Pseudogap state in superconductors: Extended Hartree approach to time-dependent Ginzburg-Landau theory

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It is well known that conventional pairing fluctuation theory at the Hartree level leads to a normal-state pseudogap in the fermionic spectrum. This pseudogap arises from bosonic degrees of freedom which appear at T^*, slightly above the superconducting transition temperature T_c. Our goal is to extend this Hartree approximated scheme to arrive at a generalized mean-field theory of pseudogapped superconductors for all temperatures T. While an equivalent approach to the pseudogap has been derived elsewhere using a more formal Green's function decoupling scheme, in this paper we reinterpret this mean-field theory and BCS theory as well, and demonstrate how they naturally relate to ideal Bose gas condensation. Here we recast the Hartree approximated Ginzburg-Landau self-consistent equations in a T-matrix form, from which we infer that the condition that the pair propagator have zero chemical potential at all $T \leqslant T_c$ is equivalent to a statement that the fermionic excitation gap satisfies the usual BCS gap equation. This recasting makes it possible to consider arbitrarily strong attractive coupling, where bosonic degrees of freedom appear at T^* considerably above T_c. The implications for transport both above and below T_c are discussed. Below T_c we find two types of contributions. Those associated with fermionic excitations have the usual BCS functional form. That they depend on the magnitude of the excitation gap, nevertheless, leads to rather atypical transport properties in the strong-coupling limit, where this gap (as distinct from the order parameter) is virtually T independent. In addition, there are bosonic terms arising from noncondensed pairs whose transport properties are shown here to be reasonably well described by an effective time-dependent Ginzburg-Landau theory.

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I. INTRODUCTION

In this paper we address the physics of the pseudogap state on the basis of a beyond-BCS mean-field theory. Our approach represents a natural extension to arbitrarily strong coupling constants g of time-dependent Ginzburg-Landau (TDGL) theory when the quartic terms are treated at the Hartree approximation level. We also explore the broken-symmetry phase, incorporating the effects of a Hartree approximation to pairing (or equivalently, excitation-gap) fluctuations for $T < T_c$ in a fashion that is necessarily compatible with their description above T_c. A central assumption in our work is that the ground state belongs to the generalized BCS-like theory introduced by Leggett,[1] and further analyzed by Nozieres and Schmitt-Rink[2] and others. In this way, one arrives at a simple reinterpretation of BCS theory as a limiting case of a more general theory of ideal Bose gas condensation. In contrast to true Bose systems, here, however, fermionic degrees of freedom are never absent: the condition $\mu_{pair} = 0$, for $T \leqslant T_c$ is equivalent to the constraint imposed by the BCS gap equation on the excitation gap $\Delta(T)$.

Several key points need to be made. The observation of a pseudogap ($T > T_c$) deriving from Hartree approximated treatment of "pairing" fluctuations dates back to the early 1970's, with the work of Abeles and co-workers,[3] Patton,[4,5] and Hassing and Wilkins.[6] Pairing fluctuations can be alternatively viewed as fluctuations in the magnitude of the fermionic excitation gap, that is, "gap fluctuations." Most importantly, at the Hartree level, these are to be distinguished from order-parameter fluctuations, as will be discussed in more detail below. Indeed, bosonic degrees of freedom, associated with fermion pairs, and the existence of a fermionic excitation gap are two sides of the same coin. The gap or pseudogap in the fermionic excitation spectrum reflects the fact that extra energy is needed to break bosons, or fermion pairs, apart. When approached from above, T_c is suppressed (relative to its strict BCS counterpart) by the existence of this normal-state pseudogap, as a consequence of the lowering of the fermionic density of states near E_F. When approached from below, one readily sees that the superfluid density will not vanish at T^*, where the fermionic gap disappears, but instead at a lower temperature T_c, as a consequence of additional (beyond BCS) bosonic excitations of the condensate.

In conventional (i.e., long coherence length ξ) three-dimensional superconductors, these noncondensed pairs are virtually undetectable. However, there are two mechanisms for enhancing their effects: reduced dimensionality, often with disorder (see the recent review[7] by Larkin and Varlamov which addresses Gaussian pair fluctuations) and increased attractive coupling constant,[2,8,9] associated with short ξ. If by "the fluctuation regime" we mean the temperature range where bosonic degrees of freedom are present, then in the second mechanism one is effectively extending this regime up to T^*, which could be as much as an order of magnitude higher than T_c. In the vicinity of Bose condensation there are divergences in various transport properties, but even away from T_c, closer to T^*, bosonic contributions may dominate over their fermionic counterparts, leading to highly unconventional transport characteristics, as will be outlined here.

The Hartree approximation[6,10,11] and its T-matrix extension[4,5] are previously well established routes for obtaining a pseudogap state in homogeneous superconductors.[3] Throughout this paper we call the latter "extended Hartree theory." Nevertheless, there is considerable emphasis in the current literature on approaches to the pseudogap based on an alternative phase fluctuation scenario.[12] The contrast between these two schemes should be emphasized. In the former case one is dealing with noncondensed bosons, which are associated with fluctuations in the fermionic *excitation gap* $\Delta(r,t)$ (say, measured relative to the spatially uniform or mean-field order parameter). In the latter case, one is dealing with fluctuations in the phase of the complex *order parameter* $\Delta_{sc}(r,t)$. At a beyond-BCS level these two channels are distinguishable ($\Delta \neq \Delta_{sc}$), but, as in any mean-field theory, the order parameter is taken to be spatially uniform. By contrast, the phase fluctuation scheme is based on order-parameter fluctuations around the strict BCS state ($\Delta = \Delta_{sc}$). Generally, order-parameter fluctuation approaches are appropriate when the separation between the mean-field onset temperature T^* and the actual instability temperature T_c is small. If this temperature difference is not small, it is more appropriate to derive an improved mean-field theory first[13] and then append fluctuation effects, if they are called for. This is the philosophy of the present approach in which at a mean-field level, the separation between T^* and T_c can be arbitrarily tuned via the size of the attractive coupling constant.

Finally, we emphasize a contrast between the present approach and previous work by Nozieres and Schmitt-Rink[2] and by Randeria.[8] Both groups studied the effects of preformed pairs or noncondensed bosons, presuming a BCS-like ground state with arbitrary attractive coupling g. We will assume this ground state as well. These other studies were conducted at the level of Gaussian pair fluctuations, so that pseudogap effects were absent in the pair propagator. Moreover, an approximate form for the number equation was assumed as well.[14] Essential to the physics we will be discussing is a pseudogap in the fermionic spectrum, which helps to decouple the bosonic and fermionic degrees of freedom. This pseudogap leads to relatively long-lived bosons, without the need to invoke unphysically strong coupling. A gap in the fermionic spectrum inhibits the decay of bosons into fermions, simply because there are very few low-energy fermionic states to decay into. As a consequence, bosonic degrees of freedom are evident over a much wider temperature range, above and below T_c, once g is beyond the weak-coupling regime.

In this paper, we make no contact with experiments in high-temperature superconductors. The goal here is to demonstrate the simple physics, as well as the genesis of pair fluctuation approaches, as distinguished from phase fluctuation schemes. It is useful to emphasize that we are focusing in the present paper on the nature of the superconductivity (below T_c) and its implications above T_c. While we presume a generalized BCS ground state (albeit, with arbitrary g and self-consistent fermionic chemical potential), other ground states have been contemplated in the cuprate literature which accommodate the physics of the Mott insulator to varying degrees. The strongest support for the relevance of our viewpoint (which effectively sidesteps Mott effects) is the anomalously short coherence length ξ. This suggests a breakdown of strict BCS physics, which, at the very least, needs to be understood and characterized, on its own, independent of Mott effects. To support this breakdown is the highly non-BCS temperature dependence of the measured[15,16] excitation gap Δ—not so different from that shown in Fig. 3 below. Although transport calculations and interpretations of transport data are often based on a BCS ground state, rather little attention has been paid in the past to the anomalous behavior of $\Delta(T)$ and its implications for transport. In this paper we address this omission.

II. ABOVE T_c AT WEAK COUPLING

A. Overview of Hartree approximated Ginzburg-Landau theory

The Ginzburg-Landau (GL) free-energy functional in momentum space is given by[6]

$$F[\Psi]=\frac{N(0)V}{\beta^2}\sum_Q |\Psi_Q|^2(\epsilon+a|\Omega_n|+\xi_1^2q^2) + \frac{1}{2\beta^2}\sum_{Q_i} b_{Q_1Q_2Q_3}\Psi^*_{Q_1}\Psi^*_{Q_2}\Psi_{Q_3}\Psi_{Q_1+Q_2-Q_3}, \tag{1}$$

where Ψ_Q are the Fourier components of the order parameter $\Psi(\mathbf{r},t)$, $Q=(i\Omega_n,\mathbf{q})$, $\epsilon=(T-T^*)/T^*$, $a=\pi/8T$, ξ_1 is the GL coherence length, T^* is the critical temperature when feedback effects from the quartic term are neglected, $\beta=1/T$ (k_B is set to 1), and $N(0)$ is the density of states at the Fermi level in the normal state. We approximate the quartic term so that only paired terms are included in the last addend of Eq. (1) leading to

$$\frac{1}{2\beta^2}\sum_{123} b_{123}\Psi_1^*\Psi_2^*\Psi_3\Psi_{1+2-3} \approx \frac{1}{2\beta^2}\left(\sum_{1\neq 2} b_{12}|\Psi_1|^2|\Psi_2|^2+\sum_1 b_{11}|\Psi_1|^4\right). \tag{2}$$

That there is no factor of 2 in the first term on the right-hand side of the above expression reflects the fact that we use Hartree rather than the Hartree-Fock approximation. As found elsewhere,[6] $b_{ij}=b_{Q_iQ_j}$ can be approximated by $b_0\delta_{\Omega_i0}\delta_{\Omega_j0}$ where $b_0=[N(0)V/\pi^2]\frac{7}{8}\zeta(3)$. To further simplify the quartic term, we apply the mean-field approximation, writing $|\Psi_{\mathbf{q}0}|^2=\langle|\Psi_{\mathbf{q}0}|^2\rangle+\delta|\Psi_{\mathbf{q}0}|^2$ and neglecting in Eq. (2) terms of order $(\delta|\Psi_{\mathbf{q}0}|^2)^2$. This leads to

$$\frac{b_0}{\beta^2}\sum_{\mathbf{q}}\left(|\Psi_{\mathbf{q}0}|^2-\frac{1}{2}\langle|\Psi_{\mathbf{q}0}|^2\rangle\right)\sum_{\mathbf{q}'}\langle|\Psi_{\mathbf{q}'0}|^2\rangle. \tag{3}$$

The contribution $\langle|\Psi_{\mathbf{q}0}|^2\rangle$ is determined self-consistently via

$$\langle|\Psi_{\mathbf{q}0}|^2\rangle=\frac{\int D\Psi e^{-\beta F[\Psi]}|\Psi_{\mathbf{q}0}|^2}{\int D\Psi e^{-\beta F[\Psi]}} \tag{4}$$

when we replace the quartic term in Eq. (1) by Eq. (3). It follows that

$$\langle|\Psi_{\mathbf{q}0}|^2\rangle=\frac{1}{N(0)VT}\left[\epsilon+\frac{b_0}{N(0)V}\sum_{\mathbf{q}'}\langle|\Psi_{\mathbf{q}'0}|^2\rangle+\xi_1^2q^2\right]^{-1}. \tag{5}$$

If we sum Eq. (5) over $\mathbf{q}$ and identify $\Sigma_{\mathbf{q}}\langle|\Psi_{\mathbf{q}0}|^2\rangle$ with $\beta^2\Delta^2$, we obtain a self-consistency equation for the energy "gap" (or pseudogap) Δ above T_c

$$\beta^2\Delta^2=\sum_{\mathbf{q}}\frac{1}{N(0)VT}\left[\epsilon+\frac{b_0}{N(0)V}\beta^2\Delta^2+\xi_1^2q^2\right]^{-1}, \tag{6}$$

$$\beta^2\Delta^2=\sum_{\mathbf{q}}\frac{1}{N(0)VT}\frac{1}{-\overline{\mu}_{pair}(T)+\xi_1^2q^2}, \tag{7}$$

where

$$\overline{\mu}_{pair}(T)=-\epsilon-\frac{b_0}{N(0)V}\beta^2\Delta^2. \tag{8}$$

Note that the critical temperature is renormalized downward with respect to T^* and satisfies

$$\overline{\mu}_{pair}(T_c)=0. \tag{9}$$

B. Introduction to T matrix: $T\approx T_c$, small $\Delta(T_c)$

Equations (7)–(9) are the central equations derived from the Hartree approximated GL scheme. They describe how the excitation gap $\Delta(T)$ and the quantity $\overline{\mu}_{pair}$ behave above, but near T_c. We now rewrite these equations using a T-matrix approach.

A central quantity in T-matrix schemes is the pair susceptibility. Here we take this function to be of the form[9]

$$\chi(Q)=\sum_K G_0(Q-K)G(K)\varphi^2_{\mathbf{k}-\mathbf{q}/2}, \tag{10}$$

where G_0 is the bare Green's function and G is the full or dressed Green's function which depends on the self-energy $\Sigma(K)$ given by

$$\Sigma(K)=\sum_Q t(Q)G_0(Q-K)\varphi^2_{\mathbf{k}-\mathbf{q}/2}. \tag{11}$$

Here and throughout we use the convention $\Sigma_Q \equiv T\Sigma_{i\Omega_n}\Sigma_{\mathbf{q}}$. In the above expressions $\varphi_{\mathbf{k}}$ represents a generalized (for example) s- or d-wave function symmetry factor. While there are two other T-matrix approaches in the literature, one[2,8] in which χ is related to G_0G_0, and one which is even more widely used[17] in which χ is related to GG, only the GG_0 scheme is simply connected to the Hartree-GL approach.[6]

To compare with GL theory we expand these equations to first order in the self-energy correction.[18] The T matrix can be written in terms of the attractive coupling constant g as

$$t(Q)=\frac{g}{1+g\chi_0(Q)+g\,\delta\chi(Q)}, \tag{12}$$

where

$$\chi_0(Q)=\sum_K G_0(Q-K)G_0(K)\varphi^2_{\mathbf{k}-\mathbf{q}/2}. \tag{13}$$

Defining

$$\Delta^2=-\sum_Q t(Q) \tag{14}$$

we arrive at (see Appendix A)

$$\Sigma(K)\approx -G_0(-K)\Delta^2\varphi^2_{\mathbf{k}}. \tag{15}$$

The results of Appendix A can be used to derive a self-consistency condition on Δ^2 in terms of the quantity $\delta\chi(0)$ (first order in Σ), which satisfies

$$\delta\chi(0)=-b_0(\beta\Delta)^2, \tag{16}$$

implying that

$$\delta\chi(0)=-\frac{b_0}{N(0)T}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\epsilon+\xi_1^2q^2-\delta\chi(0)/N(0)}, \tag{17}$$

which coincides with the condition derived earlier in Eq. (6). Finally, defining

$$\tilde{\mu}_{pair}(T)\equiv\frac{1}{N(0)t(0)}=-\epsilon+\frac{\delta\chi(0)}{N(0)} \tag{18}$$

we may interpret the vanishing of $\tilde{\mu}_{pair}$ as the condition that at $T=T_c$

$$\tilde{\mu}_{pair}(T_c)=0. \tag{19}$$

It follows from Eqs. (16), (18), and (8) that

$$\overline{\mu}_{pair}=\tilde{\mu}_{pair}, \tag{20}$$

so the above condition for T_c is in agreement with that found earlier [Eq. (9)] and the effect of a finite $\Delta(T_c)$ (self-energy correction) is a shift downward in the critical temperature relative to its value (given by T^*) in the $\Delta(T_c)=0$ limit.

III. BELOW T_c: WEAK COUPLING

A. Hartree approximated GL theory for small $\Delta(T_c)$, $T\approx T_c$

The left-hand side of Eq. (5) may be interpreted as the number density of bosons of momentum q. Since $\overline{\mu}_{pair}(T_c)=0$, the $q=0$ level becomes macroscopically occupied once the system enters the superconducting region, at $T=T_c$. To support this assertion we investigate the behavior of $\bar{\mu}_{pair}(T)$ for $T\approx T_c^-$. We separate out the $q=0$ term in Eq. (6) and write

$$\Delta^2=\frac{T}{N(0)V}\frac{1}{-\bar{\mu}_{pair}}+\sum_{q\neq 0}\frac{T}{N(0)V}\frac{1}{-\bar{\mu}_{pair}+\xi_1^2 q^2}. \tag{21}$$

Equation (8) leads to another constraint on Δ, which yields

$$\Delta^2=\frac{T^2}{c}\left(-\frac{T-T^*}{T^*}-\bar{\mu}_{pair}\right), \tag{22}$$

where $c=b_0/[N(0)V]$. Differentiating both the above equations with respect to T one obtains the following expression for $d\bar{\mu}_{pair}/dT$:

$$\frac{d\bar{\mu}_{pair}}{dT}=\frac{\dfrac{\Delta^2}{T}-\dfrac{T^2}{cT^*}}{\dfrac{T}{N(0)V\bar{\mu}_{pair}^2}+\sum_{q\neq 0}\dfrac{T}{N(0)V}\dfrac{1}{(-\bar{\mu}_{pair}+\xi_1^2q^2)^2}+\dfrac{T^2}{c}}. \tag{23}$$

The numerator of Eq. (23) is negative in the vicinity of T_c, since $c\Delta^2/T^2 \ll T/T^*$. At T_c, the first term in the denominator diverges ($1/V\bar{\mu}_{pair}$ is a finite number in the thermodynamic limit), and as also found elsewhere[6] $d\bar{\mu}_{pair}/dT=0$ ($T=T_c$). Since $\bar{\mu}_{pair}$ cannot be positive [that would make the right-hand side of Eq. (5) negative], and its derivative is negative or zero, we conclude that $\bar{\mu}_{pair}$ must remain zero in the vicinity of, but below, T_c, where the GL description is applicable:

$$\bar{\mu}_{pair}(T)=0,\quad T\approx T_c^-. \tag{24}$$

This implies, following Eq. (8),

$$\Delta^2(T)=-\frac{\epsilon}{b_0}\frac{N_0V}{\beta^2}. \tag{25}$$

This result was previously obtained in Ref. 11. It is important because it shows that when the system reaches T_c the excitation gap $\Delta(T)$ assumes the BCS or mean-field value.

We notice strong analogies with Bose-Einstein condensation (BEC) in an ideal Bose gas. Here Δ^2 plays the role of N, the total number of bosons, which below T_c contains two components, one associated with the condensate $\Delta^2_{q=0}$ and the other with the noncondensed pairs $\Delta^2_{q\neq 0}$. The latter are like the excited states of the BEC system. We write

$$\Delta^2=\Delta^2_{q=0}+\Delta^2_{q\neq 0}. \tag{26}$$

It follows from Eqs. (6), (26), and (24) that

$$\beta^2\Delta^2_{q\neq 0}=\int\frac{d^3q}{(2\pi)^3}\frac{1}{N_0VT}\frac{1}{\xi_1^2q^2}. \tag{27}$$

Just as in the ideal Bose gas problem, $\Delta^2(T)$ is constrained (via the pair chemical-potential condition), $\Delta_{q\neq 0}(T)$ is constrained through the self-consistent Hartree condition and thus one may deduce the condensate weight, or superconducting order parameter $\Delta_{q=0}(T)$.

B. Behavior near $T=0$ where $\Delta(T)$ is no longer small

A useful observation can be made at this time. We have just shown that in the Hartree theory, $\Delta(T)$ assumes the BCS value in the vicinity of, but below, T_c. One expects on very general grounds that sufficiently far away from T_c, pair "fluctuation" effects are irrelevant and that the system is described by strict BCS theory. [In this regime $\Delta(T)$ is no longer a small parameter.] Thus we may infer that everywhere below T_c

$$\Delta(T)=\Delta_{BCS}(T),\quad T\leq T_c, \tag{28}$$

so that the excitation gap is given by the BCS value.[10] What is different from strict BCS theory, however, is that

$$\Delta(T_c)\neq 0. \tag{29}$$

This is the sole effect of pair fluctuations below T_c. Nevertheless it has important consequences, because it reflects the presence of noncondensed bosons below T_c, which, in turn, mirror their normal-state counterparts.

C. T-matrix scheme below T_c

We now show that Eq. (28) is connected to the ideal Bose gas condition at all temperatures below T_c. This follows from the analysis in Appendix B in which it is shown that the BCS gap equation is associated with a divergence of the T matrix defined in Eq. (12), at $Q=0$ for *all* temperatures below T_c. Thus $t_{pg}^{-1}(0)=0$ implies

$$\mu_{pair}(T)=0,\quad T\leq T_c. \tag{30}$$

In order to satisfy this gap equation, together with an independent constraint on the number of finite momentum pairs, one must incorporate a broken symmetry $\Delta_{sc}\neq 0$, which we now reformulate within our T-matrix theory. Below T_c

$$\Sigma(K)=\sum_Q t(Q)G_0(Q-K)\varphi^2_{k-q/2} \tag{31}$$

can be decomposed into two contributions: the "pseudogap" and superconducting components via

$$t_{pg}(Q)=\frac{g}{1+g\chi(Q)},\quad Q\neq 0 \tag{32}$$

$$t_{sc}(Q)=-\frac{\Delta_{sc}^2}{T}\delta(Q), \tag{33}$$

$$t=t_{pg}+t_{sc}. \tag{34}$$

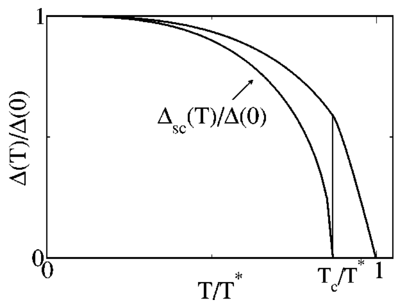

FIG. 1. Schematic plot of the full excitation gap Δ and order parameter Δ_{sc} vs reduced temperature T/T^* for the weak-coupling case.

By defining

$$\Delta_{pg}^2 \equiv -\sum_Q t_{pg}(Q) \tag{35}$$

and using the divergence of $t(Q)$ at $Q=0$ to evaluate Eq. (31), we retrieve the BCS-like self-energy

$$\Sigma(K) \approx -G_0(-K)\Delta^2\varphi_{\mathbf{k}}^2 \tag{36}$$

with

$$\Delta^2 = \Delta_{sc}^2 + \Delta_{pg}^2 . \tag{37}$$

For small $\Delta(T_c)$ and $T \approx T_c$, Eqs. (30), (35), and (37) are manifestly equivalent to their GL counterparts (with μ_{pair} corresponding to $\overline{\mu}_{pair}$,[19] Δ_{sc} to $\Delta_{q=0}$ and Δ_{pg} to $\Delta_{q\neq 0}$); they give a complete[9] description of the system by determining the full excitation gap Δ, order parameter Δ_{sc}, and the amplitude of propagating pairs (the pseudogap) Δ_{pg}.

IV. EXTENDED HARTREE APPROXIMATION: ARBITRARILY STRONG COUPLING *g*

We have demonstrated above that there is a simple *T*-matrix scheme involving the pair susceptibility (GG_0) which is equivalent to Hartree approximated GL theory both above and below T_c. (Moreover, there is a natural extension down to $T=0$.) This equivalence has been proven provided we restrict ourselves to small $\Delta(T_c)$, where GL approaches are applicable (see Fig. 1).

There is no reason, however, that the *T*-matrix scheme cannot be considered for arbitrary coupling constant *g*, or equivalently large $\Delta(T_c)$ where conventional GL theory is no longer appropriate. In this regime the separation between T^* and T_c can be extremely large. In this way, a pseudogap in the fermionic spectrum occurs at a temperature much higher than the Bose condensation temperature T_c (see Fig. 2). As in conventional GL theory, this *pseudogap reflects bosonic degrees of freedom*. Once bosons are metastable it takes a finite excitation energy to create fermions from them.

All the appropriate *T*-matrix equations have been presented above. [See Eqs. (10) and (32)–(37).] Note that the only technical difference between the cases of weak and strong coupling is in the details of the expression for the *T* matrix itself (see Appendix B). There is, however, an important additional constraint that needs to be appended on the fermionic chemical potential μ.

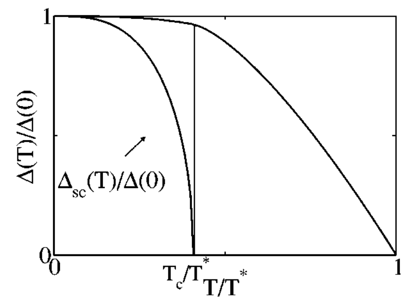

FIG. 2. Schematic plot of the full excitation gap Δ and order parameter Δ_{sc} vs reduced temperature T/T^* for the strong-coupling case.

The resulting equations greatly simplify at and below T_c because of the vanishing of the pair chemical potential. We summarize the above discussion by presenting expressions (appropriate to the superconducting state) for the gap and chemical potential μ,

$$g^{-1} + \sum_{\mathbf{k}} \frac{1-2f(E_{\mathbf{k}})}{2E_{\mathbf{k}}}\varphi_{\mathbf{k}}^2 = 0, \tag{38}$$

$$n = 2\sum_{\mathbf{k}} [f(E_{\mathbf{k}}) + v_{\mathbf{k}}^2\{1-2f(E_{\mathbf{k}})\}]. \tag{39}$$

The former follows from the vanishing of $t_{pg}^{-1}(0)$ in Eq. (32) and the latter from $n=2\Sigma_K G(K)$. The quantity $v_{\mathbf{k}}$ is the coherence factor $v_{\mathbf{k}}^2 = \frac{1}{2}(1-\epsilon_{\mathbf{k}}/E_{\mathbf{k}})$ with $\epsilon_{\mathbf{k}} = k^2/(2m) - \mu$ and $E_{\mathbf{k}}$ is the fermionic excitation energy which depends on the magnitude of the superconducting order parameter Δ_{sc} and the pseudogap energy scale Δ_{pg}:

$$E_k = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta^2(\mathbf{k})}, \tag{40}$$

where $\Delta^2(\mathbf{k}) = \Delta_{pg}^2(\mathbf{k}) + \Delta_{sc}^2(\mathbf{k}) = \Delta^2\varphi_{\mathbf{k}}^2$. The important quantity $\Delta_{pg}(T)$ here is deduced following Eq. (35).

At a physical level the parameter that should be associated with strong or weak coupling is T^*/T_c. However, T^* is frequently difficult to quantify. An alternative parameter for characterizing the strength of the coupling is $\alpha \equiv \Delta(T_c)/\Delta(T=0)$. This is more readily accessed experimentally. It should be seen from Fig. 1 that the BCS limit

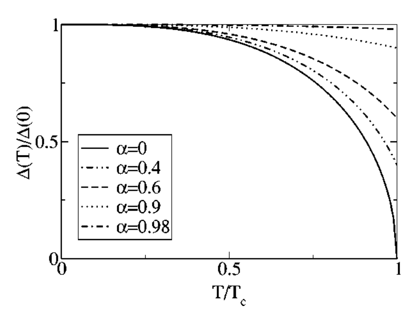

FIG. 3. Implications of the ideal Bose condensation condition ($\mu_{pair}=0$, when $T \leq T_c$) for the excitation gaps. Each of the curves corresponds to different coupling constant strengths.

corresponds to $\Delta(T_c)/\Delta(0)\approx 0$, whereas (following Fig. 2) in the strong-coupling regime $\Delta(T_c)/\Delta(0)\approx 1$. Figure 3 indicates how the temperature dependence of the gap varies with the strength of the coupling, or alternatively with α. This is an important plot because it epitomizes, perhaps, more than any other how dramatic are the differences from strict BCS theory. The challenge, then, is to accommodate the results in this figure into transport and other calculations within the superconducting phase. We thus turn to dynamical effects associated with noncondensed bosons.

V. EFFECTIVE TDGL THEORY FOR STRONG-COUPLING LIMIT

A. Overview

In this section we investigate the applicability of an effective time-dependent Ginzburg-Landau theory as a basis for characterizing the *dynamics* of our excitation-gap or pairing fluctuations. This discussion is relevant to transport calculations associated with bosonic degrees of freedom. Quite generally, we may view TDGL as a generic equation of motion for describing bosons interacting with a reservoir of fermionic pairs. This equation of motion can be associated with the long wavelength, low-frequency behavior of the pair propagator or T matrix, and it has a validity both above and below T_c, as long as bosonic degrees of freedom are relatively stable.

In the presence of a pseudogap, the situation is complicated by the fact that the order parameter is generally different from the excitation gap. This is illustrated clearly in Figs. 1 and 2. Thus, below T_c, one has to take care to distinguish between the fluctuations in each of the two channels. By contrast, above T_c bosonic degrees of freedom are uniquely associated with fluctuations in the fermionic excitation gap. Order-parameter fluctuations in the presence of a finite gap Δ at T_c have been addressed elsewhere.[20] It is conceptually straightforward, but difficult to implement. Such fluctuations also alter critical exponents within a generally narrow "critical fluctuation" regime.

We will not consider these order parameter fluctuations here, but rather presume that Δ_{sc} is spatially uniform, as in a mean-field theoretic approach. The bosons of interest to us are noncondensed pairs which in turn lead to fluctuations in the fermionic excitation gap. A variation in the number of pairs leads to a variation in the excitation gap Δ, simply because one has to pull apart the constituents of the pairs to create fermions. Moreover, our mean-field theory (associated with a BCS-like ground state[1]) provides a considerable simplification, because pairing fluctuation bosons are essentially free; they interact with fermions but not directly with each other. Once there are boson-boson interactions (beyond-mean-field theory), then necessarily there is important coupling to order-parameter fluctuations.

Fermionic degrees of freedom also contribute to transport and thermodynamics, but there is no simple phenomenology (or counterpart of TDGL) for addressing these terms. When the boson contributions to transport are small (say, as in the thermal conductivity), the fermionic terms cannot be neglected, and these have to be computed diagrammatically, as will be discussed in more detail below. In other instances the bosonic contributions are singular,[7] or nearly so, in the vicinity of T_c and TDGL calculations are appropriate for $T_c<T<T^*$, as well as for $T\leqslant T_c$. In the remainder of this section we focus only on the bosonic terms.

B. TDGL above and below T_c

At a microscopic level, one can expand the T matrix for noncondensed pairs at small q,Ω

$$t_{pg}^{-1}(\mathbf{q},\Omega)=a_0'\left[\frac{a_1}{a_0'}\Omega^2+\left(1+i\frac{a_0''}{a_0'}\right)\Omega-\frac{q^2}{2M}+\mu_{pair}\right]. \tag{41}$$

In order to be consistent with the linear dynamics of TDGL, the quadratic terms in Ω are neglected in what follows. The coefficients in Eq. (41) are in general T dependent, although, significantly below T^*, the most important temperature dependence is associated with μ_{pair}, which is zero at and below T_c and negative above T_c. The temperature T^* corresponds to the onset of a resonant structure in the T matrix. This onset temperature is reliably computed without including pseudogap effects—thus, at the Gaussian approximation level. Resonance effects[21] (reflecting the initial formation of metastable pairs) enter via the ratio a_0'/a_0'', which (at fixed T near T^*) can be increased by increasing the coupling constant g. The larger is this parameter the more pronounced are pair resonances. Stated alternatively, as the coupling constant g increases the propagating component a_0' becomes increasingly more important than the diffusive term a_0''. Indeed, similar observations were made by Randeria.[8]

We now proceed to the more detailed analysis of the coefficient a_0'', slightly above and at all T below T_c. Well above T_c (but below T^*) more detailed numerical calculations are required,[22,23] and these demonstrate that a_0'' increases with T as pseudogap effects diminish. At our leading-order approximation as in Eq. (36), we do not distinguish between lifetimes associated with the condensed and noncondensed bosons. For the purposes of computing T_c and the various energy gaps below T_c, this has been shown to be a reasonable approximation,[22,23] but it clearly misses important physics associated with the onset of true phase coherence.[24] In this approximation the finite momentum pairs are extremely long-lived [see Eqs. (B1) and (D1)]. Consequently, a_0'' is zero. Formally, this result comes from the fact that the imaginary part of the pair susceptibility $\Gamma(0,\Omega)$ has a higher power than linear dependence on Ω. If, instead we introduce (as in Appendix D) a finite lifetime to distinguish the contributions from condensed and noncondensed bosons

$$\Sigma(K)=\frac{\Delta_{sc}^2\varphi_{\mathbf{k}}^2}{\omega+\epsilon_{\mathbf{k}}}+\frac{\Delta_{pg}^2\varphi_{\mathbf{k}}^2}{\omega+\epsilon_{\mathbf{k}}+i\gamma}, \tag{42}$$

we find a nonvanishing[22] TDGL coefficient a_0'' above T_c.

$$a_0''=\frac{N(0)\gamma}{\Delta_{pg}^2}. \tag{43}$$

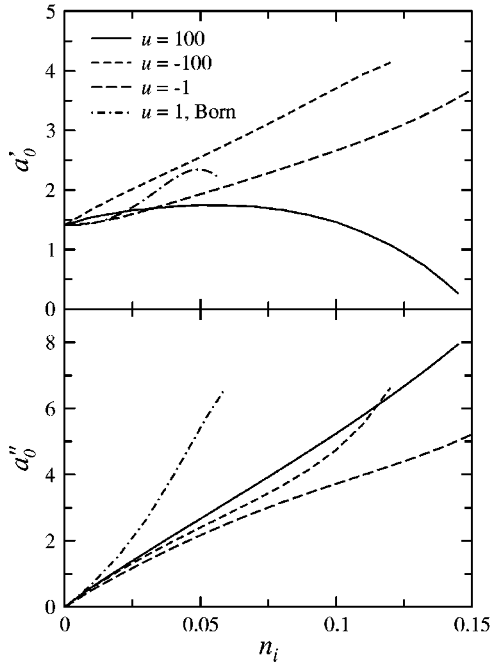

FIG. 4. TDGL coefficients a_0' and a_0'' vs impurity concentration n_i, for several values of scattering strength u from the unitary to the Born limit, in a self-consistent T-matrix treatment (Ref. 25), where the nonmagnetic s-wave scattering potential is assumed as given by $u(\mathbf{x}) = u\,\delta(\mathbf{x})$.

Below T_c, however, the existence of a condensate leads to $a_0'' = 0$, even for the more general self-energy of Eq. (42). These arguments are presented in Appendix D. However, the presence of even a small amount of disorder is sufficient to restore a linear in frequency imaginary term in the TDGL expansion of the T matrix. These observations were made by Chen and Schrieffer.[25] In Fig. 4, we present a figure from their work which illustrates how the frequency-dependent contributions to the T matrix evolve with impurity concentration, for different scattering strengths.

In this way one establishes an effective TDGL description for the noncondensed boson dynamics both above and below T_c. It should be stressed, however, that the character of this theory changes on either side of T_c.

The observation that the pairs live very much longer than anticipated by, say, Gaussian level calculations is a consequence of the fermionic pseudogap.[22,23] In this way, the diffusive component (i.e., the parameter a_0'') remains small for an extended range of temperatures above T_c, and becomes even smaller (impurity limited) below T_c. This reflects the fact that as the fermions acquire a larger gap, the bosons live longer and the two degrees of freedom become progressively more distinct. These same effects are underlined by our earlier observation that T_c, as distinguished from T^*, must be computed by including the feedback effects of the fermionic pseudogap on the bosonic propagator. This is precisely reflected in Eqs. (38) and (39) which, together with Eq. (37), must be solved to determine T_c. Because of the possibility of a large separation between T^* and T_c (as g is progressively increased) one may extend the simple dynamics associated with TDGL theory to describe bosonic transport at temperatures below T^*, *not just those limited to the immediate vicinity of Bose condensation*, T_c. This provides some microscopic support for a recent phenomenological approach[26] which addresses Nernst and other-normal state transport coefficients in underdoped cuprates.

VI. GENERALIZED APPROACH TO TRANSPORT: T-MATRIX THEORY BELOW T_c

We turn now to transport properties below T_c, beginning with the superfluid density. It should be clear from Figs. 1 and 2 that the order parameter and the excitation gap are to be distinguished in the superconducting state. We can thus say that the same T_c as calculated in Sec. II via the self-consistency conditions at the (extended) Hartree-GL level will show up, for example, in $\rho_s(T)$. In particular the superfluid density must necessarily couple to the pair fluctuations in the superconducting state in such a fashion that $\rho_s(T)$ reflects the superconducting order parameter, rather than the fermionic excitation gap. This coupling of pair fluctuations to $\rho_s(T)$ can be contrasted with the way in which collective (phase) mode contributions enter into $\rho_s(T)$ at the BCS level. These terms are only required to preserve general gauge invariance and these bosons do not affect ρ_s when it is computed in the transverse gauge. The pair fluctuations we discuss here are necessary for a consistent calculation of the superfluid density, even in the transverse gauge.

In this section we decompose the transport contributions into two types of excitations of the condensate: fermionic and bosonic. It is well known[7] that bosonic contributions to transport coefficients in the (less self-consistent) Gaussian-TDGL theory of pairing fluctuations are associated with Aslamazov-Larkin diagrams. The lowest-order T-matrix theory introduces additional diagrams called the ''Maki-Thompson'' and ''density-of-states'' contributions. These latter two may be viewed as *fermionic* contributions.

Given the self-energy and form of self-consistent T matrix, Ward identities can be used to characterize the fermionic and bosonic contributions to transport, at the extended Hartree level. Again, Aslamazov-Larkin (interpreted as associated with noncondensed bosons) and Maki-Thompson diagrams (interpreted as associated with fermionic excitations) are present, but both contain bare as well as dressed Green's functions. The fact that the combination GG_0 appears plays an important role throughout our discussion. This pair susceptibility is related to the usual Gor'kov F function.

Because of the form of the pair susceptibility and the fact that below T_c, $\mu_{pair} = 0$, we show below that the fermionic terms combine to yield the usual BCS contributions to transport, which now depend on the full gap Δ, as distinct from the order parameter Δ_{sc}. That the fermionic contributions conspire to be of the BCS form may seem natural at one level, but on another level this is highly nontrivial since at very strong coupling Δ is essentially T independent below T_c, as can be seen from Fig. 3. Thus, the fermionic contributions to transport are nearly T independent, in striking contrast to what is found in the BCS or weak-coupling limit. To demonstrate how the bosonic and fermionic contributions en-

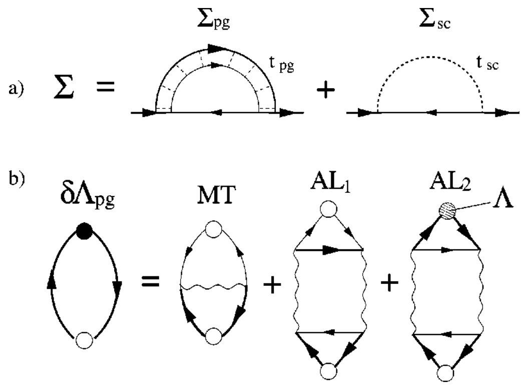

FIG. 5. (a) Self-energy contributions and (b) response diagrams for the vertex correction corresponding to Σ_{pg}. Heavy lines are for dressed, while light lines are for bare Green's functions. Wavy lines indicate t_{pg}.

ter, we begin with a formulation of the superfluid density in the presence of general self-energy effects.

A. Superfluid density: General formalism in transverse gauge

The electromagnetic response in our extended Hartree theory is constructed from the two additive contributions to the self-energy shown in Fig. 5(a). The superconducting self-energy Σ_{sc} contains an anomalous reversal of the fermion line due to the creation of a condensed pair. Moreover, in contrast to conventional fluctuation diagrams, the quantity t_{pg} depends on one full and one bare Green's function. Our aim is to derive a consistent formulation of electrodynamics using a machinery that includes self-energy effects Σ_{pg}. In contrast to, say, the usual treatment of superconducting state generalized Ward identities at the BCS level, the matrix Green's function (Nambu-Gor'kov) approach is not appropriate here since Σ_{pg} does not have a counterpart in the anomalous channel. Indeed, the Nambu-Gor'kov scheme would seem to be problematic at the outset, since it is not clear whether to associate Δ or Δ_{sc} with the "F" function. The choice of our particular pair susceptibility (simply related to F without the gap prefactor) is a way of circumventing this problem.

We define the superfluid density n_s in terms of the magnetic London penetration depth λ_L as

$$\lambda_L^{-2}=\frac{\mu_0 e^2 n_s}{m}, \tag{44}$$

where μ_0 is the magnetic permittivity. For convenience, we will set $\mu_0=e=c=1$. Note on a lattice, n/m should be replaced by

$$\left(\frac{n}{m}\right)_{\mu\nu}\equiv 2\sum_{\mathbf{k}}\frac{\partial^2\epsilon_{\mathbf{k}}}{\partial k_\mu \partial k_\nu}n_{\mathbf{k}} \quad (\mu=x,y,z), \tag{45}$$

where $n_{\mathbf{k}}$ is the fermion density distribution in momentum space. For the quasi-two-dimensional(2D) square lattice, the in-plane mass tensor is diagonal, and $(n/m)_{xx}=(n/m)_{yy}$.

We consider the in-plane penetration depth which is expressed in terms of the local (static) electromagnetic response kernel $K(0)$ in linear-response theory,[27]

$$\lambda_L^{-2}=K_{xx}(0)=\left(\frac{n}{m}\right)_{xx}-P_{xx}(0), \tag{46}$$

where K is defined by

$$J_\mu(Q)=P_{\mu\nu}A_\nu(Q)-\left(\frac{n}{m}\right)_{\mu\nu}A_\nu(Q)=-K_{\mu\nu}(Q)A_\nu(Q), \tag{47}$$

and the current-current correlation function

$$\begin{aligned}P_{\mu\nu}(Q)&=\int_0^\beta d\tau\, e^{i\Omega_n\tau}\langle j_\mu(\mathbf{q},\tau)j_\nu(-\mathbf{q},0)\rangle\\ &=-2\sum_K \mathbf{\Lambda}_\mu^{EM}(K,K+Q)G(K+Q)\mathbf{\lambda}_\nu\\ &\quad\times(K+Q,K)G(K).\end{aligned} \tag{48}$$

Here the bare vertex is given by

$$\mathbf{\lambda}(K,K+Q)=\vec{\nabla}_{\mathbf{k}}\epsilon_{\mathbf{k}+\mathbf{q}/2}=\frac{1}{m}\left(\mathbf{k}+\frac{\mathbf{q}}{2}\right), \tag{49}$$

where, for simplicity, we have used the dispersion for jellium in the last step, although the generalization to the lattice is straightforward. The electromagnetic vertex can be written in terms of the corrections coming from the two self-energy components as

$$\mathbf{\Lambda}^{EM}=\mathbf{\lambda}+\delta\mathbf{\Lambda}_{pg}+\delta\mathbf{\Lambda}_{sc}, \tag{50}$$

where $\delta\mathbf{\Lambda}_{pg}$ is the pseudogap term.

For illustrative purposes we specialize to the case $\varphi=1$ (as occurs in s-wave pairing) in this section. The general, $\varphi\neq 1$ case is discussed in Appendix C. We introduce the superconducting vertex contribution given by

$$\delta\mathbf{\Lambda}_{sc}(K+Q,K)=\Delta_{sc}^2 G_0(-K-Q)G_0(-K)\mathbf{\lambda}(K,K+Q). \tag{51}$$

At the $Q\to 0$ limit, it satisfies

$$-\delta\mathbf{\Lambda}_{sc}(K,K)=\frac{\partial\Sigma_{sc}(K)}{\partial\mathbf{k}}. \tag{52}$$

This equation should be contrasted with the $T>T_c$ Ward identity

$$\delta\mathbf{\Lambda}_{pg}(K,K)=\frac{\partial\Sigma_{pg}(K)}{\partial\mathbf{k}}. \tag{53}$$

The difference in sign between Eqs. (52) and (53) is fundamental and arises from the anomalous nature of the Σ_{sc} diagram. Indeed, Eq. (53) is equivalent to the statement that above T_c the paramagnetic and diamagnetic current contributions to the $Q=0$ response precisely cancel. This cancellation appears in the superfluid density calculation presented in the following section. By contrast Eq. (52) expresses the failure of this cancellation for the superconducting component, which is naturally associated with a Meissner effect.

The vertex $\delta\mathbf{\Lambda}_{pg}$ may be decomposed into Maki-Thompson (MT) and two types of Aslamazov-Larkin (AL_1, AL_2) diagrams, whose contribution to the response is shown here in Fig. 5(b). We write

$$\delta\mathbf{\Lambda}_{pg} \equiv \delta\mathbf{\Lambda}_{MT} + \delta\mathbf{\Lambda}_{AL}^1 + \delta\mathbf{\Lambda}_{AL}^2(\mathbf{\Lambda}). \tag{54}$$

Using conventional diagrammatic rules one can see that the MT term has the same sign reversal as the anomalous superconducting diagram. Here, however, the pairs in question are noncondensed and their internal dynamics (via t_{pg} as distinguished from t_{sc}) requires additional AL_1 and AL_2 terms as well, which will ultimately be responsible for the absence of a Meissner contribution from this normal-state self-energy effect.

Note that the AL_2 diagram is specific to the extended Hartree scheme, in which the field couples to the full G appearing in the T matrix through a vertex $\mathbf{\Lambda}$. It is important to distinguish the vertex $\mathbf{\Lambda}$ from the full electromagnetic (EM) vertex of Eq. (50). In particular, we write

$$\mathbf{\Lambda} = \boldsymbol{\lambda} + \delta\mathbf{\Lambda}_{pg} - \delta\mathbf{\Lambda}_{sc}, \tag{55}$$

where the sign change of the superconducting term (relative to $\mathbf{\Lambda}^{EM}$) is a direct reflection of the sign in Eq. (52).

We now show that for $\varphi = 1$ there is a precise cancellation between the MT and AL_1 pseudogap diagrams at $Q=0$. Analogous results for $\varphi \neq 1$ are presented in Appendix C. This cancellation follows directly from the Ward identity

$$Q \cdot \lambda(K, K+Q) = G_0^{-1}(K) - G_0^{-1}(K+Q), \tag{56}$$

which implies

$$Q \cdot [\delta\Lambda_{AL}^1(K,K+Q) + \delta\Lambda_{MT}(K,K+Q)] = 0 \tag{57}$$

so that $\delta\mathbf{\Lambda}_{AL}^1(K,K) = -\delta\mathbf{\Lambda}_{MT}(K,K)$ is obtained exactly from the $Q \to 0$ limit.

Similarly, it can be shown that

$$Q \cdot \Lambda(K,K+Q) = G^{-1}(K) - G^{-1}(K+Q). \tag{58}$$

The above result can be used to infer a relation analogous to Eq. (57) for the AL_2 diagram, leading to

$$\delta\mathbf{\Lambda}_{pg}(K,K) = -\delta\mathbf{\Lambda}_{MT}(K,K), \tag{59}$$

which expresses this pseudogap contribution to the vertex entirely in terms of the Maki-Thompson diagram shown in Fig. 5(b). It is evident that $\delta\mathbf{\Lambda}_{MT}$ is simply the pseudogap counterpart of $\delta\mathbf{\Lambda}_{sc}$, satisfying

$$-\delta\mathbf{\Lambda}_{MT}(K,K) = \frac{\partial \Sigma_{pg}(K)}{\partial \mathbf{k}}. \tag{60}$$

Therefore, one observes that for $T \leqslant T_c$

$$\delta\mathbf{\Lambda}_{pg}(K,K) = \frac{\partial \Sigma_{pg}(K)}{\partial \mathbf{k}}, \tag{61}$$

which establishes that Eq. (53) applies to the superconducting phase as well. As expected, there is no direct Meissner contribution associated with the pseudogap self-energy. Although the derivation becomes more complicated for general $\varphi_{\mathbf{k}}$, Eq. (53) is obtained for both the superconducting and normal states.

B. Superfluid density: *T*-matrix approximation

The final expression for the superfluid density is obtained by rewriting Eq. (45) by integration by parts,

$$\begin{aligned}\left(\frac{n}{m}\right)_{\alpha\beta} &= 2\sum_K \frac{\partial^2 \epsilon_{\mathbf{k}}}{\partial k_\alpha \partial k_\beta} G(K) \\ &= -2\sum_K \frac{\partial \epsilon_{\mathbf{k}}}{\partial k_\alpha}\frac{\partial G(K)}{\partial k_\beta} \\ &= -2\sum_K G^2(K)\frac{\partial \epsilon_{\mathbf{k}}}{\partial k_\alpha}\left(\frac{\partial \epsilon_{\mathbf{k}}}{\partial k_\beta} + \frac{\partial \Sigma_{pg}}{\partial k_\beta} + \frac{\partial \Sigma_{sc}}{\partial k_\beta}\right).\end{aligned} \tag{62}$$

Note here the surface term vanishes in all cases.

By inserting Eqs. (62) and (48) in Eq. (46) one can see that the pseudogap contribution to n_s/m drops out by virtue of Eq. (61). The in-plane superfluid is isotropic and is given by

$$\frac{n_s}{m} = 2\sum_K G^2(K)\frac{\partial \epsilon_{\mathbf{k}}}{\partial k_x}\left[\delta\Lambda_{sc}(K,K)_x - \frac{\partial \Sigma_{sc}(K)}{\partial k_x}\right]. \tag{63}$$

Equation (63) can be readily evaluated using the superconducting vertex and the superconducting self-energy $\Sigma_{sc}(K) = -\Delta_{sc}^2 G_0(-K)\varphi_{\mathbf{k}}^2$ associated with our GG_0-based T-matrix approach. In addition, we introduce an approximation in our evaluation of G via Eq. (36), to find

$$\begin{aligned}\frac{n_s}{m} = 2\sum_{\mathbf{k}} \frac{\Delta_{sc}^2}{E_{\mathbf{k}}^2}\left[\frac{1+2f(E_{\mathbf{k}})}{2E_{\mathbf{k}}} + f'(E_{\mathbf{k}})\right] \\ \times\left[\left(\frac{\partial \epsilon_{\mathbf{k}}}{\partial k_x}\right)^2 \varphi_{\mathbf{k}}^2 - \frac{1}{4}\frac{\partial \epsilon_{\mathbf{k}}^2}{\partial k_x}\frac{\partial \varphi_{\mathbf{k}}^2}{\partial k_x}\right].\end{aligned} \tag{64}$$

This result includes the general $\varphi_{\mathbf{k}}$ factor for the $\mathbf{k}$ dependence of the gap and order parameter, whereas it has been neglected above in Eqs. (51)–(61), where it substantially complicates the analysis.

Note that in the absence of a pseudogap (i.e., at weak coupling), $\Delta_{sc} = \Delta$. Then Eq. (64) is just the usual BCS formula. More generally, we can define a relationship

$$\left(\frac{n_s}{m}\right) = \frac{\Delta_{sc}^2}{\Delta^2}\left(\frac{n_s}{m}\right)^{BCS}, \tag{65}$$

where $(n_s/m)^{BCS}$ is just (n_s/m) with the overall prefactor Δ_{sc}^2 replaced with Δ^2 in Eq. (64). Obviously, in the pseudogap phase, $(n_s/m)^{BCS}$ does not vanish at T_c. That the results are so similar to their BCS counterparts is due to our Hartree treatment of pairing fluctuations.

Finally, from Eqs. (37) and (65) we may write

$$\left(\frac{n_s}{m}\right)=\left(1-\frac{\Delta_{pg}^2}{\Delta^2}\right)\left(\frac{n_s}{m}\right)^{BCS}. \tag{66}$$

Here the first term represents the contribution to n_s from the usual fermions, albeit with an unusual T dependence of the gap (see Fig. 3). The second term indicates that n_s is additionally depressed by bosonic pair excitations which ensure that n_s vanishes prematurely at T_c, rather than at T^*.

Finally, from Eq. (53) we can infer that the self-energy approximation of Eq. (36) implies an approximation on $\delta\mathbf{\Lambda}_{pg}$, so that

$$\delta\mathbf{\Lambda}_{MT}(K,K)\approx\Delta_{pg}^2 G_0(-K)G_0(-K)\boldsymbol{\lambda}(K,K). \tag{67}$$

We build on this internal consistency argument in what follows away from $Q=0$.

C. $Q\neq 0$ electrodynamics

The real $\Omega\neq 0$ part of the in-plane optical conductivity can be expressed as

$$\sigma(\Omega)=\Omega^{-1}\mathrm{Im}\,P_{xx}(i\Omega_n\to\Omega+i0^+), \tag{68}$$

which is related to the superfluid density through the f-sum rule

$$\frac{n_s}{m}+\frac{2}{\pi}\int_0^\infty\sigma(\Omega)\,d\Omega=\left(\frac{n}{m}\right)_{xx}. \tag{69}$$

Just as for the superfluid density, the optical conductivity involves the same set of Maki-Thompson and Aslamazov-Larkin diagrams. In this section we will regroup terms so as to identify explicit fermionic and bosonic contributions,

$$\mathbf{\Lambda}^{EM}\equiv\boldsymbol{\lambda}+\delta\mathbf{\Lambda}_{fermions}+\delta\mathbf{\Lambda}_{bosons}. \tag{70}$$

Similarly, it follows that the optical conductivity contains two contributions,

$$\sigma(\Omega)=\sigma^{fermions}(\Omega)+\sigma^{bosons}(\Omega), \tag{71}$$

where $\sigma^{fermions}$ comes from the $\boldsymbol{\lambda}+\delta\mathbf{\Lambda}_{fermions}$ portion of the vertex.

It is not unreasonable to take Eq. (67) a step further and apply it to general Q, so that (below T_c)

$$\delta\mathbf{\Lambda}_{MT}(K,K+Q)\approx\Delta_{pg}^2 G_0(-K-Q)G_0(-K)\boldsymbol{\lambda}(K+Q,K). \tag{72}$$

In effect what this approximation is saying is that for fermionic degrees of freedom the bosons enter primarily as an excitation-gap contribution. This approximation is justified by the same reasoning that leads to Eq. (36), using the divergence of $t_{pg}(Q)$ at $Q=0$.

From Eq. (72) and Eq. (52) it follows that the pseudogap and superconducting condensate terms add in a natural way to introduce the full excitation gap Δ:

$$\begin{aligned}\delta\mathbf{\Lambda}_{fermions}&(K,K+Q)\\&\equiv\delta\mathbf{\Lambda}_{MT}(K,K+Q)+\delta\mathbf{\Lambda}_{sc}(K,K+Q)\\&=\Delta^2 G_0(-K-Q)\boldsymbol{\lambda}(K+Q,K)G_0(-K).\end{aligned} \tag{73}$$

This term, combined with the density-of-states contribution from $\boldsymbol{\lambda}$, indicates that the fermionic contributions to general transport coefficients are to be calculated within a BCS framework, but with the full gap Δ, which is to be distinguished from the order parameter. Though these contributions to transport are formally similar to BCS theory, their T dependence may differ considerably due to the weak T dependence of Δ in the strong pseudogap regime.

To characterize the direct contribution from bosonic degrees of freedom to transport, which are associated with AL diagrams, we must treat their full dynamics. We define

$$\delta\mathbf{\Lambda}_{bosons}(K,K+Q)\equiv\delta\mathbf{\Lambda}_{AL}^1(K,K+Q)+\delta\mathbf{\Lambda}_{AL}^2(K,K+Q) \tag{74}$$

and turn now to the bosonic contribution to conductivity.

D. TDGL approach to bosonic transport

We have just seen that to a good approximation the contributions to the EM response of diagrams other than the AL_1 and AL_2 terms enter as in BCS theory but with the full gap Δ. Equation (66) allows us to separate out the BCS contribution with full gap, called $(n_s/m)^{BCS}$, from the additional ‘‘bosonic’’ contribution. The BCS terms also satisfy a sum rule relating the ac conductivity and superfluid density, analogous to Eq. (69),

$$\left(\frac{n_s}{m}\right)^{BCS}+\frac{2}{\pi}\int_0^\infty\sigma^{fermions}(\Omega)\,d\Omega=\left(\frac{n}{m}\right)_{xx}. \tag{75}$$

Using these sum rules along with Eqs. (66) and Eq. (71), we can deduce that the integrated conductivity of the bosons is well approximated by

$$\frac{2}{\pi}\int_0^\infty d\Omega\;\sigma^{bosons}(\Omega,T)=\frac{\Delta_{pg}^2}{\Delta^2}\left(\frac{n_s}{m}\right)^{BCS}(T). \tag{76}$$

The bosons make a maximum contribution at T_c. At this temperature, bosons can account for as much as 90% of the spectral weight in a strongly pseudogapped superconductor with $T^*/T_c=10$, while their contribution vanishes in the weak-coupling limit $T^*\to T_c$. The boson weight vanishes at $T=0$ at all couplings as a consequence of the condensation of all bosonic pairs. This pairing fluctuation bosonic contribution is not limited to a narrow region near T_c, but extends well into the superconducting state.

We define the bosonic response P_{boson} as the contribution to P given by $\delta\mathbf{\Lambda}_{bosons}$. These terms each involve a pair of T matrices, and to leading order in frequency, P_{boson} may be written as

$$P_{boson}(Q) \equiv -2\sum_K [\delta\mathbf{\Lambda}_{AL}^1(K,K+Q) + \delta\mathbf{\Lambda}_{AL}^2(K,K+Q)]G(K+Q) \times \mathbf{\lambda}(K+Q,K)G(K) \approx \sum_P \mathbf{\Lambda}_t(P,P)t_{pg}(P+Q)\tilde{\mathbf{\Lambda}}_t(P,P)t_{pg}(P). \tag{77}$$

Here $\mathbf{\Lambda}_t$ is the vertex for t_{pg} approximated as

$$\mathbf{\Lambda}_t(P,P) \approx 2\frac{a_0'}{M}\mathbf{p}, \tag{78}$$

where we have used the T-matrix expansion of Eq. (41). The quantity $\tilde{\mathbf{\Lambda}}_t = z\mathbf{\Lambda}_t(P,P)$ where $z=1$ in the normal state[5] and is modified in the superconducting state. Reasonable estimates of $z(T)$ below T_c may be obtained from Eq. (69).

If we presume Eq. (77) holds for a range of low frequencies we may infer a simple expression for the (in-plane) ac conductivity

$$\sigma^{bosons}(\Omega) = \frac{1}{2}z a_0'^2(2e)^2\sum_{\mathbf{p}}\left(\frac{p_x}{M}\right)^2 \times \int \frac{dE}{2\pi}\tilde{A}(\mathbf{p},E)\tilde{A}(\mathbf{p},E+\Omega)\frac{b(E)-b(E+\Omega)}{\Omega}, \tag{79}$$

where we now use $\tilde{A}(\mathbf{p},\Omega) = -2\,\mathrm{Im}\,t_{pg}(\mathbf{p},\Omega+i0)$ for the bosonic spectral function. More generally, at higher frequencies the internal fermionic structure via the Q dependence of the boson vertex must be included. This structure results from the individual coupling of radiation to each constituent fermion in the pair. However, it is reasonable to assume that the compositeness of the pairs will not be resolved by radiation of wavelengths larger than the pair size or frequencies below the pair-breaking energy Δ. For $\mathbf{q}=0$ and $\Omega<\Delta$, then we argue the bosonic vertex functions are well approximated by the velocity $\mathbf{p}/M$ (or a constant multiple thereof). Indeed, for calculations of the ac conductivity, Varlamov and co-worker[28] have argued that an analogous approximation (for Gaussian fluctuations) is valid near T_c where the pole structure of the T matrix causes its frequency dependence to dominate that of the Green's functions at the vertices. At the extended Hartree level this pole structure is present at and below T_c due to the vanishing of μ_{pair}. Relative to Gaussian theory, the Q dependence of the boson vertices will be further suppressed for $\Omega<\Delta$ through the appearance of the gapped fermion propagator G in all vertex subdiagrams.

The result derived above for the bosonic contribution to the ac conductivity is essentially the same as would obtain from true bosons, except for the constant factor $a_0'^2$. To see this we review the ac conductivity in a system of free bosons of charge e^*, mass M^*, and chemical potential μ^*, in contact with a reservoir of fermion pairs. This calculation[26] represents a generalization of standard TDGL-like schemes, away from the Bose condensation temperature. The simplest physical picture that allows for an exactly solvable conductivity in the presence of quantum dissipation assumes that the bosons interact with a reservoir of *localized*[26] fermion pairs (treated as having ideal gas Bose-Einstein statistics). This gives rise to a boson self-energy $\Sigma_B(\Omega)$ without introducing vertex corrections to the electromagnetic response. Such a model yields[26] an ac conductivity given by

$$\sigma^0(\Omega) = \frac{1}{2}(e^*)^2\sum_{\mathbf{p}}\left(\frac{p_x}{M^*}\right)^2 \times \int \frac{dE}{2\pi}\tilde{A}(\mathbf{p},E)\tilde{A}(\mathbf{p},E+\Omega)\frac{b(E)-b(E+\Omega)}{\Omega}. \tag{80}$$

Here $b(E)$ is the Bose statistical function and $\tilde{A} = -2\,\mathrm{Im}\,B$ is the boson spectral function. The boson propagator is given by $B(\mathbf{q},\Omega)^{-1} = \Omega - q^2/2M^* + \mu^* - \Sigma_B(\Omega)$. The boson vertex here is the velocity $\mathbf{p}/M^*$. The boson self-energy arises from scattering processes into and out of the thermal reservoir.

To compare with the pairing theory, we note that while the boson self-energy in the above model arises from scattering into the reservoir, the self-energy of bosons in the pairing model (whose imaginary part we may regard as $\mathrm{Im}\,t_{pg}^{-1}$) arises from pair dissociation and recombination processes. We note, however, that fundamental differences between fermion pairs and true bosons remain in the analytic structure of the respective T matrix and boson propagator B. For true bosons, the real and imaginary parts of Σ_B obey Kramers-Kronig relations and vanish in the high-energy limit. The propagator B then reduces to its bare form $(\Omega - q^2/2M^* + \mu^*)^{-1}$. The T matrix has the structure $t_{pg}^{-1} = g^{-1} + \chi$ where the pairing susceptibility χ satisfies the same causality constraints. The vanishing of χ in the high-energy limit leaves the asymptote $t_{pg} \to g$ and the T matrix loses all energy and momentum structure due to the dissociation of all pairs. However, this difference is not expected to be relevant for conductivity calculations done below the pair-breaking energy scale.

We now consider the evaluation of Eq. (79) using the expanded T matrix of Eq. (41), neglecting the Ω^2 term. Dissipation in σ at nonzero frequencies requires $a_0'' \neq 0$, which below T_c requires impurity scattering, as discussed in Appendix D. Here, we focus on the behavior of the conductivity as a function of the ratio $\nu \equiv a_0''/a_0'$, which in turn enters $\Gamma(\mathbf{q},\Omega)$ in the expanded T-matrix approximation [Eq. (B2)] as $\nu\Omega$. We note here that in the TDGL approach for Gaussian fluctuations near T_c, this ratio is typically large [of order (E_F/T_c)] due to the fact that a_0' is a measure of particle-hole asymmetry. As shown earlier in Fig. 4, ν tends in the extended Hartree theory to increase rapidly from zero, becoming of order unity with the introduction of a modest concentration of impurities.

TDGL calculations are typically done in the classical regime $\Omega \ll T$, which allows the simplification $[b(E)-b(E+\Omega)]/\Omega \to T/[E(E+\Omega)]$ in Eq. (79). The conductivity below T_c is then found to have the characteristic $\Omega^{-1/2}$ dependence[28] for all values of ν:

$$\sigma^{boson}(\Omega)\rightarrow \frac{2}{3\pi}ze^2T\left(\frac{2M}{\Omega}\right)^{1/2}\left(\frac{1+\nu^2}{\nu}\right)^{1/2}. \tag{81}$$

To compute the conductivity in the superconducting state up to $\Omega\sim\Delta$, however, requires the inclusion of quantum statistical factors when $\Delta \gg T_c$, in which case the ν parameter affects the frequency dependence of σ^{boson}. While the low-frequency limiting behavior is $\Omega^{-1/2}$ independent of ν, σ falls faster than $\Omega^{-1/2}$ outside the classical regime before crossing over to $\Omega^{1/2}$ behavior at higher frequencies. This crossover may extend over a large frequency range, depending on the value of ν, so that generally σ is characterized by $\Omega^{-1/2}$ at low frequencies and a long high-frequency tail which cannot be integrated to infinity. Since the integrated bosonic weight is finite, it is reasonable to expect that this expression for the conductivity is cut off above the pair-breaking scale [at several times $\Delta(0)$], where the TDGL formulation is known to break down.

VII. CONCLUSIONS

This paper is based on the observation that the BCS ground-state wave function has a more general validity.[1,29] By increasing the strength of the attractive interactions, superconductivity in this state progressively takes on the character of Bose-Einstein condensation. That the same $T=0$ wave function can apply to a system where pair formation and pair condensation are associated with different energy scales provides support for the notion that there exists a mean-field theory of a more general nature than that of strict BCS theory. In the more general case the various energies of the BCS picture are no longer degenerate: $T^*\neq T_c$ and $\Delta \neq \Delta_{sc}$.

In this paper we have shown that this generalization of BCS physics is to be found in a treatment[4,6] of pair fluctuation effects at the Hartree level. This establishes that the excitation gap $\Delta(T_c)$ is nonzero, or equivalently that there is a pseudogap, even in the weak-coupling limit, as was observed experimentally many years ago.[3] Going beyond this previous work and into the broken-symmetry phase, a self-consistent analysis leads to the condition that the pair chemical potential $\mu_{pair}=0$ for $T\leqslant T_c$, and that this is equivalent to the statement that the excitation gap $\Delta(T)$ assumes the BCS value at and below T_c. It follows from the fact that the order parameter is necessarily zero at T_c, that the excitation gap and the superconducting order parameter are distinguishable below T_c. This behavior is schematically illustrated in Figs. 1 and 2.

Because it exists in conjunction with fermionic degrees of freedom, the ideal Bose character found here supports true superconductivity. This is demonstrated here by calculations of the superfluid density. More generally, transport properties contain two types of contributions from both fermionic and bosonic excitations. At finite temperatures, the fermionic terms are well approximated by the usual BCS contributions to transport *but with a highly non-BCS-like, often temperature-independent gap* $\Delta(T)$.

At this extended Hartree level, fluctuations in the order parameter Δ_{sc} and in the excitation gap Δ represent distinct channels for bosonic effects in the superconducting state. It is more convenient to decompose these into ‘‘condensed’’ and ‘‘noncondensed’’ bosons. The latter are the focus of the present paper, and we refer to these as ‘‘pair fluctuations’’ as distinct from fluctuations of the order parameter. Order-parameter fluctuations were discussed in the context of our extended Hartree theory in earlier work.[20] Noncondensed bosons are present above and below T_c, but absent (like fermionic excitations) at $T=0$. These long-lived, metastable states, even above T_c (albeit, below T^*), are associated with our extended Hartree treatment that introduces a pseudogap in the fermionic spectrum. This ‘‘gap’’ then inhibits dissociation of the bosons into fermions.

The dynamics of these noncondensed bosons is reasonably described by time-dependent Ginzburg-Landau theory. While the microscopic character of this TDGL changes abruptly from above to below T_c, this approach offers a very powerful technique[26,30,31] for addressing the pair fluctuation or bosonic contributions to transport. In this paper we have applied these results to calculations of the ac conductivity. One could equally well address the thermal conductivity and we can anticipate some of the ensuing conclusions. When both bosonic and fermionic excitations are present (i.e., away from $T=0$), we expect that the Wiedemann-Franz law is violated. Because of the associated soft energy scale, bosonic contributions to the thermal conductivity are considerably smaller than their counterparts for the electrical conductivity. In this way the thermal conductivity κ in the superconducting state should be well approximated by considering only BCS-like contributions, but with the anomalous temperature-dependent excitation gap shown in Fig. 3. Moreover, because the bosonic contributions are unimportant in κ, we expect to recover the well-known universal result[32] for this property. This is not necessarily true for the electrical conductivity.

In this paper we have not made reference to specific physical systems where our mean-field theory may have some applicability. In addition to short coherence length superconductors, (among these the high-temperature cuprates), the present picture may also be relevant[33–35] to fermionic superfluidity in atomic trap experiments.

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APPENDIX A: *T* MATRIX FOR SMALL Δ AT WEAK COUPLING

If we replace the dressed Green's function in Eq. (10) by the bare Green's function G_0, we obtain

$$t_0(\mathbf{q},\Omega_n)=\frac{g}{1+g\chi_0(Q)}=-\frac{1}{N(0)}\frac{1}{\epsilon+a|\Omega_n|+\xi_1^2q^2}. \tag{A1}$$

The self-energy $\Sigma(K)$ is then given by

$$\Sigma(K)=\sum_{Q} t(Q)G_0(Q-K)\varphi^2_{\mathbf{k}-\mathbf{q}/2}. \tag{A2}$$

To calculate the pair susceptibility to the first order in self-energy, we write the dressed Green's function in Eq. (10) as $G=G_0+G_0\Sigma G_0$, which gives the first-order correction

$$\delta\chi(Q)=\sum_{K} G_0(Q-K)G_0(K)\Sigma(K)G_0(K). \tag{A3}$$

Since $t(Q)$ is sharply peaked around $Q=0$ we disregard the momentum dependence of G_0 in Eq. (A2), and obtain

$$\Sigma(P)\approx -G_0(-K)\Delta^2\varphi^2_{\mathbf{k}}, \tag{A4}$$

where

$$\Delta^2=-\int \frac{d^3q}{(2\pi)^3}t(\mathbf{q},\omega_n=0). \tag{A5}$$

Putting this back into Eq. (A3) we find

$$\delta\chi(0)=-b_0\beta^2\Delta^2=b_0/T^2\int \frac{d^3q}{(2\pi)^3}t(\mathbf{q},\omega_n=0), \tag{A6}$$

where $b_0/T^2=\Sigma_K G_0^2(K)G_0^2(-K)\varphi^2_{\mathbf{k}}=N(0)7\xi(3)/8(\pi T)^2$ (this last equality holds only for *s*-wave pairing). Using the expression for the *T* matrix corrected by $\delta\chi(0)$ via Eq. (12)

$$t(Q)=-\frac{1}{N(0)}\frac{1}{\epsilon+a|\Omega_n|+\xi_1^2q^2-\delta\chi(0)/N(0)}, \tag{A7}$$

the self-consistency equation reads

$$\delta\chi(0)=-\frac{b_0}{N(0)T}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\epsilon+\xi_1^2q^2-\delta\chi(0)/N(0)}. \tag{A8}$$

APPENDIX B: *T* MATRIX FOR ARBITRARY Δ, BELOW T_c FOR GENERAL COUPLING

From Eq. (36) and $G_0(P)=1/(i\omega_n-\epsilon_{\mathbf{p}})$ we find $G(P)=(i\omega_n+\epsilon_{\mathbf{p}})/[(i\omega_n)^2-E^2_{\mathbf{p}}]$. Here $\epsilon_{\mathbf{p}}$ is the electron normal-state dispersion measured with respect to the chemical potential μ, while $E_{\mathbf{p}}=\sqrt{\epsilon^2_{\mathbf{p}}+\Delta^2\varphi_p^2}$. Thus after performing the Matsubara sum and analytical continuation $i\Omega_n\rightarrow\Omega+i0^+$, Eq. (10) becomes

$$\chi(\mathbf{q},\Omega)=\sum_{\mathbf{k}}\left[\frac{1-f(E_{\mathbf{k}})-f(\epsilon_{\mathbf{k}-\mathbf{q}})}{E_{\mathbf{k}}+\epsilon_{\mathbf{k}-\mathbf{q}}-\Omega-i0^+}u^2_{\mathbf{k}} -\frac{f(E_{\mathbf{k}})-f(\epsilon_{\mathbf{k}-\mathbf{q}})}{E_{\mathbf{k}}-\epsilon_{\mathbf{k}-\mathbf{q}}+\Omega+i0^+}v^2_{\mathbf{k}}\right]\varphi^2_{\mathbf{k}-\mathbf{q}/2}. \tag{B1}$$

In the long wavelength, low-frequency limit one can expand the inverse *T* matrix as

$$t^{-1}_{pg}(\mathbf{q},\Omega)=g^{-1}+\chi(\mathbf{q},\Omega) =a_1\Omega^2+a_0'\left(\Omega-\frac{q^2}{2M}+\mu_{pair}+i\Gamma_{\mathbf{q},\Omega}\right). \tag{B2}$$

The linear contribution ($\mathbf{q}$) is absent due to the inversion symmetry ($\mathbf{q}\leftrightarrow-\mathbf{q}$) of the system.

In the weak-coupling limit, the ratio a_0'/a_1 is vanishingly small; when the system has exact particle-hole symmetry (e.g., a 2D tight-binding band at half-filling with a nearest-neighbor hopping), a_0' vanishes. In this case the dispersion determined via

$$t^{-1}_{pg}(\mathbf{q},\Omega)=0 \tag{B3}$$

is linear in q, $\Omega_{\mathbf{q}}\sim cq$, which shows up in the dispersion, *only in the very weak coupling limit* or where there is exact particle-hole symmetry.

In the absence of particle-hole symmetry, as g increases, a_0'/a_1 increases, thus $a_0'\Omega$ gradually dominates and we find the important result: $\Omega_{\mathbf{q}}\sim q^2$. For any finite g and arbitrarily small q, the dispersion is always quadratic, at the lowest energies.

We are interested in the moderate- and strong-coupling cases, where we can drop the $a_1\Omega^2$ term in Eq. (B2), and hence we have

$$t_{pg}(\mathbf{q},\Omega)=\frac{a_0'^{-1}}{\Omega-\Omega_{\mathbf{q}}+\mu_{pair}+i\Gamma_{\mathbf{q},\Omega}}, \tag{B4}$$

where

$$\Omega_{\mathbf{q}}\equiv\frac{q^2}{2M} \tag{B5}$$

is quadratic. This defines the effective pair mass *M*. Below T_c, we have

$$\mu_{pair}(T)=0,\quad T\leqslant T_c. \tag{B6}$$

Using Eq. (B1) this condition can be shown to be equivalent to Eq. (38), the familiar BCS gap equation.

APPENDIX C: REVISITING THE SUPERFLUID DENSITY WITH $\varphi\neq 1$

The electromagnetic response vertex is given by Eq. (50). For a general $\varphi\neq1$, the superconducting vertex contribution is given by

$$\delta\mathbf{\Lambda}_{sc}(K+Q,K)=\Delta^2_{sc}G_0(-K-Q)G_0(-K) \times\boldsymbol{\lambda}(K,K+Q)\varphi_{\mathbf{k}}\varphi_{\mathbf{k}+\mathbf{q}}. \tag{C1}$$

This can be obtained by proper vertex insertion to the superconducting self-energy in Fig. 5(a).

At the $Q\rightarrow0$ limit, it satisfies

$$\frac{\partial \Sigma_{sc}(K)}{\partial \mathbf{k}} = -\delta \mathbf{\Lambda}_{sc}(K,K) - \Delta_{sc}^2 G_0(-K) \nabla_{\mathbf{k}} \varphi_{\mathbf{k}}^2$$
$$\equiv \delta \mathbf{\Lambda}'_{sc}(K,K). \tag{C2}$$

The minus sign reflects the fact that $\delta\mathbf{\Lambda}_{sc}$ alone does not satisfy the generalized Ward identity in the transverse gauge, as a consequence of the Meissner effect. To recover full gauge invariance, the collective mode contribution has to be included. Nevertheless, one can define, in the transverse gauge,

$$\mathbf{\Lambda}^{BCS} \equiv \boldsymbol{\lambda} + \delta \mathbf{\Lambda}'_{sc}. \tag{C3}$$

This is not the EM vertex of BCS theory, but it automatically satisfies the generalized Ward identity in the BCS case. This circumvents the complication of collective modes, which ultimately make no contribution to the transverse response.

The vertex $\delta\mathbf{\Lambda}_{pg}$ is given by Eq. (54). Now we define the full vertex, which is distinct from the EM vertex $\mathbf{\Lambda}^{EM}$,

$$\mathbf{\Lambda}(K,K_+) = [\boldsymbol{\lambda} + \delta\mathbf{\Lambda}_{pg} + \delta\mathbf{\Lambda}'_{sc}](K,K_+),$$
$$\delta\mathbf{\Lambda}'_{sc}(K,K_+) = -\delta\mathbf{\Lambda}_{sc}(K,K_+) - \Delta_{sc}^2 G_0(-K)\nabla_{\mathbf{k}}\varphi_{\mathbf{k}}^2, \tag{C4}$$

where the sign change of the superconducting term (relative to $\mathbf{\Lambda}^{EM}$) is a direct reflection of the sign in Eq. (C2). The temporal component of Λ is defined similarly, but with no correction coming from derivatives of $\varphi_{\mathbf{k}}$.

Invoking the Ward identity for the bare Green's function G_0, Eq. (56), we obtain

$$Q \cdot \mathrm{MT}_{pg}(K,K_+) = -[\Sigma_{pg}(K_+) - \Sigma_{pg}(K)]$$
$$- \sum_P t_{pg}(P) G_0(P-K) \mathbf{q} \cdot \nabla_{\mathbf{k}} \varphi_{\mathbf{k}-\mathbf{p}/2}^2 \tag{C5}$$

to linear order in Q. Taking $Q \to 0$, we obtain

$$\frac{\partial \Sigma_{pg}(K)}{\partial \mathbf{k}} = -\mathrm{MT}_{pg}(K,K) + \sum_P t_{pg}(P) G_0(P-K) \nabla_{\mathbf{k}} \varphi_{\mathbf{k}-\mathbf{p}/2}^2$$
$$\approx -\mathrm{MT}_{pg}(K,K) - \Delta_{pg}^2 G_0(-K) \nabla_{\mathbf{k}} \varphi_{\mathbf{k}}^2. \tag{C6}$$

This is a generalization of Eq. (60) to a general $\varphi \neq 1$.

The AL_1 diagram can be similarly evaluated using Eq. (56). It can also be shown that the generalized Ward identity

$$Q \cdot \Lambda(K,K+Q) = G^{-1}(K) - G^{-1}(K+Q) \tag{C7}$$

holds to linear order in Q. This allows the evaluation of the AL_2 diagram, with the result

$$Q \cdot (\mathrm{AL}_1 + \mathrm{AL}_2)(K,K_+)$$
$$= 2[\Sigma_{pg}(K_+) - \Sigma_{pg}(K)]$$
$$+ \sum_P t_{pg}(P) G_0(P-K) \mathbf{q} \cdot \nabla_{\mathbf{k}} \varphi_{\mathbf{k}-\mathbf{p}/2}^2 \tag{C8}$$

valid to linear order in Q. Adding these results, the terms depending on the derivative of φ cancel, leading to

$$Q \cdot \delta\Lambda_{pg}(K,K+) = \Sigma_{pg}(K_+) - \Sigma_{pg}(K) \tag{C9}$$

valid to linear order in Q. Taking $Q \to 0$, we obtain Eq. (61) in the text.

Note that the various vertex terms in the above equations contain corrections from the derivative of φ at $Q=0$ only in the spatial components and not in the scalar or temporal component.

APPENDIX D: IMAGINARY PART OF THE INVERSE T-MATRIX BOSONIC LIFETIME NEAR T_c AND BELOW

In this appendix, for definiteness, we assume d-wave pairing so that $\varphi_{\mathbf{k}} = 2\cos(2\phi)$, where ϕ is the polar angle. By taking the imaginary part of the inverse T matrix using Eq. (B1) we obtain the expression

$$\Gamma_{\mathbf{q},\Omega} = \frac{\pi}{a'_0} \sum_{\mathbf{k}} \{[1 - f(E_{\mathbf{k}}) - f(\epsilon_{\mathbf{k}-\mathbf{q}})] u_{\mathbf{k}}^2 \delta(E_{\mathbf{k}} + \epsilon_{\mathbf{k}-\mathbf{q}} - \Omega)$$
$$+ [f(E_{\mathbf{k}}) - f(\epsilon_{\mathbf{k}-\mathbf{q}})] v_{\mathbf{k}}^2 \delta(E_{\mathbf{k}} - \epsilon_{\mathbf{k}-\mathbf{q}} + \Omega)\} \varphi_{\mathbf{k}-\mathbf{q}/2}^2. \tag{D1}$$

For small $\Omega \ll T$, and setting $\mathbf{q}=0$ we can expand the Fermi functions to first order in Ω:

$$\Gamma_{0,\Omega} = -\frac{\pi}{a'_0} \Omega \sum_{\mathbf{k}} [f'(E_{\mathbf{k}}) u_{\mathbf{k}}^2 \delta(E_{\mathbf{k}} + \epsilon_{\mathbf{k}} - \Omega)$$
$$+ f'(E_{\mathbf{k}}) v_{\mathbf{k}}^2 \delta(E_{\mathbf{k}} - \epsilon_{\mathbf{k}} + \Omega)] \varphi_{\mathbf{k}}^2$$
$$= \frac{a''_0}{a'_0} \Omega. \tag{D2}$$

Here $\Gamma(0,\Omega)$ reflects the rate of decay of noncondensed bosons into a bare and a dressed fermion. The δ functions in the above expression determine the energies of these fermions $\epsilon_{\mathbf{k}}$ and $E_{\mathbf{k}}$ to be of order $(\Omega^2 + \Delta^2 \varphi_{\mathbf{k}}^2)/\Omega$. At temperatures of interest ($T < T_c$ and $T \gtrsim T_c$), away from the nodes (where $\varphi_{\mathbf{k}}=0$) and for $\Omega < \Delta$ this energy is large. The only appreciable contribution then comes from parts of the Fermi surface near the nodes (of angular width of the order of $\sqrt{\Omega T/\Delta^2}$), giving a higher power than linear in Ω dependence of $\Gamma(0,\Omega)$ and $a''_0 = 0$.

This result is improved upon by including the effects of fermion-boson scattering. Quite generally in the absence of impurity scattering one finds that

$$a''_0 = -\frac{1}{2} \sum_{\mathbf{k}} \varphi_{\mathbf{k}}^2 A(\mathbf{k}, -\epsilon_{\mathbf{k}}) f'(\epsilon_{\mathbf{k}}), \tag{D3}$$

where $A(\mathbf{k},E) = -2\,\mathrm{Im}G(\mathbf{k}, i\omega_n \to E + i0^+)$ is the fermion spectral function. Fermion-boson scattering broadens the spectral function and $A(\mathbf{k},-\epsilon_{\mathbf{k}})f'(\epsilon_{\mathbf{k}})$ is generally nonzero in the normal state for all $\mathbf{k}$, implying that $a''_0 \neq 0$. One way to implement this is to introduce an inverse lifetime γ into the expression for the pseudogap self-energy

$$\Sigma(K)=\frac{\Delta_{sc}^2\varphi_{\mathbf{k}}^2}{\omega+\epsilon_{\mathbf{k}}}+\frac{\Delta_{pg}^2\varphi_{\mathbf{k}}^2}{\omega+\epsilon_{\mathbf{k}}+i\gamma} \tag{D4}$$

leading to the following expression for the spectral function $A(\mathbf{k},\omega)$, above T_c:

$$A(\mathbf{k},\omega)=\frac{2\Delta_{pg}^2\varphi_{\mathbf{k}}^2\gamma}{(\omega^2-E_{\mathbf{k}}^2)^2+\gamma^2(\omega-\epsilon_{\mathbf{k}})^2} \tag{D5}$$

As is evident from Eq. (D5), the quasiparticle peaks are broadened as a result of a nonzero γ. Thus, above T_c the imaginary part of the inverse T matrix (or, equivalently, of the pair susceptibility) is now

$$\mathrm{Im}\chi(\mathbf{q},\Omega)=\Gamma(\mathbf{q},\Omega)=\gamma\Delta_{pg}^2\sum_{\mathbf{k}}\frac{1-f(\Omega-\epsilon_{\mathbf{k-q}})-f(\epsilon_{\mathbf{k-q}})}{[(\Omega-\epsilon_{\mathbf{k-q}})^2-E_{\mathbf{k}}^2]^2+\gamma^2(\Omega-\epsilon_{\mathbf{k-q}}-\epsilon_{\mathbf{k}})^2}\varphi_{\mathbf{k}}^2\varphi_{\mathbf{k-q}/2}^2. \tag{D6}$$

Since the quasiparticle peaks are broadened, the boson can now decay into states near the Fermi surface not only near the nodes but everywhere else on the Fermi surface. $\Gamma(0,\Omega)$ is now linear in Ω and the coefficient of proportionality is easily found to be

$$\frac{\partial}{\partial\Omega}\mathrm{Im}t^{-1}(0,\Omega)|_{\Omega=0}=-\frac{\gamma\Delta_{pg}^2}{T}\sum_{\mathbf{k}}\frac{f'(\epsilon_{\mathbf{k}})\varphi_{\mathbf{k}}^4}{\Delta_{pg}^4\varphi_{\mathbf{k}}^4+4\gamma^2\epsilon_{\mathbf{k}}^2}. \tag{D7}$$

The second term in the denominator of the summand in the above expression can be safely neglected if γ is taken to be much smaller than Δ_{pg}, since the main contribution to the sum comes from the vicinity of the Fermi surface (where $\epsilon_{\mathbf{k}}$ is small). The only place where this term is not negligible in comparison with $\Delta_{pg}^4\varphi_{\mathbf{k}}^4$ is in a small region near the nodes (angular width $\propto\sqrt{\gamma T/\Delta_{pg}}$). However, this correction can also be neglected, and therefore

$$a_0''=\frac{\gamma}{4T\Delta_{pg}^2}\sum_{\mathbf{k}}\frac{1}{\cosh^2\dfrac{\epsilon_{\mathbf{k}}}{2T}}, \tag{D8}$$

which, assuming that $T\ll\mu$, after a standard procedure gives, as expected, a nonzero result

$$a_0''=\frac{N(0)\gamma}{\Delta_{pg}^2}. \tag{D9}$$

Below T_c however $A(\mathbf{k},-\epsilon_{\mathbf{k}})$ in Eq. (D3) vanishes due to the sharpness of the superconducting self-energy, as is easily checked by inspecting the expression for the spectral function below T_c following from the self-energy Eq. (D4),

$$A(\mathbf{k},\omega)=\frac{2\Delta_{pg}^2\varphi_{\mathbf{k}}^2\gamma(\omega+\epsilon_{\mathbf{k}})^2}{(\omega+\epsilon_{\mathbf{k}})^2(\omega^2-E_{\mathbf{k}}^2)^2+\gamma^2(\omega^2-\epsilon_{\mathbf{k}}^2-\Delta_{sc}^2\varphi_{\mathbf{k}}^2)^2}. \tag{D10}$$

Thus we can infer that $a_0''=0$. Nevertheless, a small amount of impurity scattering will restore a nonvanishing a_0'', since it is likely to produce a nonzero $A(\mathbf{k},-\epsilon_{\mathbf{k}})$ in Eq. (D3). In the d-wave case, the impurity renormalized imaginary part of the pair susceptibility has the following form:[25]

$$\chi''(\mathbf{q},\Omega+i0^+)=-\sum_{\mathbf{k}}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathrm{Im}\,G^R(\omega,\mathbf{k})A_0(\Omega-\omega,\mathbf{q}-\mathbf{k})\times[f(\Omega-\omega)-f(\omega)]\varphi_{\mathbf{k-q}/2}^2, \tag{D11}$$

where both $\mathrm{Im}G^R$ (the spectral function of the full Green's function G) and A_0 (the spectral function of the bare Green's function G_0) are dressed with the impurity self-energy. To see the order of magnitude of this effect, we make a crude approximation of this formula by assuming that only the full G is dressed by impurities but not G_0 which is equivalent to the statement that expression (D3) with renormalized A is valid. We model the impurity self-energy in the Born limit by $\Sigma_{imp}(\omega)=-is|\omega|$ (as is reasonable close to the Fermi surface in the d-wave case), where s is a dimensionless constant parametrizing the concentration of scatterers. Assuming $s\ll1$ (clean limit), we obtain

$$A(\mathbf{k},-\epsilon_{\mathbf{k}})=-2\frac{s|\epsilon_{\mathbf{k}}|\Delta^2\varphi_{\mathbf{k}}^2}{\Delta^4\varphi_{\mathbf{k}}^4+4s^2\epsilon_{\mathbf{k}}^4}, \tag{D12}$$

which, in conjunction with Eq. (D3) and identifying $-s|\epsilon_{\mathbf{k}}|$ with γ yields an expression identical to Eq. (D7). Using reasoning similar to that leading to Eq. (D9), we arrive at the following estimate for a_0'' in the presence of impurities:

$$a_0''=2\ln2\,N(0)\frac{sT}{\Delta^2}. \tag{D13}$$

More complete numerical results are shown in Fig. 4.

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