i$. $H_0$ is the on-site single particle energy where diagonal disorder is introduced by drawing $\epsilon_i$ randomly from a uniform distribution in the interval $[-W/2, W/2]$. Here $W$ measures strength of disorder. The second term in $H_1$ is due to nearest neighbor hopping and the presence of a magnetic field, the last term in $H_1$ is due to Zeeman energy. Here $g_s = (1/2)g_\mu_B$ (with $g = 4$) is the Lande $g$ factor, phase $\phi_{ij} = \int \mathbf{A} \cdot d\mathbf{l}/\phi_0$, $\phi_0 = h/e$ is the quantum of flux, and the spin-Hall conductance and its fluctuation are in units of $e/4\pi$. In $H_2$ the last term is the SOI that involves next nearest sites of indices $i, j$ with $k$ the common nearest neighbor of $i$ and $j$, and $\mathbf{d}_{ik}$ describes a vector pointing from $k$ to $i$.

We use the four-probe device schematically shown in the inset of Fig. 1 to investigate USCF in QSHE. The four probes are exact extensions from the central scattering region; i.e., the probes are graphene nanoribbons. The number of sites in the scattering region is denoted as $N = n_x \times n_y$, where there are $n_z = 8 \times n + 1$ sites on $n_x = 4 \times n$ chains. We apply external bias voltages $V_i$ with $i = 1, 2, 3, 4$ at the four different probes as $V_i = (v/2, 0, -v/2, 0)$. The spin-Hall conductance $G_{SH}$ can be calculated from the multiprobe Landauer-Buttiker formula [4] $G_{SH} = \langle e/8\pi \rangle [(T_{21,1} - T_{21,1}) - (T_{21,3} - T_{21,3})]$, where the transmission coefficient is given by $T_{2\sigma,1} = \text{Tr}(\Gamma_{2\sigma,1} G^r G^a)$ with $G^{r,a}$ being the retarded and the advanced Green functions of the central disordered region which can be evaluated numerically. The quantities $\Gamma_{i\sigma}$ are the linewidth functions describing coupling of the probes and the scattering region and are obtained by calculating self-energies $\Sigma'$ of the semi-infinite leads using a transfer matrices method [14]. The spin-Hall conductance fluctuation is defined as $\text{rms}(G_{SH}) = \sqrt{\langle G_{SH}^2 \rangle - \langle G_{SH} \rangle^2}$, where $\langle \cdots \rangle$ denotes averaging over an ensemble of samples with different configurations of the same disorder strength $W$. In the following, our numerical data are mainly collected on a system with $n = 8$, i.e., with $32 \times 65$ sites in the graphene. In the rest of the Letter, we fix units by setting energy $E$, disorder strength $W$, SOI coupling $t_{SO}$ in terms of the hopping parameter $t$, and the magnetic field in terms of magnetic flux $\phi$.

We first examine model I which has unitary symmetry. In the absence of disorder, countercirculating edge states are formed [9,10]. A transverse flow of spin current between probes 2 and 4 is generated when the bias voltage is applied across probes 1 to 3 leading to the QSHE. In the regime of QSHE, we now increase disorder strength $W$. This causes a breakdown of the integer value of $G_{SH}$ and induces sample-to-sample fluctuations of $G_{SH}$. Figure 1(a) plots the average $G_{SH}$ by calculating 5000 samples for each point on the figure; Fig. 1(b) plots the corresponding fluctuation $\text{rms}(G_{SH})$, as a function of $W$. When $W$ is increased, $G_{SH}$ decreases from its quantized value $G_{SH} = 1$ and $\text{rms}(G_{SH})$ increases. The breakdown of quantized
**GSH** is due to \( W \) that causes a direct transmission from probe 1 to 3; this is shown in Fig. 3(c) where the direct transmission \( T_{31} \) is plotted against \( W \). From \( T_{31} \) we conclude that the graphene device is in an insulating regime at small \( W \), i.e., zero or very small \( T_{31} \); it is in a diffusive regime for intermediate \( W \) and finally reentrant to the insulating regime for large \( W \). We note that for a different set of system parameters, the diffusive regime corresponds to a different range of \( W \). For a given \( E \) or \( \phi \), \( \text{rms}(G_{SH}) \) develops a “plateau” region, e.g., in the range \( W = [3, 7] \) in Fig. 1(b). This plateau is at \( \text{rms}(G_{SH}) = 0.285 \) in unit of \( e^2/4\pi \). Although the plateau range of \( W \) depends on specific values of \( E \) or \( \phi \), our results show that it is always inside the diffusive regime. From Fig. 3 we found that \( \text{rms}(G_{SH}) = 0.285 \) is always true if there is a plateau, i.e., if the diffusive transport regime is established. We therefore identify \( \text{rms}(G_{SH}) = 0.285 \) as a “universal” value. This USCF value is different from that of the conventional SHE situation [4,7] where the universal value is 0.18. Therefore QSHE and SHE belong to different universality classes due to these different statistical properties.

Next, we investigate model II that has a symplectic symmetry. For such a graphene device there is an energy gap between \(-1 < E < 1\), within which edge states exist [8]. Figure 2 plots averaged \( G_{SH} \) and \( \text{rms}(G_{SH}) \) versus \( W \) for a given set of \( E, t_{SO} \) parameter values. Five thousand samples were calculated for the disorder averaging. Similar behavior is found as that of model I. For different values of \( t_{SO} \), \( \text{rms}(G_{SH}) \) reaches a plateau at a different range of \( W \) [see Fig. 2(b)]. Amazingly, all plateaus have the same value and this value is precisely \( \text{rms}(G_{SH}) = 0.285 \). To further support this finding, Fig. 2(c) plots \( \text{rms}(G_{SH}) \) versus \( W \) for a fixed \( t_{SO} = 0.7 \) but several different values of energy \( E \). Again, the same conclusion is obtained. This indicates that there exists a transport regime where the QSHE conductance fluctuation has a universal behavior independent of disorder (albeit in a region roughly \( 2t \sim 5 \text{ eV} \)), energy, and SOI.

Now we examine the third model that has a symplectic symmetry but on the square lattice with four probes. The Hamiltonian is given in Ref. [13]. Note that the quadratic dispersion relation of this model is different from the Dirac dispersion relation in graphene. Figure 3 shows the averaged \( G_{SH} \) and \( \text{rms}(G_{SH}) \) versus \( W \) for different energies \( E \). The plateau values of \( \text{rms}(G_{SH}) \) are, again, \( \sim 0.285 \). Results of Figs. 1–3 strongly suggest that there is a universal spin-Hall conductance fluctuation in the quantized spin-Hall regime with USCF = 0.285 ± 0.005 in units of \( e^2/4\pi \). This is different from the conventional SOI induced SHE where USCF = 0.18 [4].

Very importantly, it appears that symmetry does not play a role in the QSHE regime at least for the CUE and CSE cases we have examined: both give \( \text{rms}(G_{SH}) = 0.285 \). To further support this finding, we calculated the distribution function of \( G_{SH} \) and \( P(G_{SH}) \), in the QSHE regime. Such a distribution is a Gaussian for conventional SHE in the diffusive regime [4]. For QSHE, Figs. 4(a)–4(d) plot \( P(G_{SH}) \) for four different values of \( W \) in the universal regime for model II, which has CSE symmetry. Data were collected by calculating 84,000 samples for each \( W \). The distributions are completely different from a Gaussian. We found that by using \( \ln(G_{SH}) \) as a variable and plotting \( P(\ln(G_{SH})) \), all the distributions become one-sided log-normal [see Figs. 4(e)–4(h)]. For model I, which has CUE symmetry, and the third model, our results give the same conclusion; i.e., the distribution of quantum spin-Hall conductance is a one-sided log-normal. Therefore, for the three models we investigated, not only is USCF \( \text{rms}(G_{SH}) = 0.285 \) the same, but also the distribution function is the same. This strongly indicates that in the presence
In summary, we have investigated quantized spin-Hall conductance fluctuation for three models with unitary and symplectic symmetry, respectively. Our numerical results show that three models exhibit the same universal quantized spin-Hall conductance fluctuation with the value $USCF = 0.285 \pm 0.005e/4\pi$. The fact that both Dirac dispersion relation and quadratic dispersion relation give rise to the same USCF indicates that the edge states dominate the physics in the QSHE regime. Because of the presence of edge states, the distribution of quantum spin-Hall conductance obeys one-sided log-normal distribution for three models. This strongly suggests that the quantized spin-Hall conductance fluctuation for systems with both unitary symmetry and symplectic symmetry belongs to the same universality class that is different from the usual spin-Hall conductance fluctuation in the absence of edge states.

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[3] An exceptional case to the COE, CUE, and CSE universalities is in the integer quantum Hall regime where it is the edge states that carry charge current. As disorder is increased and quantized Hall plateaus destroyed, the Hall conductance fluctuates from sample to sample. The universality class of Hall conductance fluctuation in the quantum Hall regime has not been well understood.