

计算机辅助几何设计 2023秋学期

Bézier Curves (continue)

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Matrix representations

(common in software implementations)

Homogeneous coordinates

$$[P] = \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ \dots & \dots & \dots & \dots \\ x_n & y_n & z_n & 1 \end{pmatrix}$$

Transformations

- Basic representation $[P^*] = [P][T]$

$[P^*]$ is the new coordinates matrix

$[P]$ is the original coordinates matrix, or points matrix

$[T]$ is the transformation matrix

Transformations

- Translation (2D example)

$$[T_t] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

$$[P^*] = [P][T_t]$$

Transformations

- Basic representation $[P^*] = [P][T]$

$[P^*]$ is the new coordinates matrix

$[P]$ is the original coordinates matrix, or points matrix

$[T]$ is the transformation matrix

$$[P] = \begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \\ \dots & \dots & \dots \\ x_n & y_n & 0 \end{pmatrix}$$

Transformations

- Uniform scaling

$$[T] = \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Non-uniform scaling

$$[T] = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation 2D

$$\left. \begin{aligned} x &= r \cos \alpha \\ y &= r \sin \alpha \end{aligned} \right\} \text{Original coordinates of point } P$$

$$\left. \begin{aligned} x^* &= r \cos(\alpha + \theta) \\ y^* &= r \sin(\alpha + \theta) \end{aligned} \right\} \text{The new coordinates}$$

$$[x^* \quad y^* \quad 0 \quad 1] = [x \quad y \quad 0 \quad 1] \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation about an arbitrary axis (2D)
 - Translate the fixed axis so it coincides with z-axis
 - apply to object
 - Rotate object about the axis
 - Translate object back

$$[P^*] = [P][T_t][T_r][T_{-t}]$$

Transformations

- Rotation about an arbitrary axis (2D)

Step 1: Translate the fixed axis so it coincides with z-axis

Step 2: Rotate object about the axis

Step 3: Translate the fixed axis back to the original position

$$[P^*] = [P][T_t][T_r][T_{-t}]$$

Transformations

- Scaling with an arbitrary point (x, y)

$$[P^*] = [P][T_t][T_s][T_{-t}]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ x - sx & y - sy & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation about an arbitrary point (x, y)

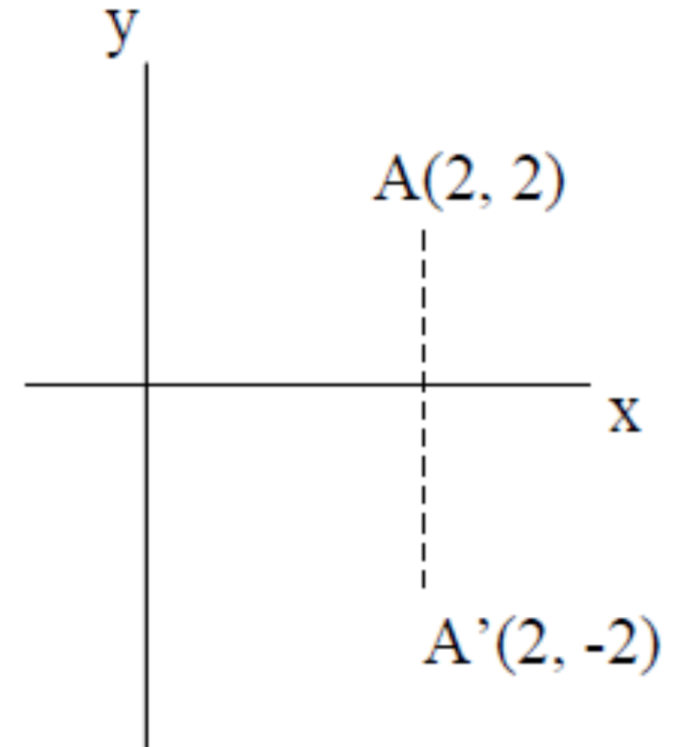
$$[T_{\text{cond}}] = [T_t][T_s][T_{-t}]$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -x & -y & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ x & y & 0 & 1 \end{pmatrix}$$

Transformations

- Mirroring about x-axis (negative scaling along y-axis)

$$[P^*] = [2 \quad 2 \quad 0 \quad 1] \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= [2 \quad -2 \quad 0 \quad 1]$$



Transformations

- Mirroring about arbitrary axis
 - Translate line to pass through origin
 - Rotate axis to coincide with x -axis
 - Mirror about x -axis
 - Rotate back
 - Translate back to original position

$$[P^*] = [P][T_t][T_r][T_m][T_{-r}][T_{-t}]$$

Transformations

- Rotation about coordinates axes (3D)

$$[T_{rz}] = \begin{pmatrix} \cos \theta & \sin \theta & 0 & 0 \\ -\sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[T_{rx}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[T_{ry}] = \begin{pmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Transformations

- Rotation θ about an arbitrary axis (3D)
 1. Translate the given line so that it will pass through the origin
 2. Rotate about the x -axis so that the line lies in the xz -plane (angle α)
 3. Rotate about the y -axis so that the line coincides with the z -axis (angle ϕ)
 4. Rotate the geometric object about the z -axis (angle θ – given rotation angle)
 5. Reverse of step 3
 6. Reverse of step 2
 7. Reverse of step 1

$$[P^*] = [P][T_t][T_r]_\alpha[T_r]_\phi[T_r]_\theta[T_r]_{-\phi}[T_r]_{-\alpha}[T_{-t}]$$

Alternatively you can use Quaternions!

Bézier Curves

- Cubic Bézier curves $f(t) = P_0B_0^{(3)} + P_1B_1^{(3)} + P_2B_2^{(3)} + P_3B_3^{(3)}$

$$B_0^{(3)}(t) = \frac{3!}{0!3!}t^0(1-t)^3 = (1-t)^3$$

$$B_1^{(3)} = \frac{3!}{1!2!}t^1(1-t)^2 = 3t(1-t)^2$$

$$B_2^{(3)} = \frac{3!}{2!1!}t^2(1-t)^1 = 3t^2(1-t)$$

$$B_3^{(3)} = \frac{3!}{3!0!}t^3(1-t)^0 = t^3$$

Bézier Curves

- Cubic Bézier curves

$$f(t) = P_0 B_0^{(3)} + P_1 B_1^{(3)} + P_2 B_2^{(3)} + P_3 B_3^{(3)}$$

- In Matrix form:

- -The curve

$$f(t) = [t^3 \quad t^2 \quad t \quad 1] \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

- -The tangent

$$f'(t) = [3t^2 \quad 2t \quad 1 \quad 0] \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

B-spline Curves (to be covered later)

- Uniform cubic B-Spline curve

$$f_i(t) = \frac{1}{6} [t^3 \quad t^2 \quad t \quad 1] \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix} \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$$

B-spline Curves (to be covered later)

- Other splines:

- Catmull-Rom

$$f_i(t) = [t^3 \quad t^2 \quad t \quad 1] \frac{1}{2} \begin{pmatrix} -1 & 3 & -3 & 1 \\ 2 & -5 & 4 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{bmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{bmatrix}$$

- Cardinal splines

$$\begin{pmatrix} -a & 2 - a & a - 2 & a \\ 2a & a - 3 & 3 - 2a & -a \\ -a & 0 & a & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Tensioned splines

$$\frac{1}{6} \begin{pmatrix} -a & 12 - 9a & 9a - 12 & a \\ 2a & a - 3 & 18 - 15a & -a \\ -3a & 0 & 3a & 0 \\ 0 & 6 - 2a & a & 0 \end{pmatrix}$$