Particle density distributions in Fermi gas superfluids: Differences between one- and two-channel models in the Bose-Einstein-condensation limit

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We discuss the differences between one- and two-channel descriptions of fermionic gases with arbitrarily tunable attractive interactions; these two cases correspond to whether molecular bosonic degrees of freedom are omitted or included. We adopt the standard ground state wave function for the fermionic component associated with the BCS to BEC crossover problem: for weak attraction the system is in the BCS state while it crosses over continuously to a Bose-Einstein-condensed (BEC) state as the interaction strength is increased. Our analysis focuses on the BEC and near-BEC limit where the differences between the one- and two-channel descriptions are most notable, and where analytical calculations are most tractable. Among the differences we elucidate are the equations of state at general T below T_c and related particle density profiles. We find a narrowing of the density profile in the two-channel problem relative to the one-channel analog. Importantly, we infer that the ratio between bosonic and fermionic scattering lengths depends on the magnetic detuning and is generally smaller than its one-channel counterpart. Future experiments will be required to determine to what extent this ratio varies with magnetic fields, as predicted here.

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I. INTRODUCTION

The recent observations [1–8] of Bose-Einstein condensation (BEC) of molecules formed from fermionic atoms are extremely exciting. Because of a Feshbach resonance, it is possible [9,10], via application of magnetic fields, to obtain an arbitrarily strong attraction between fermions, and to probe the crossover [11,12] from BEC to BCS. The resultant superfluidity of preformed (or precursor) pairs has a natural counterpart in some theories [13–17] of high T_c superconductors. Indeed, a striking and important feature of the cuprate superconductors is their pronounced precursor superconductivity, as evidenced by "pseudogap" effects, the origin of which is still under active debate.

A second, very important motivation for these experiments is based on the theoretical observation that a BCS-like ground-state wave function is capable [18] of describing both fermionic and bosonic superconductors, provided that the chemical potential of the fermions, μ, is determined self-consistently. Given the vast success of weak coupling or BCS theory, it is extremely important to formulate this extended theory at all T, and confront it with controlled experiments.

In this paper, we explore the implications of this specific ground-state wave function [18,19] in detail to address both $T=0$ and $T=T_c$, with particular emphasis on the "near-BEC" regime. Our goals are (i) to discuss in some detail the differences between the "one-channel" model, which is widely used in BCS-BEC crossover studies, and the "two-channel" model, in which the interatomic scattering processes are associated with a field-dependent Feshbach resonance; (ii) to provide analytic calculations and insights by working in a (near-BEC) regime where calculations are more tractable; and (iii) to compare with well established theories of weakly interacting Bose superfluid [20]; as well as (iv) with the measured density distributions in a trap.

Since both one- and two-channel models have been applied to the atomic Fermi gases, it is thus very important to compare their different predictions. We begin with the homogeneous case and then consider the trap configuration; from this we infer the ratio between the effective bosonic and fermionic scattering lengths, which is found to be strongly dependent on the magnetic detuning ν_0. Moreover, this ratio is generally less than the number 2.0, associated with its "one channel" counterpart. This factor of 2 is derived from the standard ground state [18] shown in Eq. (3) below.

II. ONE AND- TWO-CHANNEL MODELS

The "one-channel" model, where the BCS-BEC crossover is tuned via a single parameter (i.e., the pairing strength or scattering length), has been widely used in the context of high T_c superconductors [13]. It has also recently been applied [21] to atomic Fermi gases. In the one-channel model, no microscopic reference is made to the details of how the variable scattering length is obtained. Alternatively, in the two-channel model for cold atomic gases, this scattering length is tuned with the application of an external magnetic field. In the presence of a Feshbach resonance, one includes [9,10] two types of particles—"fermions" and "molecular bosons"—and the "two-channel" Hamiltonian contains two types of interaction effects: those associated with the direct interaction between fermions parametrized by U, and those associated with "fermion-boson" interactions, whose strength is governed by g,

$$H-\mu N=\sum_{\mathbf{k},\sigma}(\epsilon_{\mathbf{k}}-\mu)a^{\dagger}_{\mathbf{k},\sigma}a_{\mathbf{k},\sigma}+\sum_{\mathbf{q}}(\epsilon^{mb}_{\mathbf{q}}+\nu-2\mu)b^{\dagger}_{\mathbf{q}}b_{\mathbf{q}} + \sum_{\mathbf{q},\mathbf{k},\mathbf{k}'}U(\mathbf{k},\mathbf{k}')a^{\dagger}_{\mathbf{q}/2+\mathbf{k},\uparrow}a^{\dagger}_{\mathbf{q}/2-\mathbf{k},\downarrow}a_{\mathbf{q}/2-\mathbf{k}',\downarrow}a_{\mathbf{q}/2+\mathbf{k}',\uparrow} + \sum_{\mathbf{q},\mathbf{k}}[g(\mathbf{k})b^{\dagger}_{\mathbf{q}}a_{\mathbf{q}/2-\mathbf{k},\downarrow}a_{\mathbf{q}/2+\mathbf{k},\uparrow}+\text{H.c.}]. \quad (1)$$

The "one-channel" Hamiltonian is given by the first and third term only. As will become clear soon, these two different Hamiltonians describe different physical systems. It is generally assumed, as we do here, that there are no direct boson-boson interactions. Here the fermion and boson kinetic energies are given by $\epsilon_{\mathbf{k}} \equiv \hbar^2 k^2/2m$ and $\epsilon_{\mathbf{q}}^{mb} \equiv \hbar^2 q^2/2M$, respectively, and ν is an important parameter which represents the magnetic "detuning." The bosons ($b_{\mathbf{k}}^{\dagger}$) of the cold atom problem [9,10] will be referred to as Feshbach bosons (FB). Formally, these represent a separate species, not to be confused with the fermion pair ($a_{\mathbf{k}}^{\dagger} a_{-\mathbf{k}}^{\dagger}$) operators; these distinctions arise from different hyperfine states of the atomic system. The two scattering channels are sometimes referred to as "closed" and "open."

The variational ground state which we will consider here is a product of both fermionic and bosonic contributions

$$\overline{\Psi}_0 = \Psi_0 \otimes \Psi_0^B, \tag{2}$$

where the normalized fermionic wave function is the standard crossover state [18,19]

$$\Psi_0 = \Pi_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_k^{\dagger} c_{-k}^{\dagger}) |0\rangle \tag{3}$$

and the *normalized* molecular or Feshbach boson contribution Ψ_0^B is represented by

$$\Psi_0^B = e^{-\lambda^2/2 + \lambda b_0^{\dagger}} |0\rangle. \tag{4}$$

For completeness, we note here that in the "one-channel" analog theory, the ground-state wave function is that of Eq. (3) without the contribution from Ψ_0^B.

The variational parameters are, thus, $u_{\mathbf{k}}$, $v_{\mathbf{k}}$, and λ. Applying standard variational techniques on the ground-state wave function leads to a number of results which have already appeared in the literature; among these are the $T=0$ limits of Eqs. (5) and (B4), and Eq. (19), along with the result that $\lambda = \phi_m \equiv \langle b_{\mathbf{q}=0} \rangle$. Thus, this wave function is compatible with previous $T=0$ studies [11,12].

Which of the one- or two-channel descriptions is appropriate is presumably dependent on the atomic system under study, as well as the strength of the detuning. Present experiments involve rather wide Feshbach resonances in ^{6}Li and ^{40}K. We have shown elsewhere [13] that for these wide resonances and in the intermediate coupling (unitary scattering) and BCS regimes, there is little difference between the one- and two-channel systems. The differences are most evident in the BEC and near-BEC limits which we explore here. The contributions of this paper are expected to help in the ongoing debate as to whether one- or two-channel models are the more appropriate.

A. Extending conventional crossover theory to $T \neq 0$: BEC limit without Feshbach bosons

In order to facilitate the comparison between "one-channel" and "two-channel" models, here we will first present the "one-channel" model from a slightly different perspective than in previously published work [16,17]. We begin by making the important observation that for $T \leq T_c$, the variational parameters associated with the wave function of Eq. (3)—$\Delta(T)$ and $\mu(T)$—*are temperature independent in the near-BEC regime, for all* $T \leq T_c$. Indeed, this is consistent with the physical picture of well established, preformed pairs in the BEC limit, so that the fermionic energy scales are unaffected by T below T_c.

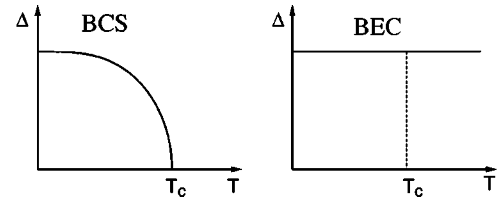

FIG. 1. Contrasting temperature dependences of Δ in the BCS and BEC regimes. Similarly, in the BEC regime μ is a constant, so that all fermionic energy scales are T-independent, as expected.

This simple physics may be schematically represented by plots of Δ versus temperature. Figure 1 contrasts the behavior in the weak coupling or BCS and strong coupling or BEC regimes. In the BCS limit, $\Delta(T)$ follows the behavior of the order parameter, Δ_{sc}, whereas in the BEC regime, pairs are preformed and there is no temperature dependence in $\Delta(T)$ on the scale of T_c. We now extend these qualitative observations to a more quantitative level.

The self-consistent equations in the BEC limit for general temperature T can then be written as

$$\frac{m}{4\pi\hbar^2 a_s} = \sum_{\mathbf{k}} \left[\frac{1}{2\epsilon_{\mathbf{k}}} - \frac{1}{2E_{\mathbf{k}}} \right], \tag{5}$$

$$n = \sum_{\mathbf{k}} \left[1 - \frac{\epsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}} \right], \quad T \leq T_c, \tag{6}$$

where a_s is the s-wave scattering length, $E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta^2}$, and Δ is, on general grounds, to be distinguished from the order parameter [22,23], Δ_{sc}. Note that we have used the $T=0$ conditions [18] in Eqs. (5) and (6), since the Fermi function $f(E_{\mathbf{k}})$ is essentially zero in the BEC limit, where $E_{\mathbf{k}}/T \sim |\mu|/T_c \gg 1$. Equations (5) and (6) (and their two channel analogs) are central to the theory presented in this paper. They show that even in the strong attraction limit, where the system can be viewed as consisting of "bosons," *the underlying fermionic constraints on* Δ *and* μ *must be respected*. These constraints do not have a natural analog in the Gross-Pitaevskii (GP) theory of true bosons.

It follows from the above equations that, just as in the $T=0$ limit [21,24], we have for general $T \leq T_c$,

$$n_{pairs} = \frac{n}{2} = Z_0 \Delta^2, \tag{7}$$

where the coefficient of proportionality

$$Z_0 \approx \frac{m^2 a_s}{8\pi\hbar^4}. \tag{8}$$

We arrive at an important physical interpretation of the BEC limit. Even though Δ or n_{pairs} is a constant in T, this constant must be the sum of two temperature-dependent terms. As in the usual theory of BEC, these two contributions correspond to condensed and noncondensed components

$$n_{pairs} = n_{pairs}^{condensed}(T) + n_{pairs}^{noncondensed}(T), \tag{9}$$

so that we may decompose the excitation gap into two contributions

$$\Delta^2 = \Delta_{sc}^2(T) + \Delta_{pg}^2(T), \tag{10}$$

where $\Delta_{sc}^2(T)$ corresponds to the condensed and $\Delta_{pg}^2(T)$ to the noncondensed (or pseudo) gap component. Each of these are proportional to the respective number of condensed and noncondensed pairs with proportionality constant Z_0. Just as in conventional BEC, at T_c,

$$n_{pairs}^{noncondensed}(T_c) = \frac{n}{2} = \sum_{\mathbf{q}} b(\Omega_q, T_c), \tag{11}$$

where $b(x)$ is the usual Bose-Einstein function and Ω_q is the dispersion of the noncondensed pairs, which will be self-consistently determined below. Thus

$$\Delta^2(T_c) = \Delta_{pg}^2(T_c) = Z_0^{-1} \sum b(\Omega_q, T_c) = \frac{n}{2} Z_0^{-1}. \tag{12}$$

We may deduce from Eq. (12) that $\Delta_{pg}^2 = -\Sigma_Q t(Q)$, if we presume that below T_c, the noncondensed pairs have propagator $t(Q) = Z_0^{-1}/(i\Omega_n - \Omega_q)$. In this way, we may rewrite Eq. (11) in the form

$$n_{pairs}^{noncondensed}(T_c) = -Z_0 \sum_{Q \neq 0} t(Q). \tag{13}$$

[For brevity, we have used a four-momentum notation as in Ref. [16]: $K \equiv (\mathbf{k}, i\omega_n)$, $Q \equiv (\mathbf{q}, i\Omega_n)$, $\Sigma_Q \equiv T\Sigma_{n,\mathbf{q}}$, where ω_n and Ω_n are odd and even Matsubara frequencies, respectively.]

This leads to a key question: how can one deduce the contribution from *noncondensed* pairs? We now work backwards to infer the dispersion Ω_q for these pairs. A fundamental requirement on noncondensed pairs in equilibrium with a Bose condensate is that their effective chemical potential satisfies $\mu_{pair}(T) = 0$, for $T \leqslant T_c$. Equation (5) can be shown [23] to be consistent with this constraint on μ_{pair} provided that the propagator for noncondensed pairs is given by

$$t(Q) = \frac{U}{1 + U\chi(Q)}, \tag{14}$$

where

$$\chi(Q) \equiv \sum_K G(K) G_0(Q - K), \tag{15}$$

and G represents the fermionic Green's function which has a self-energy $\Sigma(K) = -\Delta^2 G_0(-K)$. Here G_0 is the bare propagator. The details of this analysis are presented in Appendix A.

Another important point should be noted. This pair propagator or T-matrix differs from that first introduced by Nozieres and Schmitt-Rink [25] because here there is one dressed and one bare Green's function. In the approach of Ref. [25], both are taken as bare Green's functions. By contrast, there are other schemes in the literature [26–28] where both Green's functions are dressed. We end by noting that at small four-vector Q (and moderately strong coupling) we may expand Eq. (14) after analytical continuation to real frequency $i\Omega_n \to \Omega + i0^+$ to obtain the expected form

$$t(Q) \approx \frac{Z_0^{-1}}{\Omega - \Omega_q + \mu_{pair} + i\Gamma_Q}. \tag{16}$$

Now we can deduce directly from Eq. (14) that the dispersion of noncondensed pairs is of the form

$$\Omega_q = \hbar^2 q^2 / 2M_0^*. \tag{17}$$

In summary, this quadratic dispersion can be derived from the pair susceptibility $\chi(Q)$. In turn, the particular form for $\chi(Q)$ shown in Eq. (15) is chosen in order to be consistent with Eqs. (5) and (6). In this sense, the usual BEC constraint ($\mu_{pair} = 0$) is intimately connected to the BCS-like gap equation of Eq. (5). The details of this analysis are given in Appendix A.

B. Two-channel model: Effects of Feshbach bosons

We now extend this analysis to include Feshbach bosons [11,12]. For this situation we can write down an equation [23] equivalent to Eq. (5) with the effective scattering length $a_s \to a_s^*$ or equivalently the direct fermion interaction U replaced by $U_{eff} \equiv U + g^2/(2\mu - \nu)$. Here we define

$$U^* \equiv U_0 - \frac{g_0^2}{(\nu_0 - 2\mu)} \equiv \frac{4\pi\hbar^2 a_s^*}{m}, \tag{18}$$

where a_s^* is dependent on μ. We thus arrive back at Eqs. (5) and (6) with a_s^* appearing in place of a_s.

In this generalization of Eq. (6), n represents the number of fermions, which is to be distinguished from the total number of particles which involves both condensed and uncondensed bosons as well. Importantly, in the two-channel problem the particle number constraint involves the sum of three terms given by

$$n + 2n_b + 2n_b^0 = n^{tot}, \tag{19}$$

where $n_b^0 \equiv \phi_m^2$ is the number of molecular bosons in the condensate; this condensate is discussed in more detail in Appendix B. The number of noncondensed molecular bosons is given by

$$n_b(T) = -\sum_{Q \neq 0} D(Q), \tag{20}$$

where the Bose propagator is

$$D(Q) \equiv \frac{1}{i\Omega_n - \epsilon_q^{mb} - \nu + 2\mu - \Sigma_B(Q)} \tag{21}$$

and we choose the self-energy [23]

$$\Sigma_B(Q) \equiv -\frac{g^2 \chi(Q)}{1 + U\chi(Q)} \tag{22}$$

to be consistent with the Hugenholtz-Pines condition that bosons in equilibrium with a condensate must necessarily have zero chemical potential: $\mu_{boson}(T) = 0$ at $T \leqslant T_c$, where

$\mu_{boson}=2\mu-\nu-\Sigma_B(0)$. It follows after some simple algebra that this constraint on μ_{boson} is equivalent to Eq. (5).

In this way, $D(Q)$ may be expanded at small Q in real frequency to be of the same form as Eq. (16),

$$D(Q) \approx \frac{Z_b^{-1}}{\Omega - \Omega_q + \mu_{boson} + i\Gamma_Q}. \tag{23}$$

Importantly, *there is only one branch* Ω_q *for bosoniclike excitations*. This branch represents a hybridized mix of molecular bosons and fermion pairs. Just as there is a direct analogy between Eqs. (16) and (23), Eqs. (13) and (20) are closely connected.

III. EQUATIONS OF STATE AT $T=0$

We now rewrite our central equations (5)–(7) in the near-BEC limit to compare more directly with the case of a weakly interacting Bose gas, described by the GP theory. It can be shown that (in the absence of FB)

$$n = \Delta^2 \frac{m^2}{4\pi\sqrt{2m|\mu|}\hbar^3}, \tag{24}$$

which, in conjunction with the expansion of Eq. (5),

$$\frac{m}{4\pi\hbar^2 a_s} = \left(\frac{2m}{\hbar^2}\right)^{3/2} \frac{\sqrt{|\mu|}}{8\pi}\left[1 + \frac{1}{16}\frac{\Delta^2}{\mu^2}\right], \tag{25}$$

yields

$$\mu = -\frac{\hbar^2}{2ma_s^2} + \frac{a_s \pi n \hbar^2}{m}. \tag{26}$$

These equations hold at all $T \leqslant T_c$. At $T=0$, these equations have been shown [21,24] to be equivalent to the results of GP theory where one identifies an effective interpair scattering length $a_B=2a_s$ via $n_B=\mu_B/(4\pi a_B \hbar^2/M_B)$. Here $n_B=n/2$ represents the number density of pairs, $\mu_B=2\mu+\hbar^2/ma_s^2$ is the "bare" chemical potential of the pairs, and $M_B \approx 2m$ the pair mass. We emphasize that *the value of 2 for the scattering length ratio is entirely dictated by the assumed form for the ground state, Eq. (3)*.

We now show that in the presence of Feshbach bosons, this equation of state is no longer that of GP theory and, moreover, there are important implications for the ratio of the bosonic to fermionic scattering lengths. As a result of the Bose condensate n_b^0 in Eq. (19) one finds an extra term in the number equation which is discussed in more detail in Appendix B,

$$n^{tot} = \Delta^2\left[\frac{m^2}{4\pi\sqrt{2m|\mu|}\hbar^3} + 2\frac{(1-U_0/U^*)^2}{g_0^2}\right]. \tag{27}$$

Combining the gap and number equation yields

$$\frac{m}{4\pi\hbar^2 a_s^*} = \left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\sqrt{|\mu|}}{8\pi}\left[1 + \frac{1}{16\mu^2} \times \frac{n^{tot}}{\frac{1}{16\pi\sqrt{|\mu|}}\left(\frac{2m}{\hbar^2}\right)^{3/2} + 2\frac{(1-U_0/U^*)^2}{g_0^2}}\right]. \tag{28}$$

Solving for μ in terms of a_s^*, one finds a new equation of state [29]

$$\mu \approx -\frac{\hbar^2}{2ma_s^{*2}} + \frac{2\pi^2}{m}\frac{g_0^2\hbar^2}{U_0^2} n^{tot} a_s^{*4} \tag{29}$$

to lowest order in a_s^*. The second term in the above equation derives from the molecular boson condensate contribution. The first term in Eq. (27) contributes only a term of order a_s^{*7} to the equation of state. This behavior should be contrasted with the situation when FB are absent, where $a_B=2a_s$. It should also be emphasized that the fermionic scattering length in a model without FB is an independent experimental parameter, while here a_s^* depends on μ, and must be obtained self-consistently.

IV. CALCULATIONS AT T_c

We turn now to a calculation of T_c, which requires that we determine Ω_q [via Eq. (11), along with the T-matrix of Eq. (14)] as a function of the scattering length a_s. We address the one-channel case first. The general expression for $1/M_0^*$ in the near BEC limit is given by

$$\frac{1}{M_0^*} = \frac{1}{Z_0\Delta^2}\sum_{\mathbf{k}}\left[\frac{1}{m}v_{\mathbf{k}}^2 - \frac{4E_{\mathbf{k}}\hbar^2k^2}{3m^2\Delta^2}v_{\mathbf{k}}^4\right], \tag{30}$$

where we have used Eqs. (17) and (16). After expanding to lowest order in na_s^3,

$$M_0^* \approx 2m\left(1 + \frac{\pi a_s^3 n}{2}\right). \tag{31}$$

Equation (11) reflects the fact that, in the near-BEC limit, and at T_c, all fermions are constituents of uncondensed pairs. It then follows that $(M_0^* T_c)^{3/2} \propto n=$const, which, in conjunction with Eq. (31), implies

$$\frac{T_c - T_c^0}{T_c^0} = -\frac{\pi a_s^3 n}{2}. \tag{32}$$

Here T_c^0 is the transition temperature of the ideal Bose gas with $M_0=2m$. This downward shift of T_c follows the effective-mass renormalization, much as expected in a Hartree treatment of GP theory at T_c. Here, however, in contrast to GP theory for a homogeneous system with a contact potential [20], there is a nonvanishing renormalization of the effective mass.

In the presence of Feshbach bosons, the (inverse) residue in the T-matrix is replaced by

$$Z = Z_0 + Z_g \approx \frac{n}{2\Delta^2}, \tag{33}$$

where

$$Z_g = \frac{g^2}{[U(2\mu - \nu) + g^2]^2} \tag{34}$$

is derived from $U_{eff}^{-1}(Q)$, and the $\hbar^2 q^2$ coefficient $B_1 \equiv 1/(2M_1^*)$ in $\Omega_{\mathbf{q}}$ is such that

$$B = \frac{B_0 Z_0 + \frac{1}{2M} Z_g}{Z} \approx \frac{1}{4m}\left[1 - \frac{2g_0^4}{U_0^4}\pi^3 n a_s^{*9}\right]. \tag{35}$$

Here Z_0 and B_0 are the appropriate counterparts when FB are absent. Since Z_0 is proportional to the fermionic contribution to the density, it is very small in the BEC limit. Using the same reasoning as in the previous case, we conclude that the ratio T_c/B is constant with varying coupling. *Thus, in the two-channel case as well*, T_c *follows the behavior of the inverse effective mass* with, to leading order, a very weak dependence on scattering length: $(T_c^0 - T_c)/T_c^0 \propto a_s^{*9}$.

V. SUMMARIZING THE DIFFERENCES BETWEEN ONE- AND TWO-CHANNEL SYSTEMS

In this section, we summarize the differences and similarities we have found thus far between the one- and two-channel descriptions of the atomic Fermi gases at the level of the ground state equation: Eq. (3).

(i) The gap equations [Eq. (5)] are essentially equivalent in the two cases, except that for the two-channel system, an effective interaction U_{eff} enters in place of U. This is equivalent to the statement that the scattering length a_s is replaced by the μ-dependent quantity we have defined as a_s^*.

(ii) In both cases, the full excitation gap Δ is to be distinguished from the order parameter $\tilde{\Delta}_{sc}$. Thus, there are noncondensed fermion pairs and the associated pseudogap Δ_{pg} in both cases. When two channels are present, the noncondensed fermion pairs are strongly admixed with the Feshbach bosons. A physical probe will, in general, only couple to this single hybridized bosoniclike excitation. The degree of fermion pair and FB admixture will vary with detuning.

(iii) The number equations [Eqs. (6) and (19)] in the two models contain an important difference. The condensate contribution for Feshbach bosons (FB) appears as a separate term in the number equation, along with the contribution from noncondensed FB. Importantly, the Cooper condensate Δ_{sc} and its noncondensed analog Δ_{pg} appear more directly in the gap equation. There is, of course, a single physical condensate or order parameter $\tilde{\Delta}_{sc}$ in the problem as discussed in Appendix B.

(iv) Finally, there is a crucial difference which we explore in the remainder of this paper. This difference is associated with the fact that in the BEC limit, when there are two channels available, the system consists entirely of Feshbach bosons; in this way, the number equation is satisfied without a population of fermions. The absence of fermions means that there is no medium for the interaction between bosons, and they are even more "ideal" than in the one-channel case. This demonstrates that *for the particular ground-state ansatz considered here, and in the BEC limit, the "one-channel" and "two-channel" models describe very different physics*.

VI. PARTICLE DENSITY PROFILES IN TRAPS

These differences between the equations of state for the one- and two-channel models will have physical implications in the density profiles of particles in a trap. We now introduce the harmonic trapping potential $V(r) = \frac{1}{2}m\omega^2 r^2$, which is treated in the Thomas-Fermi (TF) approximation. In this approximation, one replaces μ with $\mu(r) = \mu - V(r)$. In contrast to the uniform case, here $\mu_{pair}(r,T)$ becomes nonzero beyond a critical radius $R_c(T)$, where $R_c(T_c) = 0$. In this way at T_c, only the center of the trap is superfluid, while at $T=0$ all of the trap contains condensed states. To obtain the $T=0$ density profile, $n(r)$, we insert $\mu(r)$ into Eq. (28) and solve for $n(r)$ [here $n(r)$ refers to the sum of both fermion and molecular boson contributions]. The solution is

$$n(r) = 16\mu^2(r)\left[\frac{\hbar}{\sqrt{2m|\mu(r)|}a_s^*} - 1\right] \times \left[\frac{1}{16\pi\sqrt{|\mu(r)|}}\left(\frac{2m}{\hbar^2}\right)^{3/2} + 2\frac{\left(1 - \frac{U_0}{U^*}\right)^2}{g_0^2}\right], \tag{36}$$

where we use $N = \int n(r) d^3 r$ to self-consistently determine μ. Here it should be noted that a_s^* is itself a function of $\mu(r)$.

In Fig. 2(a), we plot $n(r)$ for the parameters $U_0 = -0.89$, $g_0 = -35$, $\nu_0 = -1260$, and $N = 10^5$, for the units specified in the caption. We chose parameters appropriate to ^{40}K. To connect the various energy scales which appear in the problem, typically 1 G $\approx 60 E_F$. For this value of ν_0, we are somewhat away from the deepest BEC regime as is necessary to ensure the validity of the TF approximation, and the fermionic contribution to the density is no longer negligible compared to its bosonic counterpart. Indeed, the percentage weight of the molecular boson condensate contribution (shown in the inset) indicates that by $\nu_0 \approx -400$, the Cooper pair condensate contribution is beginning to dominate. In this figure, we also show the density profile as computed in the absence of FB (dashed line), as well as the behavior at $T = T_c$ (dotted line). Our $T = T_c$ curves were computed in the absence of FB for both panels. We were unable thus far to find important differences between this distribution and that of a weakly interacting Bose gas. Here one has to solve self-consistently for $\mu_{pair}(r)$, as well.

To arrive at a meaningful comparison of the two $T=0$ cases (with and without Feshbach bosons), we used the same value for the effective two-body scattering length a_s^*, obtained from the self-consistent calculations of $n(r)$ in the presence of FB. The profile without FB is then calculated using the familiar TF result near the BEC limit [24] at a fixed fermionic scattering [30] length a_s^*. The same plots are presented with $\nu_0 = -250$ in Fig. 2(b), which is further from the BEC limit.

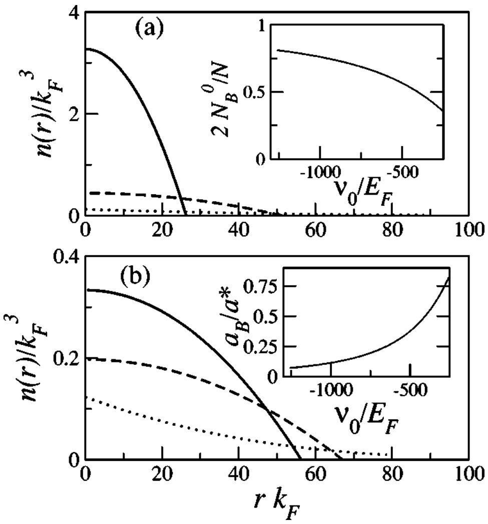

FIG. 2. Density profiles at (a) $\nu_0=-1260$ and (b) $\nu_0=-250$. Solid and dashed lines are at $T=0$ with and without Feshbach bosons (FB), respectively, and the dotted lines are at $T=T_c$. The inset to (a) plots the FB weight in the condensate. The inset to (b) shows the ratio of the interboson to interfermion scattering lengths at trap center. Units are chosen such that $k_F \equiv 1$ and $E_F = (3N)^{1/3}\hbar\omega = 1$. Hence the units for ν_0, U_0, and g_0 (which are E_F, E_F/k_F^3, and $E_F/k_F^{3/2}$, respectively) are all set to unity. For $\nu_0 = -250$, the interaction parameter $k_F a_s^* \approx 0.22$ and for $\nu_0 = -1260$ it is 0.05.

An important consequence of the one- versus two-channel problems is that for the latter, as the extreme BEC limit is approached, the number of fermions is diminished in favor of FB. This occurs due to the self-consistent adjustment of the fermionic chemical potential. Thus the interaction between the bosons mediated by the fermions is weaker than what might have been expected without consideration of FB. This weakened interaction is reflected in Fig. 2 through a comparison between the solid and dashed lines, which shows that the trap profiles are narrower in the presence of Feshbach effects. Indeed, this could have been anticipated from the above calculations in the homogeneous case, at $T=0$ and T_c, which deduced a very weak dependence of the bosonic scattering length and effective pair mass on the fermionic scattering length a_s^*. This finding is analogous to the result in a trapped atomic Bose gas [20], where one sees that a weaker repulsive interboson interaction leads to a narrowed density profile. Comparison between the lower and upper panels of Fig. 2 shows that in both cases (with and without FB), the profiles become narrower as the BEC limit is approached.

We may use the results in Fig. 2 to obtain a semiquantitative estimate of the bosonic scattering length a_B, based on a phenomenology used in experimental analysis [1–3]. We compare our results to the Thomas-Fermi approximated GP equation at $T=0$, which yields $n_B(r) = [\mu_B - V(r)]/U_B$ where the interboson interaction U_B is connected to the scattering length a_B via $U_B = 4\pi\hbar^2 a_B/M_B$. If we fit the profiles of Fig. 2 to the inverted parabola $(C_1 - C_2 r^2)$, we may infer that $a_B = M_B^2\omega^2/4\pi\hbar^2 C_2$. The ratio a_B/a_s^* is plotted versus ν_0 in the inset to the lower panel of Fig. 2(b). The same analysis applied to the profile without FB yields the familiar 2:1 ratio of the bosonic to fermionic scattering lengths. The parameter a_B is an important quantity which appears, as in experiment, to be considerably less than a factor of 2 times its fermionic counterpart.

Other estimates of the interboson scattering length

There are very detailed calculations [31] of the exact four-body atomic scattering processes which yield a fixed length ratio $a_B/a_s = 0.6$ for the bosonic to fermionic scattering lengths. It is widely agreed that this constraint should be imposed on a more complete theory of the ground state. Indeed, a weakness of the approach taken here (based on a generalized BCS ground state) is that "boson-boson" interactions are treated only approximately, in a mean-field sense. This can be viewed as being related to the inclusion of only one-fermion and two-fermion propagators (i.e, T-matrix) with no higher-order terms.

The theory of Ref. [31] is based on treating fermions in a single- (open) channel problem. This approach is presumed to be exact providing the system is sufficiently dilute and in the BEC regime so that one may consider only a four-body problem. By contrast, in the present approach, the many-body physics associated with the broken symmetry or superfluid state is included. *A comparison of the one- and two-channel calculations presented here shows that, through many-body effects, the nature of the condensate enters in an important way to determine the equation of state and, thereby, the ratio of the scattering lengths*. In the many-body context, the interboson scattering length ratio is different from the canonical value 2.0, which is based on Eq. (3) and derived following Eq. (26); this value is only appropriate to the single-channel case.

In the two-channel problem, where Feshbach bosons are present, the ratio a_B/a_s depends upon the magnetic field or detuning. This is a central point of the present paper. Future experiments will be required to determine whether, as some have presumed, the ratio of 0.6 applies to all detunings, or whether this number is variable as argued here. Furthermore, it will be of interest to repeat the few-body calculations of Ref. [31] in the presence of both open and closed channels. Although the factor of 2 associated with the single-channel calculations will be changed in a more elaborate many-body theory (i.e., beyond the simplest ground state assumed here), it is, conversely, reasonable to assume that for the few-body calculations such as in Ref. [31], the ratio a_B/a_s will change as well. Indeed, it seems reasonable to assume that, as found here, it will be smaller in the two-channel than in the one-channel cases.

VII. CONCLUSIONS

In this paper, we have shown that superfluidity of fermionic atoms in the near-BEC limit is in general different from Bose superfluidity (as described by GP theory). We have compared one- and two-channel models and find that differences from the GP picture for each are associated with the underlying fermionic character of the system. Our comparisons with GP theory were presented at both $T=0$ and $T=T_c$, both with and without Feshbach bosons. A related set of observations concerning the difference between composite and true bosons can be made on the basis of the behavior of the collective modes [32,33].

A key aspect of this work is in the comparison we have presented between one- and two-channel models. Differences arise because of the nature of the condensate. A fermionic or Cooper pair condensate contribution (Δ_{sc}) enters into the gap equation while a bosonic condensate contribution (n_b^0) enters also into the number equation. When there is an appreciable fraction of bosonic condensate (as in the near-BEC and BEC regimes), these differences will be apparent. A distinction between the one- and two-channel models is relatively unimportant once the bosonic condensate contribution is negligible.

These differences are associated with physical properties such as the equations of state and the density profiles in a trap. Feshbach bosons can, in effect, collapse much more completely to the center of a trap than can fermion pairs. This effect can be inferred from the ratio of the scattering lengths, and is related to the fact that in this limit, fermions are essentially absent, since the number equation constraint can be entirely satisfied by populating the bosonic state. As a result, these bosons are close to ideal.

For the relatively broad Feshbach resonances currently under study in lithium and potassium, differences between the one- and two-channel models are presumably important only in the BEC and near-BEC limits studied here, where there is an appreciable bosonic condensate. It will be interesting to study narrower Feshbach resonances where these differences may persist into the unitary scattering regime.

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APPENDIX A: CONNECTION BETWEEN $\chi(Q)$ AND THE STANDARD GAP EQUATION

The expansion of the T-matrix, after analytical continuation $i\Omega_n \to \Omega + i0^+$), at small four-vector $Q=(\Omega,\mathbf{q})$ can be written in the following form:

$$t^{-1}(Q) \approx Z_0(\Omega - \Omega_{\mathbf{q}} + \mu_{pair} + i\Gamma_{\mathbf{q}}). \tag{A1}$$

Therefore, the condition for the divergence of the T-matrix at zero Q is equivalent to

$$\mu_{pair} = 0 \tag{A2}$$

or

$$U^{-1} + \chi(0) = 0. \tag{A3}$$

We now show that for a proper choice of the pair susceptibility

$$\chi(Q) = \sum_K G(K)G_0(Q-K), \tag{A4}$$

Eq. (A2) is equivalent to the BCS-like gap equation of Eq. (5), provided also the Green's function is of the BCS form

$$G(K) = \frac{u_{\mathbf{k}}^2}{i\omega_n - E_{\mathbf{k}}} + \frac{v_{\mathbf{k}}^2}{i\omega_n + E_{\mathbf{k}}} = \frac{i\omega_n + \xi_{\mathbf{k}}}{(i\omega_n)^2 - E_{\mathbf{k}}^2}. \tag{A5}$$

Just as in BCS theory, we associate this dressed Green's function with self-energy,

$$\Sigma(K) = -\Delta^2 G_0(-K). \tag{A6}$$

We calculate $\chi[Q=(i\Omega_m,\mathbf{q})]$ by performing the appropriate Matsubara sums following standard procedure [34]. At $Q=0$, we have

$$\chi(0) \equiv \sum_K \frac{i\omega_n + \xi_{\mathbf{k}}}{(i\omega_n)^2 - E_{\mathbf{k}}^2}\frac{1}{-i\omega_n - \xi_{\mathbf{k}}} = \sum_{\mathbf{k}} \frac{1-2f(E_{\mathbf{k}})}{2E_{\mathbf{k}}} \tag{A7}$$

and Eq. (A3) becomes

$$U^{-1} + \sum_{\mathbf{k}} \frac{1-2f(E_{\mathbf{k}})}{2E_{\mathbf{k}}} = 0, \tag{A8}$$

which, in conjunction with the two-body scattering equation

$$\frac{m}{4\pi\hbar^2 a_s} = U^{-1} + \sum \frac{1}{2\epsilon_{\mathbf{k}}}, \tag{A9}$$

gives

$$\frac{m}{4\pi\hbar^2 a_s} = \sum_{\mathbf{k}} \left[\frac{1}{2\epsilon_{\mathbf{k}}} - \frac{1-2f(E_{\mathbf{k}})}{2E_{\mathbf{k}}}\right]. \tag{A10}$$

Thus, we have demonstrated that, if we presume Eqs. (A4) and (A6) then Eqs. (A2) and (A10) are equivalent.

APPENDIX B: CHARACTERIZING THE CONDENSATE

There is an important difference in the nature of the condensate for the two cases, with and without Feshbach bosons. In the latter case, there are two components to the condensate associated with Cooper pairs (Δ_{sc}) and condensed molecular bosons n_b^0. Moreover, the Cooper condensate enters only into the gap equation, whereas the molecular Bose condensate enters also into the number equation. As long as the bosonic condensate n_b^0 is non-negligible, this difference leads to essentially different physics between the one- and two-channel problem, as will be demonstrated below.

In this appendix, we enumerate the different relationships between the different components. The order parameter associated with Eq. (1) represents a linear combination of both paired fermions (Cooper condensate) and condensed molecules. It is given by [11,12]

$$\tilde{\Delta}_{sc} = \Delta_{sc} - g\phi_m, \tag{B1}$$

where the boson order parameter $\phi_m=\langle b_{\mathbf{q}=0}\rangle$. We have

$$\Delta_{sc} = - U \sum_{\mathbf{k}} \langle a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} \rangle, \tag{B2}$$

and $\tilde{\Delta}_{sc}$ can be written as

$$\tilde{\Delta}_{sc} = - \left[U + \frac{g^2}{(2\mu - \nu)} \right] \sum_{\mathbf{k}} \langle a_{-\mathbf{k}\downarrow} a_{\mathbf{k}\uparrow} \rangle, \tag{B3}$$

where we have used the fact [35] that

$$\phi_m = \frac{g\Delta_{sc}}{(\nu - 2\mu)U}. \tag{B4}$$

The number of condensed Feshbach bosons which enters the number equation [Eq. (19)] is given by $n_b^0 = \phi_m^2$. Thus we have

$$n_b^0 = \frac{g^2 \Delta_{sc}^2}{[(\nu - 2\mu)U]^2} = \frac{g^2 \tilde{\Delta}_{sc}^2}{[(2\mu - \nu)U + g^2]^2}. \tag{B5}$$

Using the renormalization scheme of Ref. [35], it can be shown that we can also write

$$n_b^0 = \left(1 - \frac{U_0}{U^*}\right)^2 \frac{\tilde{\Delta}_{sc}^2}{g_0^2}, \tag{B6}$$

where U^* is defined in Eq. (18). Note that Eq. (27) in the text is based on the above result at $T=0$ with $\tilde{\Delta}_{sc} \equiv \Delta$.

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