计算机辅助几何设计
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Differential Geometry of Curves

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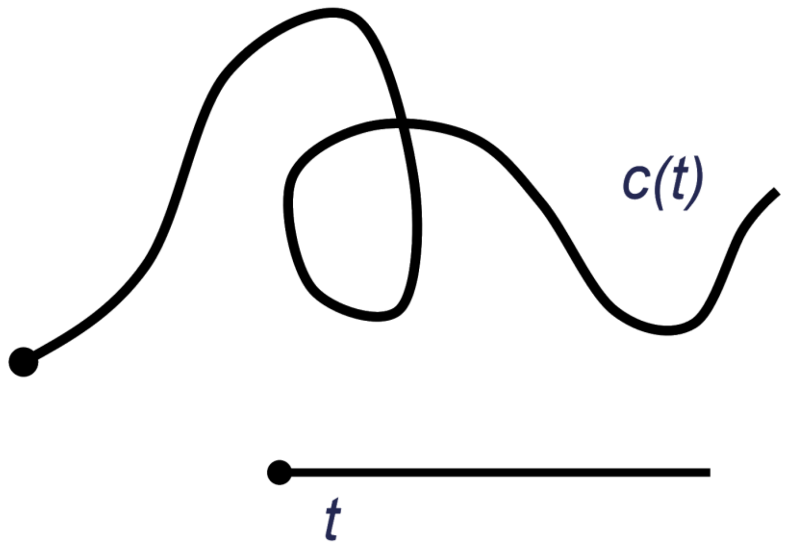
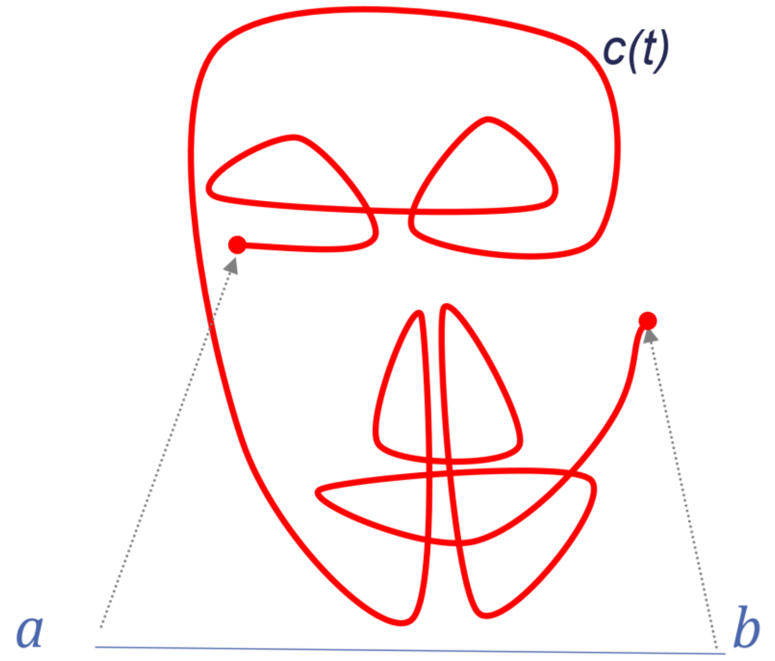
Parametric Curves

- **Parametric Curves:**
  - Think of a curve c as the path of a moving particle
  - Not always enough to know where a particle went – we also want to know when it got there $\rightarrow c(t)$
  - Parameter t is often thought of as time



Parametric Curves

- **Parametric Curves:**
  - A *parameterization of class* $C^k (k \geq 1)$ of a curve in $\mathbb{R}^n$ is a smooth map $c: I = [a, b] \subset \mathbb{R} \mapsto \mathbb{R}^n$, where c is of class C^k

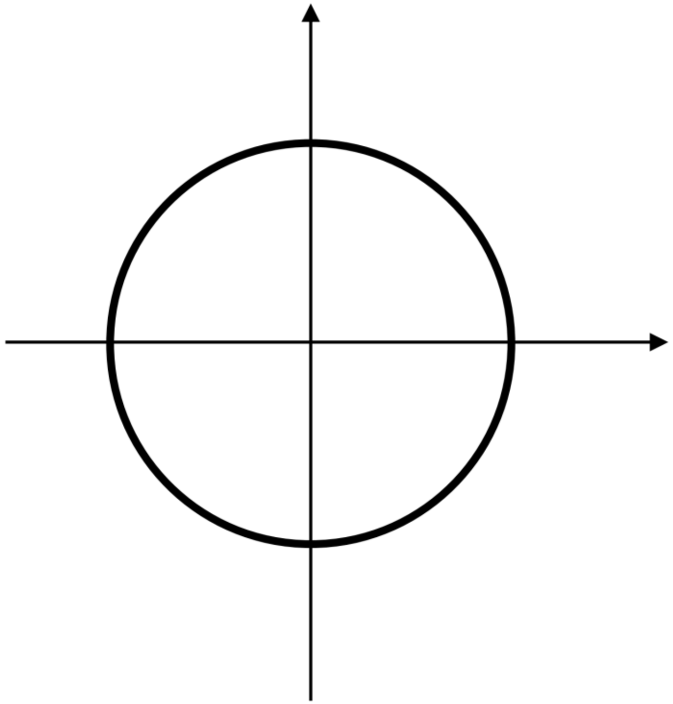


Parametric Curves

- **Parametric Curves:**
  - The image set $c(I)$ is called the ***trace*** of the curve
    - Different parameterizations can have the same trace.
  - A point in the trace, which corresponds to more than one parameter value t, is called ***self-intersection*** of the curve

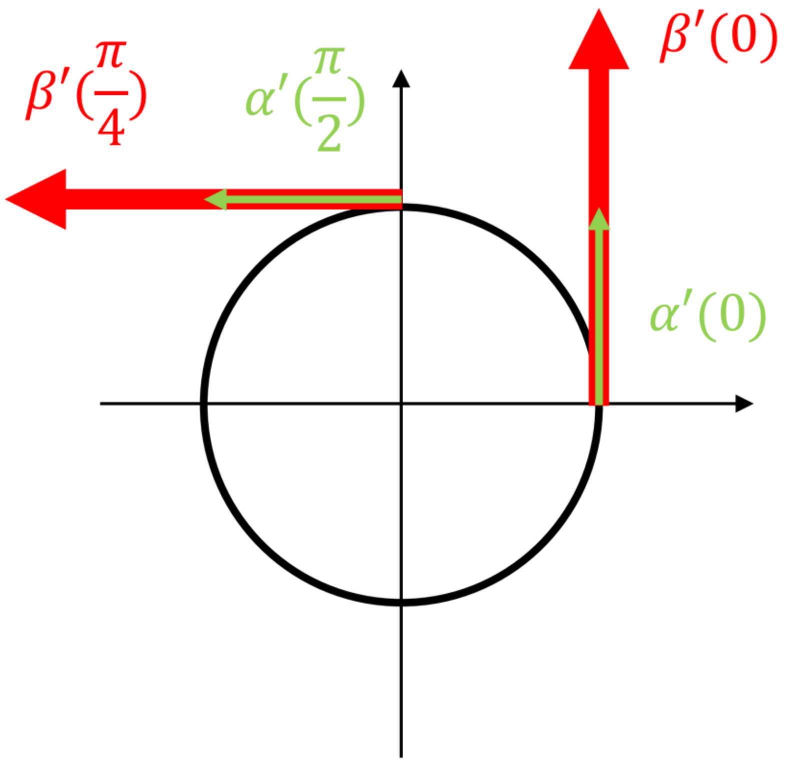
Parametric Curves: Examples

- The *positive* x-axis
  - $c(t) = (t, 0), t \in (0, \infty)$
  - $c(t) = (e^t, 0), t \in \mathbb{R}$
- **Circle**
  - $c(t) = (\cos t, \sin t), \quad t \in [0, 2\pi]$
  - $c(t) = (\cos 2t, \sin 2t), \quad t \in [0, \pi]$
  - $c(t) = (\cos t, \sin t), \quad t \in \mathbb{R}$



The velocity vector

- The derivative $c'(t)$ is called the velocity vector to the curve c at time t
  - $c'(t)$ gives the direction of the movement
  - $|c'(t)|$ gives the speed
- Example
  - $\alpha(t) = (\cos t, \sin t), \qquad t \in [0, 2\pi]$
  - $\beta(t) = (\cos 2t, \sin 2t), \qquad t \in [0, \pi]$



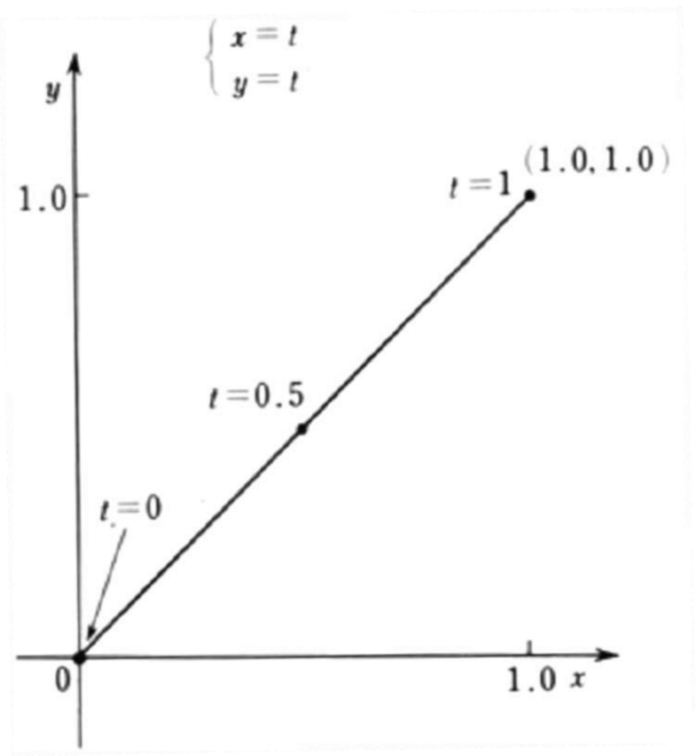
Regular parametric curves

- **Regular parametrization**
  - A parameterization is called *regular* if $c'(t) \neq 0$ for all t
  - A point at which a curve is regular is called an *ordinary* point
  - A point at which a curve is non-regular is called an *singular* point

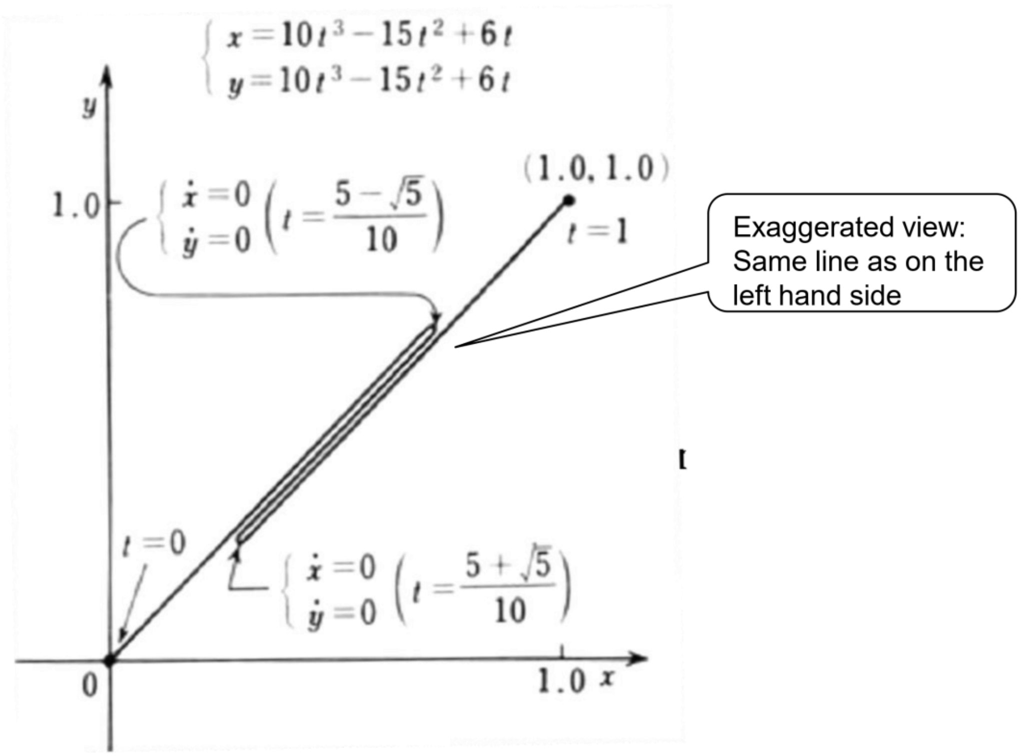
Examples: regularity

- **Examples: issues with non-regular parameterization**

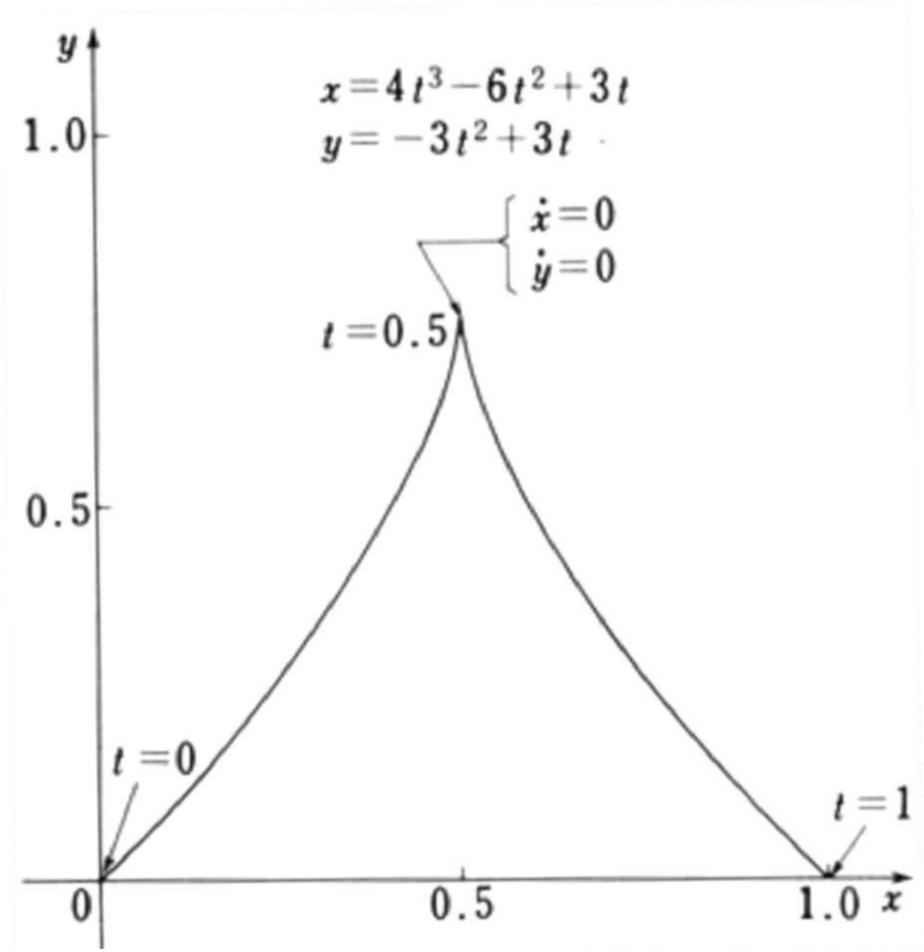
Regular parametrization



Non-regular parametrization

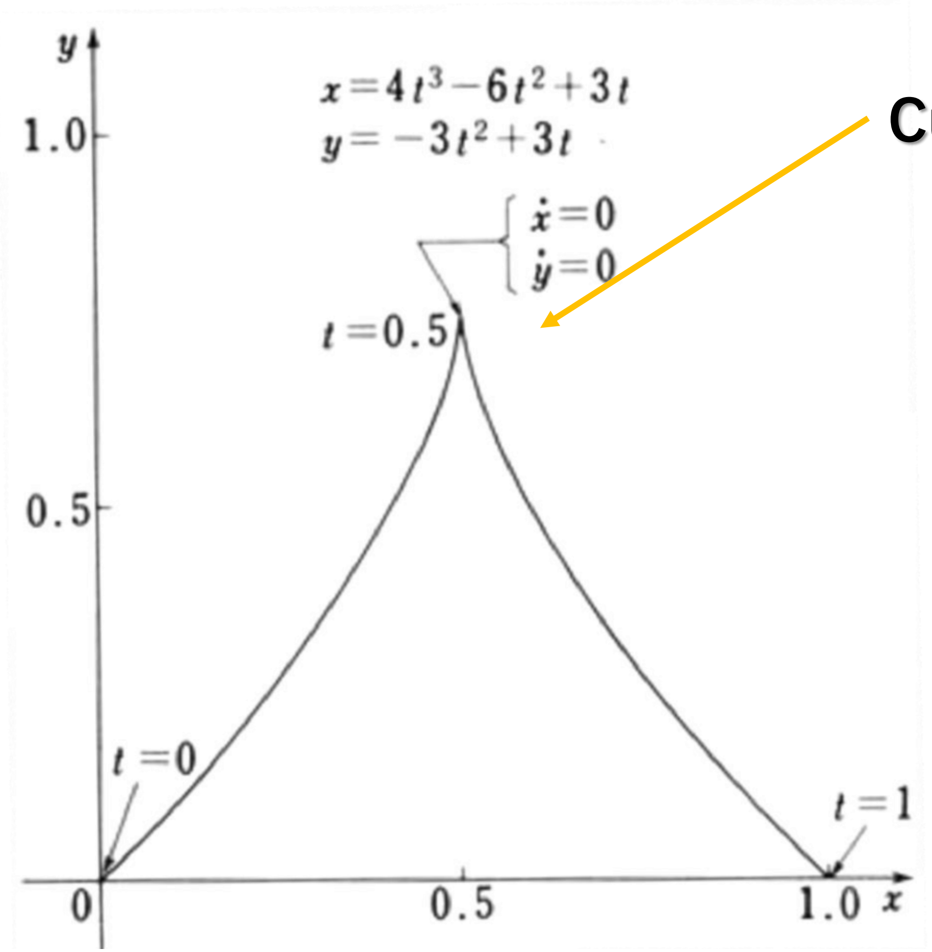


Examples: cusps



Singularities can be desired design features

Examples: cusps



Cusp

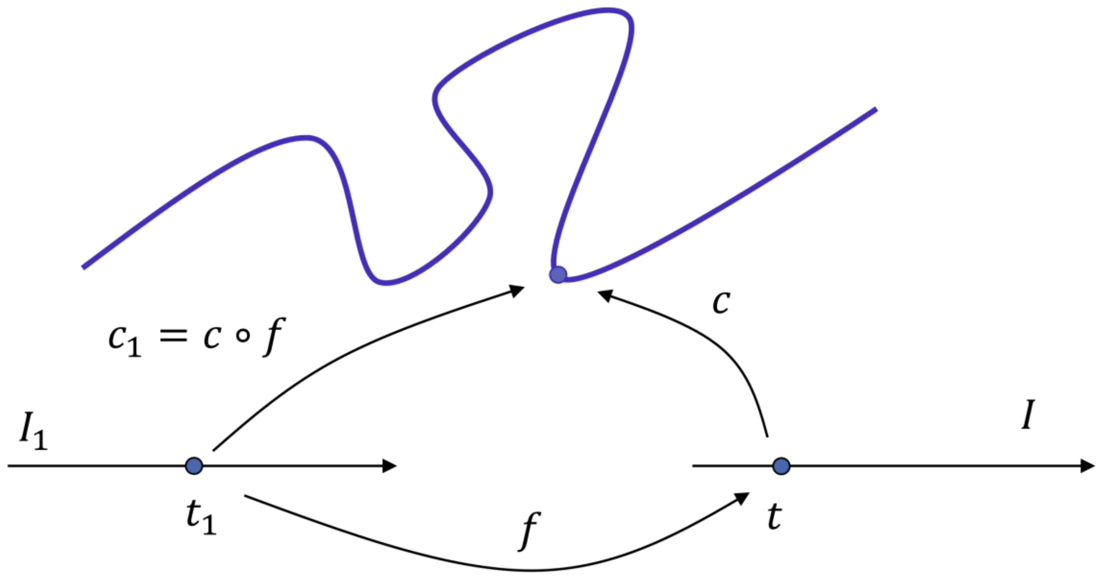
Singularities can be desired design features

Change of parameterization

- **Given a smooth regular parametrization, an allowable change of parameter is any real smooth (differentiable) function**

$$f: I_1 \to I \text{ such that } f' \neq 0 \text{ on } I_1$$

  - It is orientation preserving when $f' > 0$

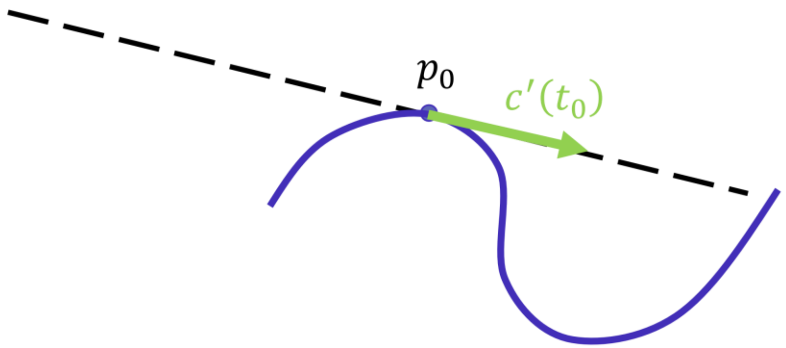


Change of parameterization

- **Parameter Transformations:**
  - We can regard a *regular curve* as a collection of regular parameterizations, any two of which are reparameterizations of each other (equivalence class)
  - We are interested in properties that are invariant under parameter transformations

Geometric observations

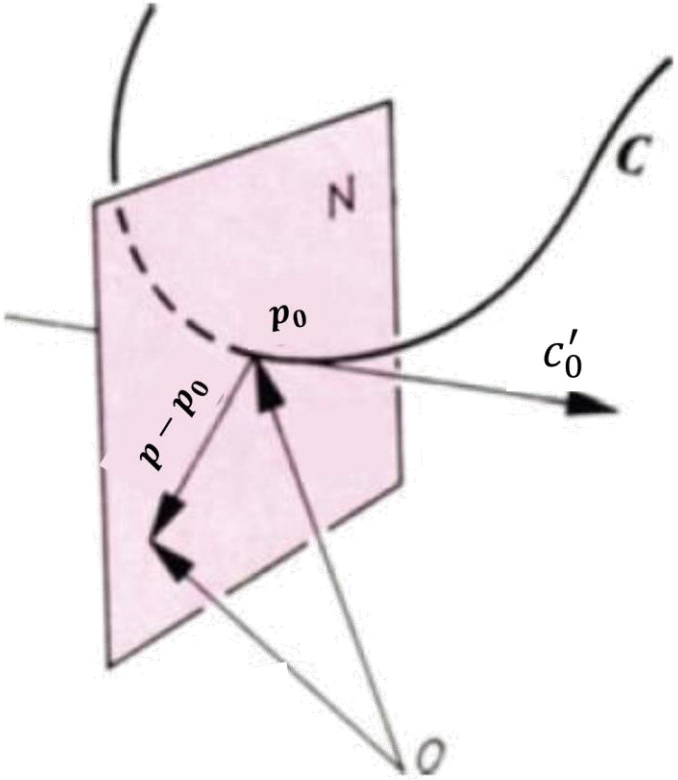
- **Tangent vector:**
  - The tangent line to a regular curve $c(t)$ at $p_0 = c(t_0)$ can be defined as points p which satisfy $p - p_0 \parallel c_0'$, where $c_0' = c'(t_0)$
  - The normalized vector $t = \frac{c'}{|c'|}$ is called the tangent vector



Geometric observations

- **The normal plane:**
  - The normal plane can be obtained as points p whose coordinates satisfy $p - p_0 \perp c_0'$

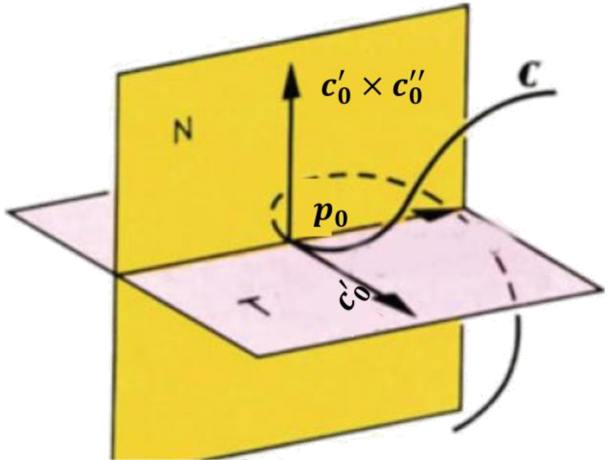
    $\Leftrightarrow (p - p_0) \cdot c_0' = 0$



Geometric observations

- **Osculating plane: 密切平面**
  - Assume the curve $c(t)$ is not a straight line. Any three arbitrary non-collinear points p_1, p_2, p_3 determine a plane
  - If p_1, p_2, p_3 tend to the same points p_0 of c, then their plane converges to a plane called the osculating plane T of c at p_0
  - The osculating plane is well defined if the first two derivatives c_0' and c_0'' at p_0 are linearly independent and is give as:

$$(c_0' \times c_0'') \cdot (p - p_0) = 0$$



Geometric observations

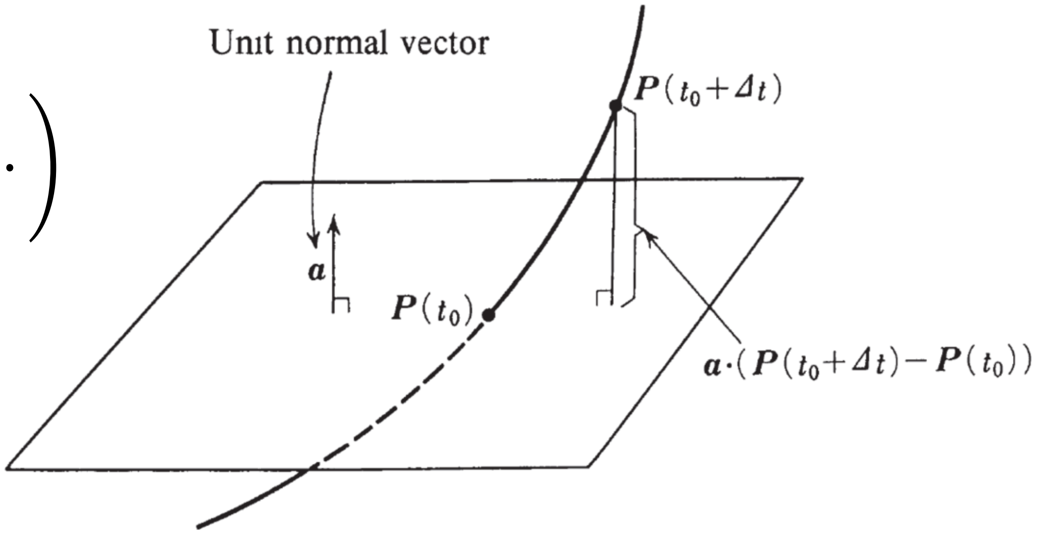
Observe the distance between $P(t_0 + \Delta t)$ and a given plane passing through $P(t_0)$ with normal vector a

$$a \cdot \big(P(t_0 + \Delta t) - P(t_0)\big) = a \cdot \left(\dot{P}(t_0)\Delta t + \frac{\ddot{P}(t_0)}{2!}\Delta t^2 + \cdots\right)$$

The distance is minimal when

$$a \cdot \dot{P}(t_0) = 0,\ a \cdot \ddot{P}(t_0) = 0$$

That is when the plane is osculating

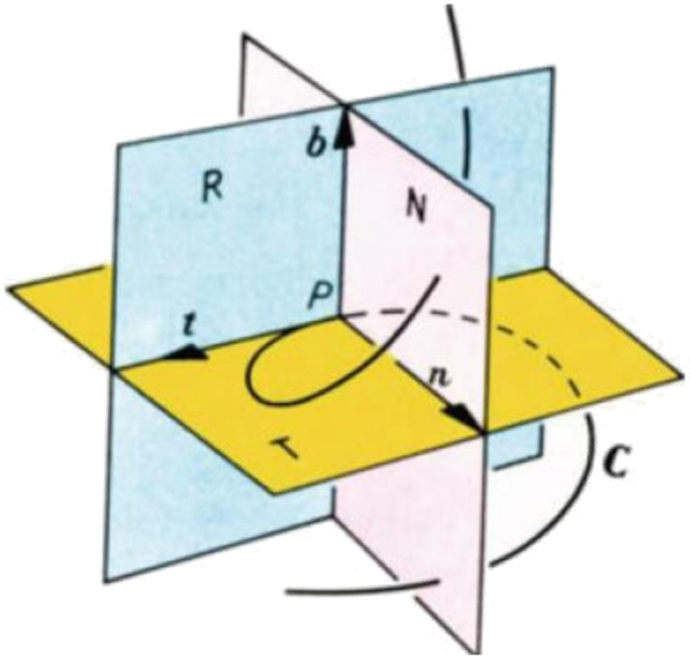


→ The osculating plane is the plane that best fits the curve at $P(t_0)$

Geometric observations

- **The rectifying plane: 从切平面**
  - The plane normal to both, the osculating plane and the normal plane, is called the rectifying plane R and can be obtained as points p whose coordinates satisfy

$$\left(c_0' \times (c_0' \times c_0'')\right) \cdot (p - p_0) = 0$$



Geometric observations

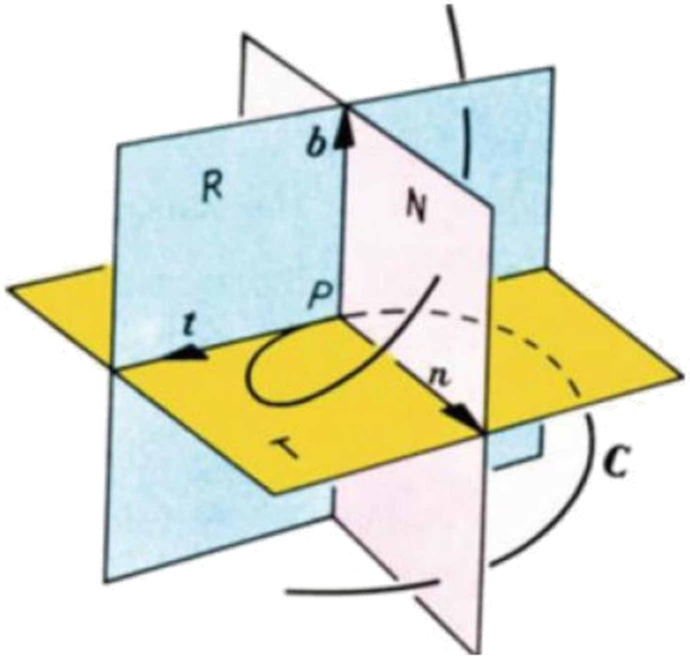
Normals: any vector in the normal plane is normal to the curve, in particular:

- The normal n lying in the osculating plane is called the principal normal at p_0.

  It has a direction $(c_0' \times c_0'') \times c_0'$

- The normal b lying in the rectifying plane is called the binormal. 副法向

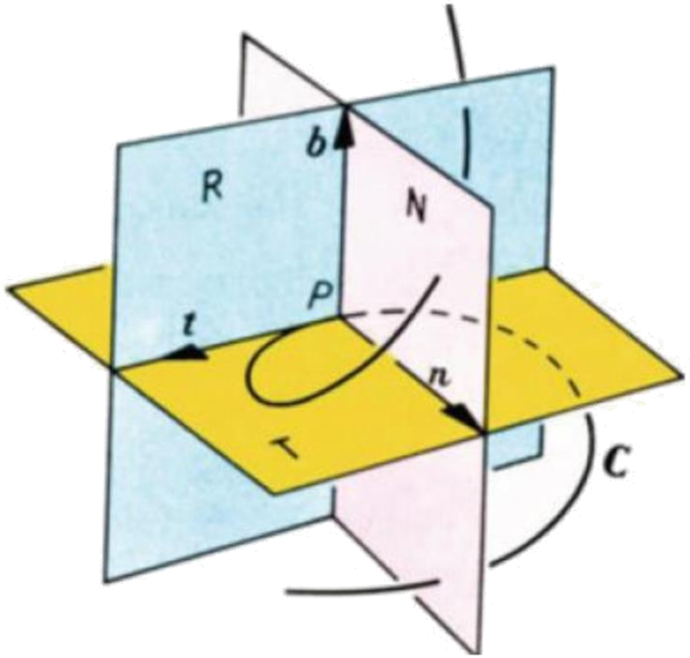
  It has a direction $c_0' \times c_0''$



The Frenet frame

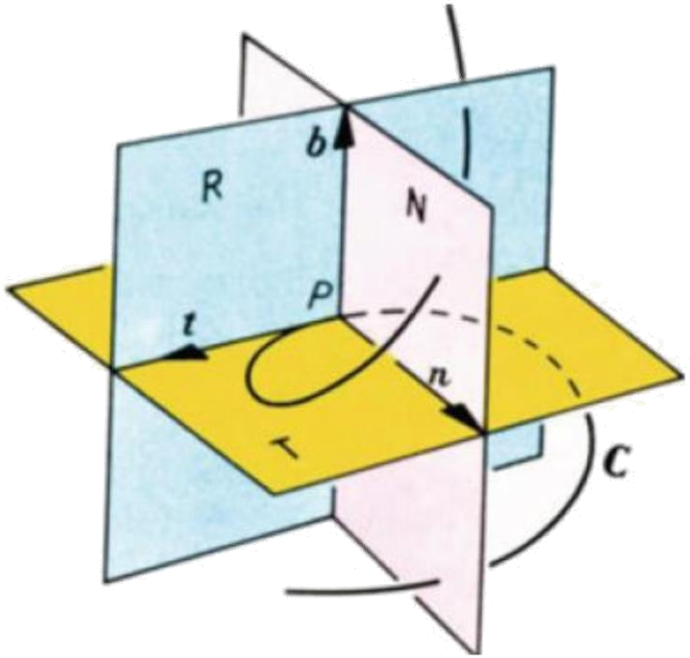
We can define a local coordinates system on the curve by three vectors

- The tangent $t = \frac{c'}{\|c_0'\|}$
- The binormal $b = \frac{c_0' \times c_0''}{\|c_0' \times c_0''\|}$
- The principal normal $n = b \times t$



The Frenet frame and associated planes

- The tangent $t = \frac{c'}{\|c_0'\|}$
  - the normal plane $(p - p_0) \cdot t = 0$
- The binormal $b = \frac{c_0' \times c_0''}{\|c_0' \times c_0''\|}$
  - the osculating plane $(p - p_0) \cdot b = 0$
- The principal normal $n = b \times t$
  - the rectifying plane $(p - p_0) \cdot n = 0$



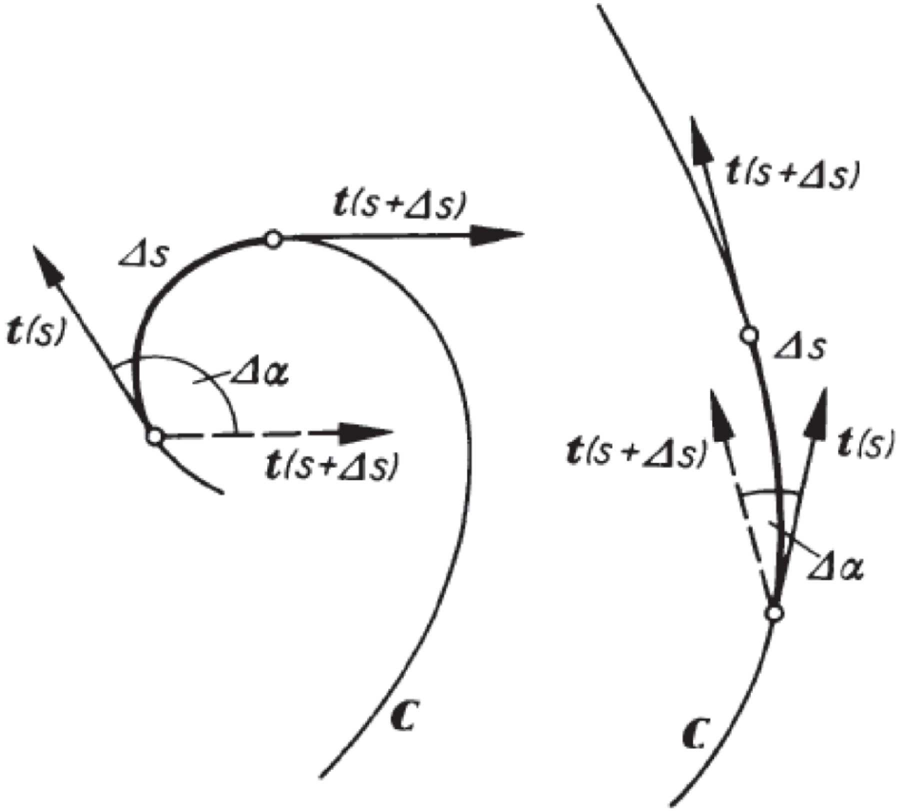
Curvature

- Common conceptions of curvature
  - Measures bending of a curve
  - A straight line does not bend → 0 curvature
  - A circle has constant bending → constant curvature

Curvature

Euler's heuristic approach for planar curves

- Variation of the tangent angle: how much does the curve differ from a straight line



Curvature for regular parameterization

The curvature is denoted by κ and defined as

$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$

Examples:

- Consider the circle $c(t) = (r\cos t\,, r\sin t\,, 0)$

  The ***curvature*** is given by

$$\kappa(t) = \frac{\|(-r\sin t, r\cos t, 0)\times(-r\cos t, -r\sin t, 0)\|}{r^3} = \frac{\|(0, 0, r^2)\|}{r^3} = \frac{1}{r}$$

- Consider the helix $c(t) = (r\cos t\,, r\sin t\,,\ at)$, the ***curvature*** is

$$\kappa(t) = \frac{r}{r^2 + a^2}$$

$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$

Special case: planar curves

- For a regular planar curve $c(t) = \big(x(t), y(t)\big)$

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$

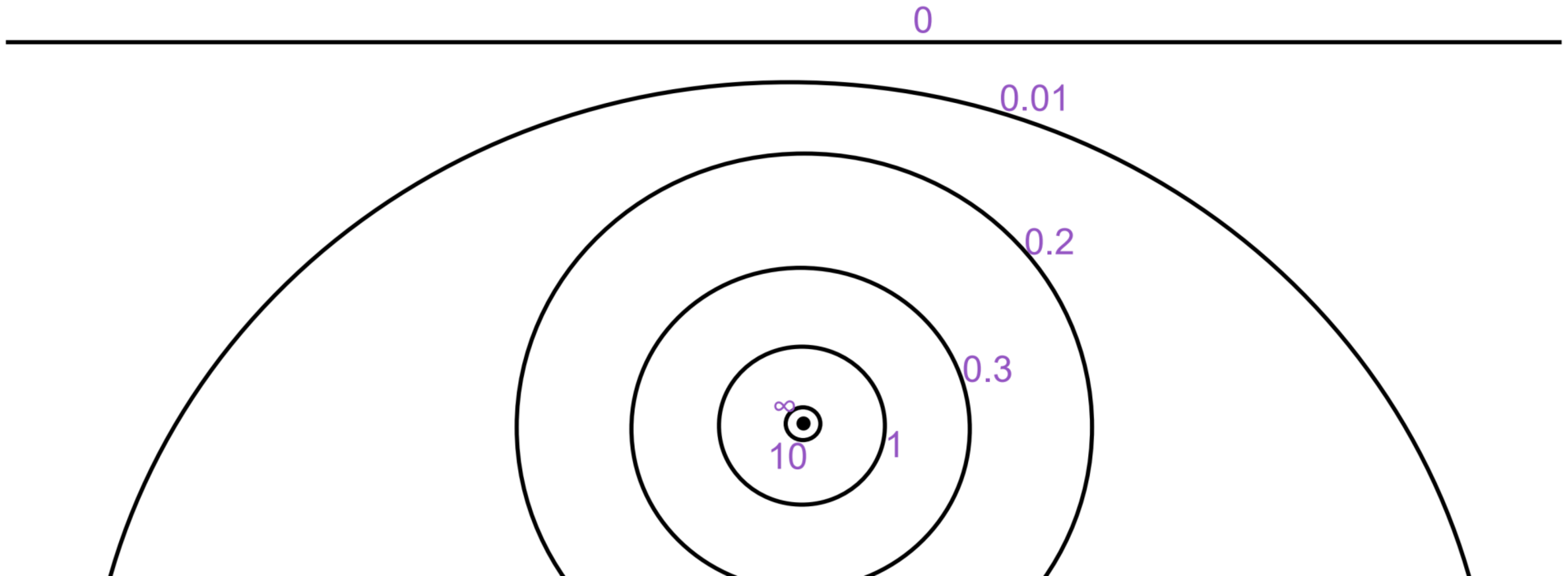
- Sometimes we talk about signed curvature, and then curvature can be allowed to be signed (negative, zero, or positive)

$$\kappa(t) = \frac{x'y'' - x''y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$

Examples

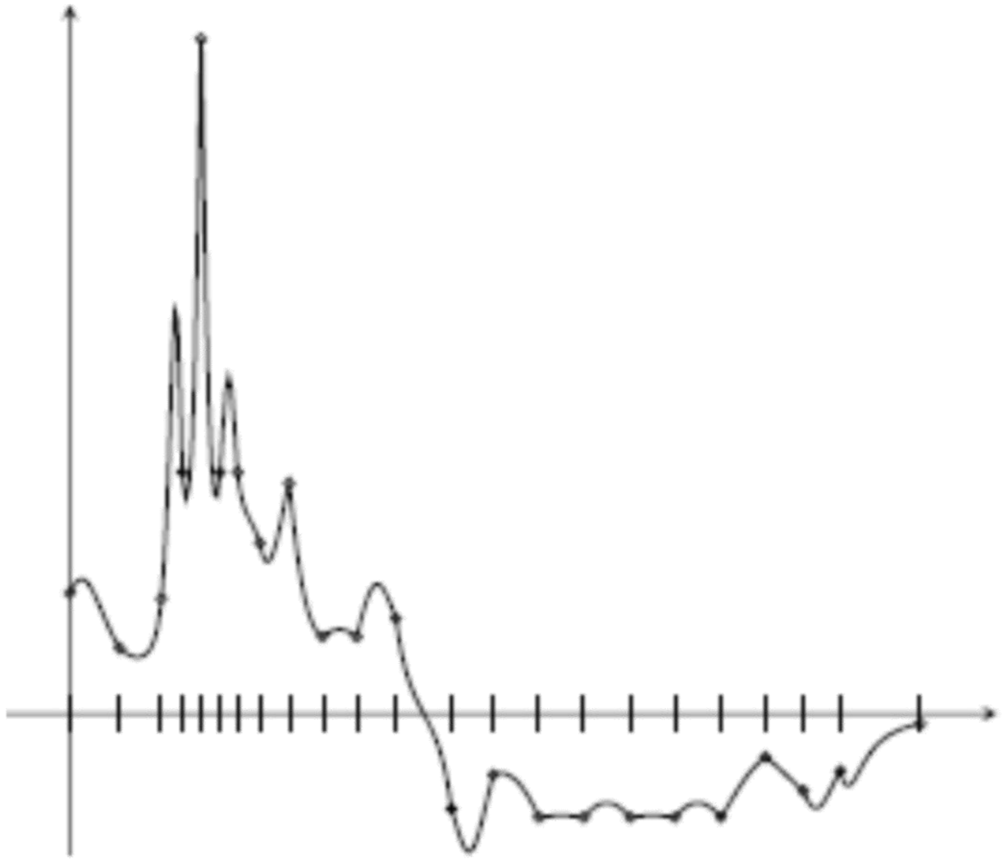
Curvature of circles

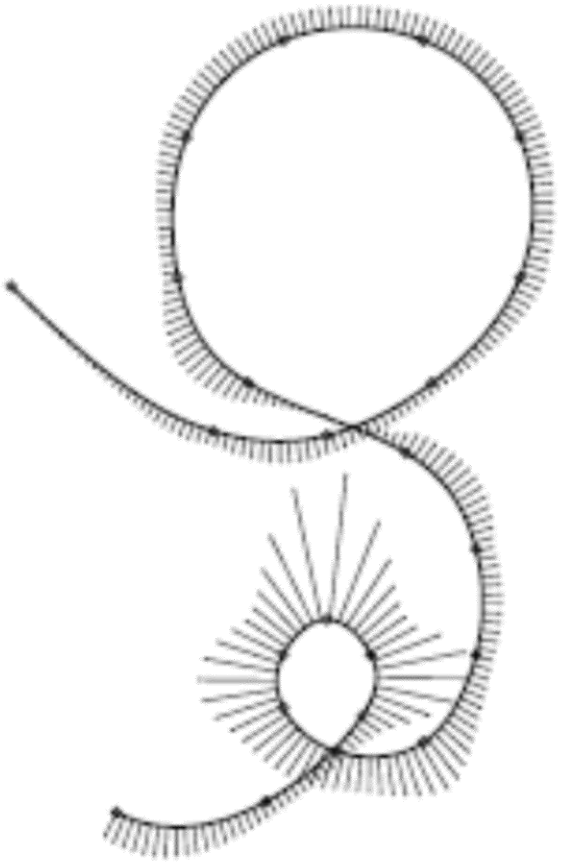
- Curvature of a circle is constant, $\kappa \equiv \frac{1}{r}$ (r = radius)
- Accordingly: define *radius of curvature* as $\frac{1}{\kappa}$



Curvature in practice

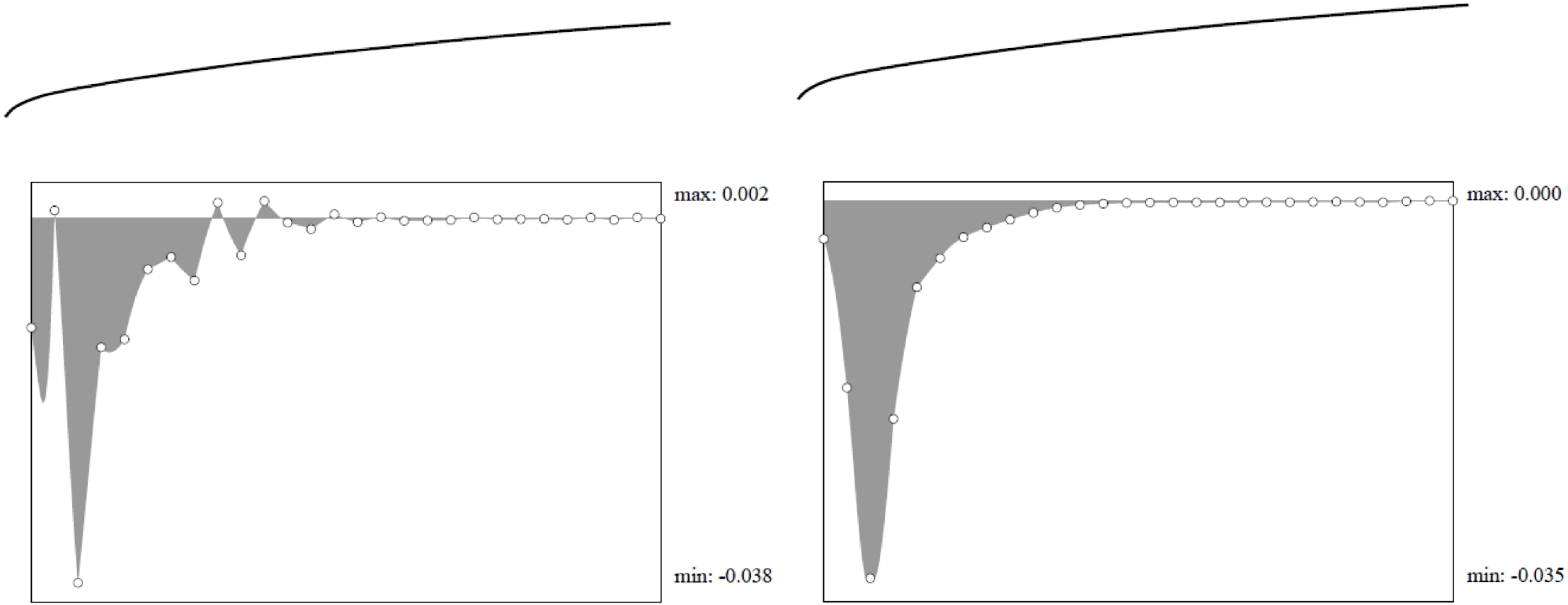
Most of commercial package allow inspecting the quality of the curvature



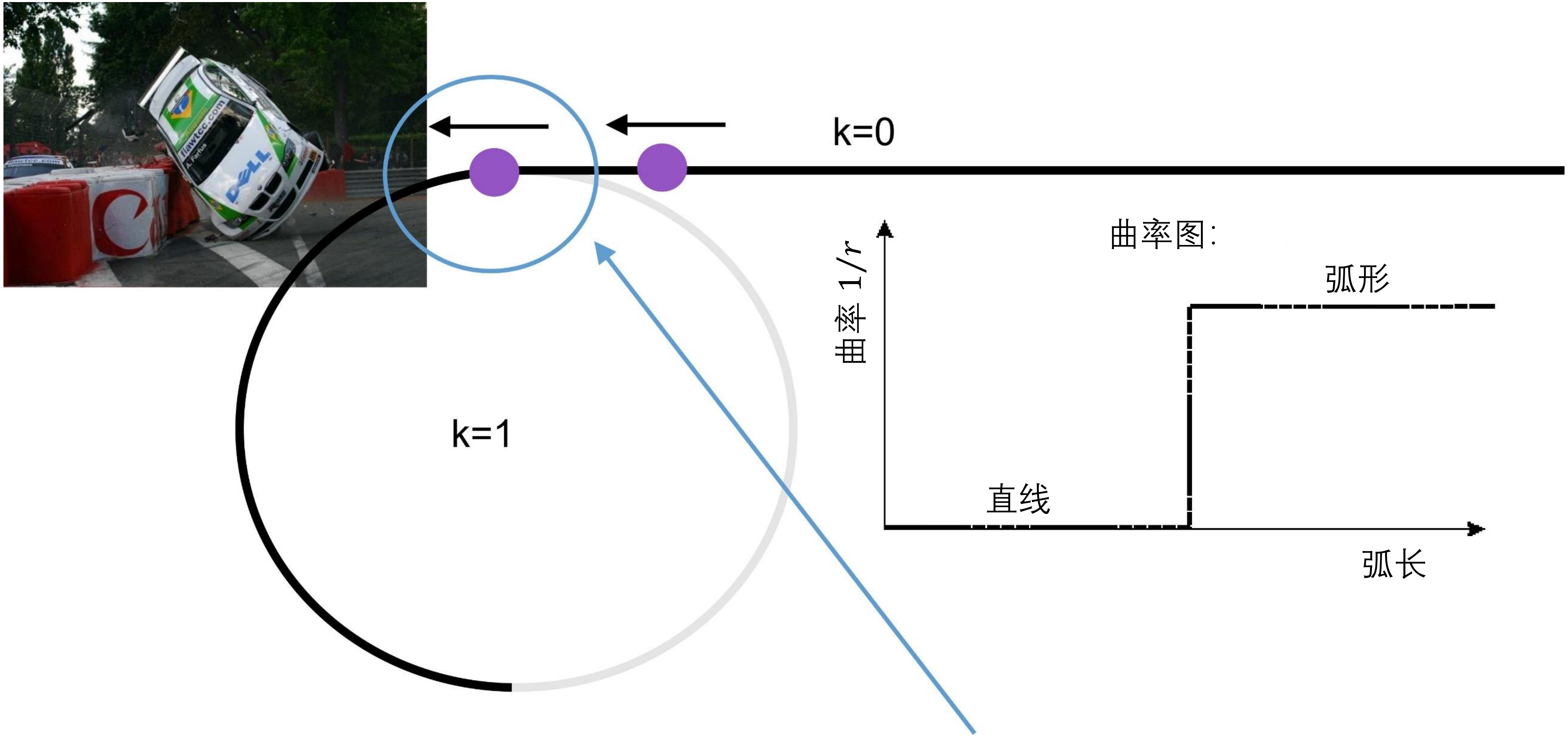


Curvature in practice

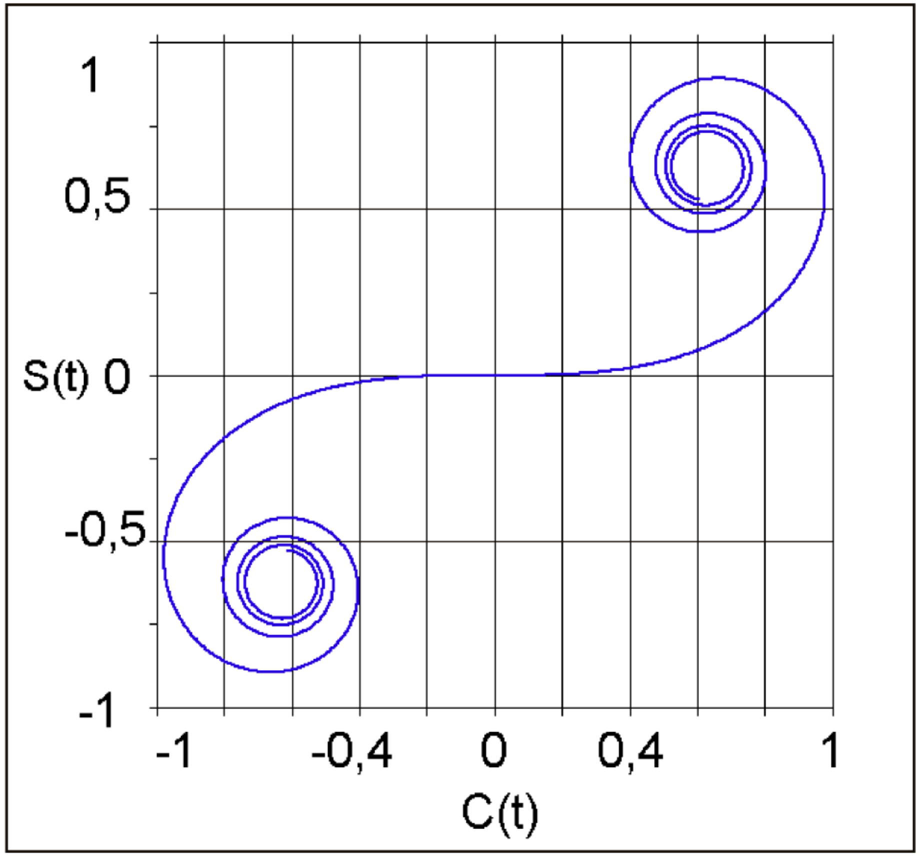
Most commercial package allow checking the quality of the curvature even meticulously!

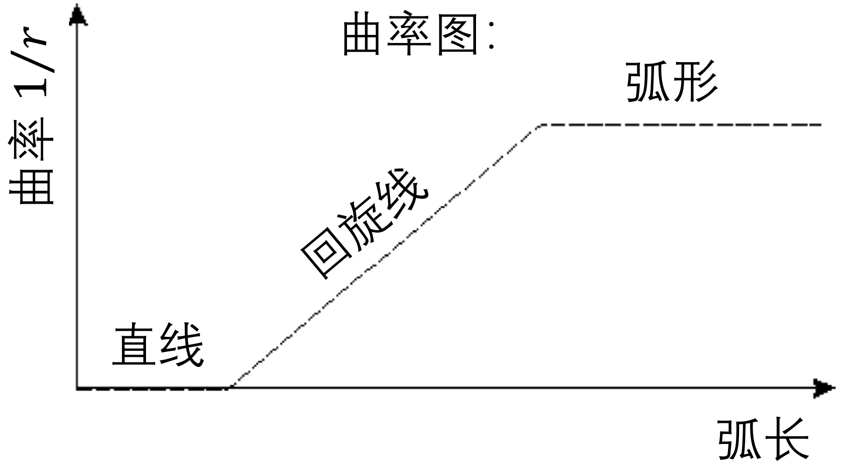


Curvature and Road Construction



Clothoide, Euler Spiral　羊角螺线





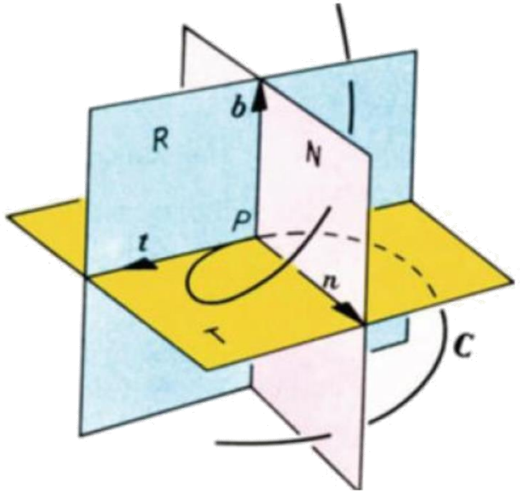
$$c(t) = \begin{pmatrix} \int_0^t \cos \frac{\pi}{2} u^2 \, du \\ \int_0^t \sin \frac{\pi}{2} u^2 \, du \end{pmatrix}$$

Torsion for regular parameterization

Definition

- The torsion τ measures the variation of the binormal vector
- (deviation of the curve from its projection on the osculating plane, can be regarded as how far is the curve is from being a planar curve) and is given by

$$\tau(t) = \frac{(c' \times c'') \cdot c'''}{\|c' \times c''\|^2}$$



Torsion

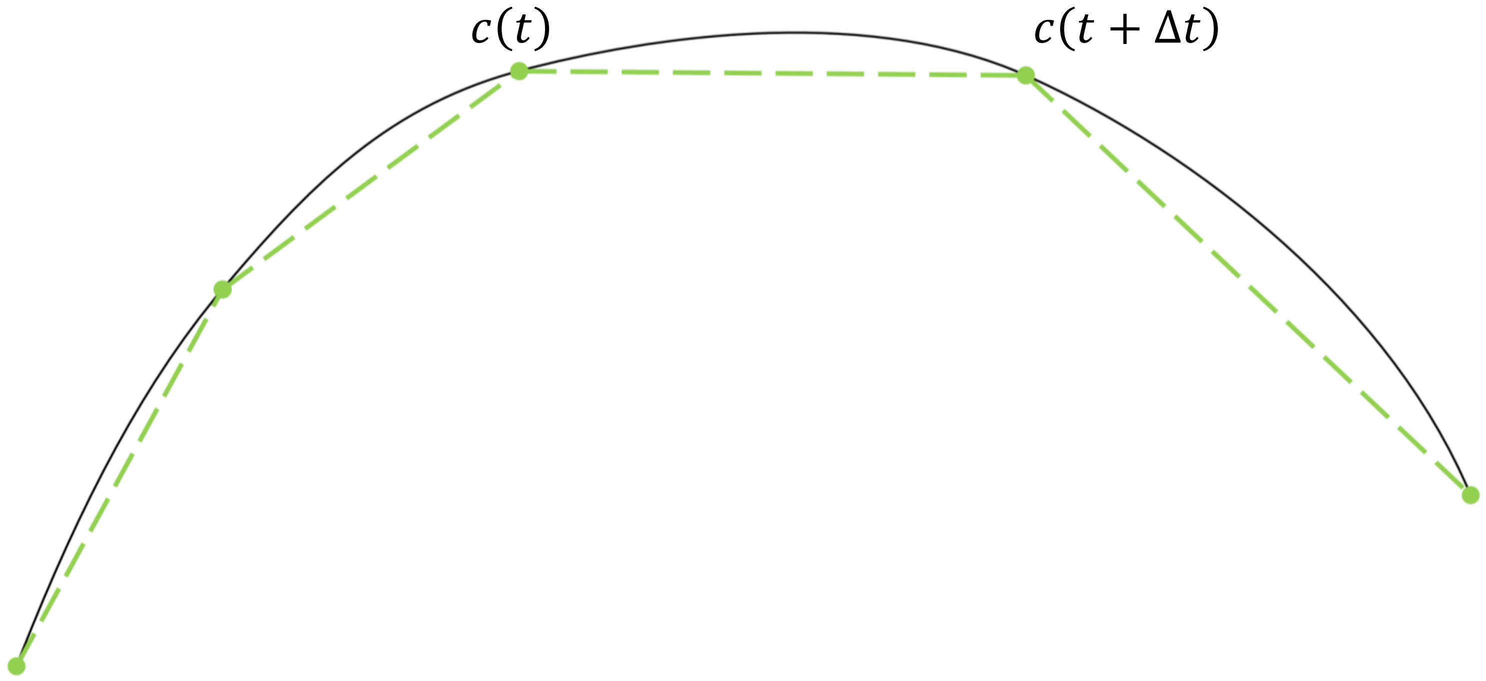
Examples:

- Torsion for a planar curve
- Torsion for a quadratic curve

Measuring lengths on curves

The arc length of a curve

- Can be regarded as the limit of the sum of infinitesimal segments along the curve



Measuring lengths on curves

The arc length of a curve

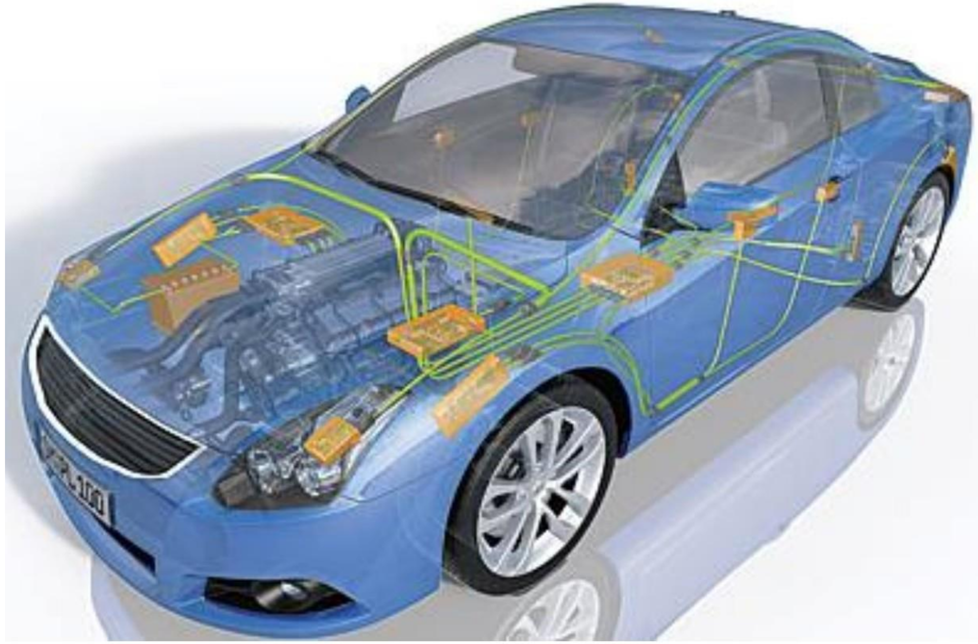
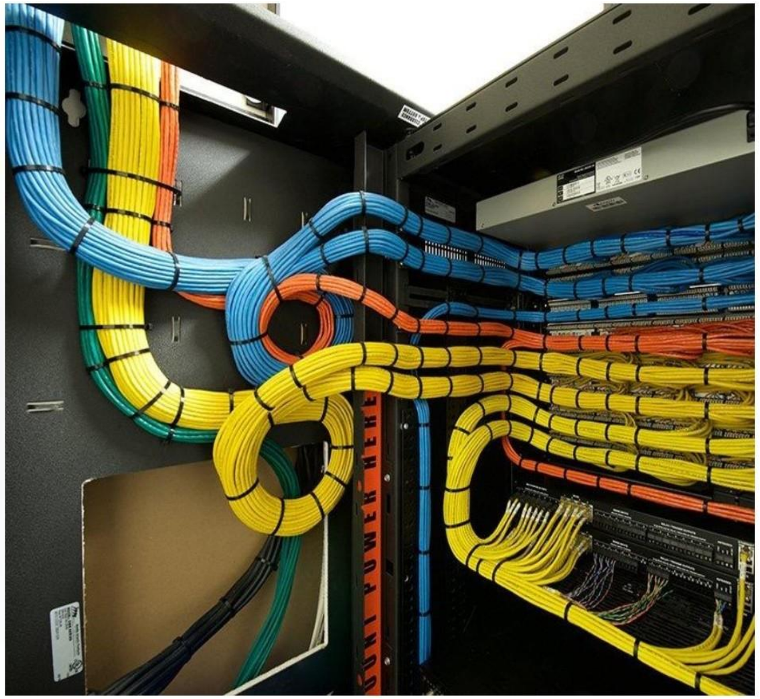
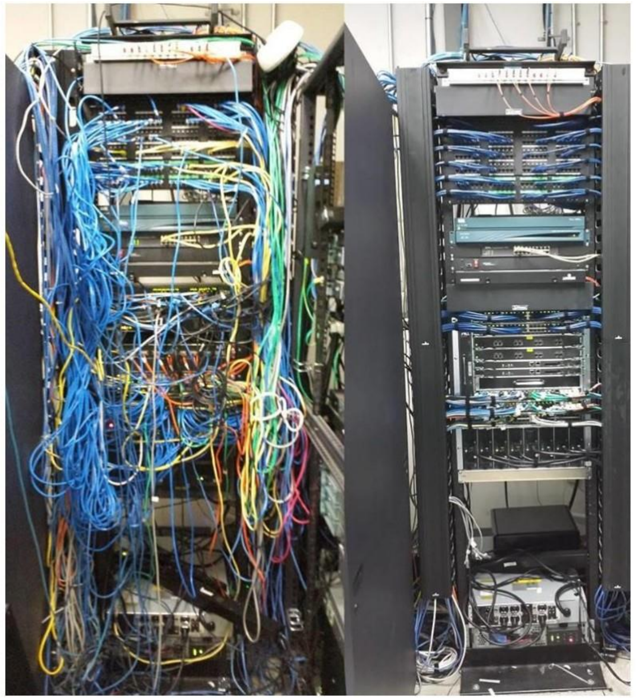
- The arc length of a regular curve C is defined as :

$$\mathrm{length}_{\mathrm{c}} = \int_a^b \|c'\| dt$$

- Independent of the parameterization (to prove this, use integration by substitution)

Measuring lengths on curves

Curve arc length matters in practice (e.g., cable routing problems)

Arc-length parametrized curves

Arc length parametrization

- Consider the portion of $c(t)$ spanned from 0 to t, the length s of this arc is a function of t:

$$s(t) = \int_0^t \|c'(u)\| du$$

- Since $\frac{ds}{dt} = \|c'(t)\| > 0$ (why?) → s can be introduced as a new parameterization

Arc length parametrization

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$$s(t) = \int_0^t \|c'(u)\| du$$

- Since $\frac{ds}{dt} = \|c'(t)\| > 0$ (why?) → s can be introduced as a new parameterization
- We have $c'(s) = \frac{dc}{ds} = \frac{dc/dt}{ds/dt} \Rightarrow \|c'(s)\|$=1
- $c(s)$ is called an *arc-length* (or *unit-speed*) *parametrized curve*, the parameter s is called the *arc length* of c or the *natural parameter*

Reparameterization by arc length

- Arc-length (or unit-speed) parameterization:
  - Any regular curve admits an arc-length parameterization
  - This does not mean that the arc-length parameterization can be computed

Examples

$$s(t) = \int_0^t \|c'(u)\| dt$$

- Find an arc-length parameterization for the Helix: $\begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$

Examples

$$s(t) = \int_0^t \|c'(u)\| du$$

- Find an arc-length parameterization for the Helix: $\begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$

$$s(t) = \int_0^t \sqrt{(-\sin u)^2 + (\cos u)^2 + 1^2} du = t\sqrt{2} \Rightarrow t = \frac{s}{\sqrt{2}}$$

The arc-length parameterized Helix: $\begin{pmatrix} \cos \frac{s}{\sqrt{2}} \\ \sin \frac{s}{\sqrt{2}} \\ \frac{s}{\sqrt{2}} \end{pmatrix}$

$$s(t) = \int_0^t \|c'(u)\| du$$

Examples

- How about the ellipse $\alpha(t) = \begin{pmatrix} 2\cos t \\ \sin t \\ 0 \end{pmatrix}$?

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- How about the ellipse $\alpha(t) = \begin{pmatrix} 2\cos t \\ \sin t \\ 0 \end{pmatrix}$?

$$s(t) = \int_0^t \sqrt{4(-\sin u)^2 + (\cos u)^2} du = \int_0^t \sqrt{4 - 3\cos^2 u}\, du$$

Does not admit any closed form antiderivative

$$s(t) = \int_0^t \|c'(u)\| dt$$

Examples

- How about $\alpha(t) = \begin{pmatrix} t \\ \frac{t^2}{2} \\ 0 \end{pmatrix}$?

$$s(t) = \int_0^t \|c'(u)\| dt$$

Examples

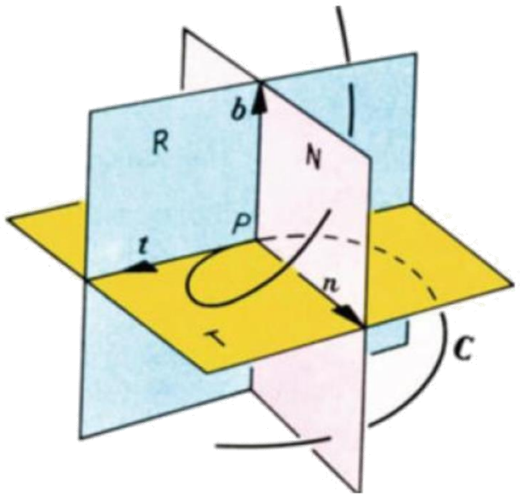
- How about $\alpha(t) = \begin{pmatrix} t \\ \frac{t^2}{2} \\ 0 \end{pmatrix}$?

$$s(t) = \int_0^t \sqrt{1+u^2} du = t\sqrt{1+t^2} + \ln\left(t + \sqrt{1+t^2}\right)$$

- No straightforward way to write t as a function of s!

Geometric consequences of Arc length parameterization

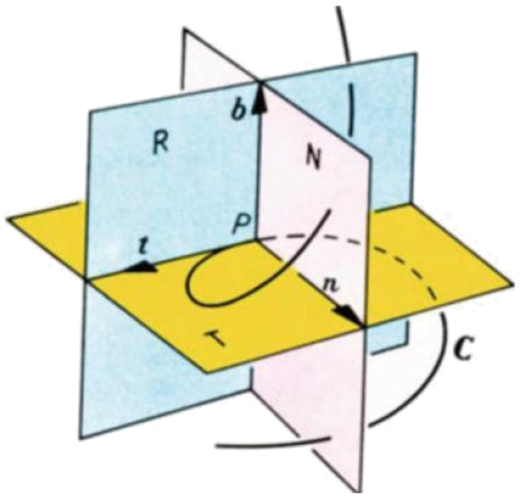
- Since $||c'(u)|| = 1$



Geometric consequences of Arc length parameterization

- Since $\|c'(u)\| = 1$, by noting that $c' \cdot c' = 1$ and taking the derivative, we have $c' \cdot c'' = 0$
- c'' is perpendicular to c' (both lives on the osculating plane)
- Therefore c'' is a direction vector of the principal normal (provided that $c'' \neq 0$)

$$\Rightarrow n = \frac{c''}{\|c''\|}$$



$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$

Curvature again

- The curvature of an arc-length parametrized curve (unit speed curve) $c(t)$ simplifies to

$$\kappa = \|c''(t)\|$$

Further mathematical formulations: Frenet Curves

Frenet Curves

- Frenet curves
  - A *Frenet curve* is an arc-length parametrized curve c in $\mathbb{R}^n$ such that $c'(s), c''(s), \dots, c^{n-1}(s)$ are linearly independent

Frenet Curves

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- Frenet frame
  - Every Frenet curve has a unique Frenet frame $e_1(s), e_2(s), \dots, e_n(s)$ that satisfies
    - $e_1(s), e_2(s), \dots, e_n(s)$ is orthonormal and positively oriented

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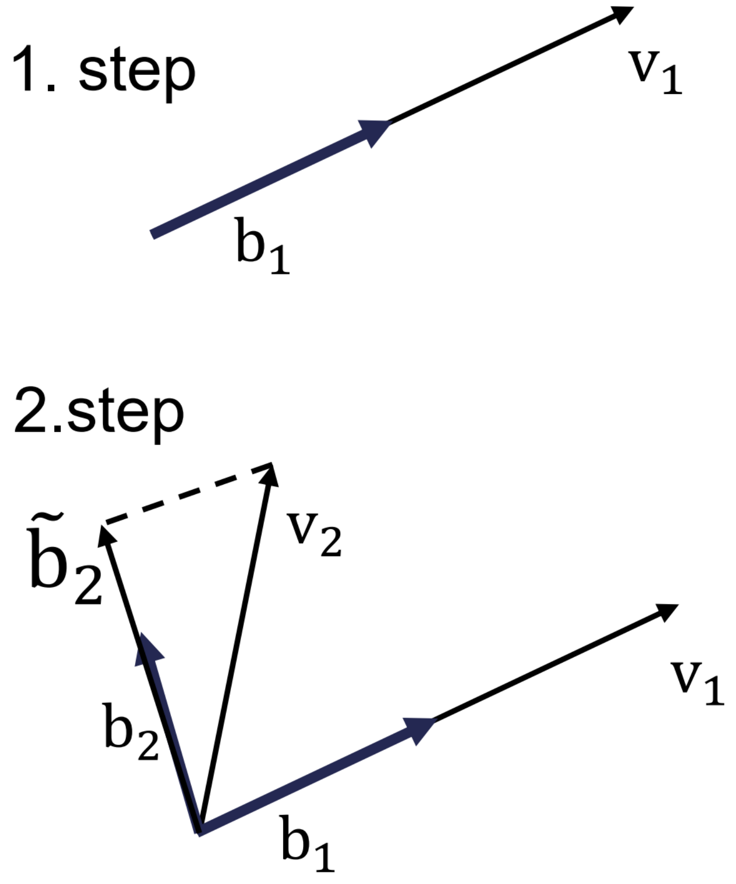
  - Apply the Gram-Schmidt process to $\{c', c'', \dots, c^n\}$

Gram-Schmidt Process: Construction of Orthonormal Bases

- Input: Linear independent set $\{v_1, v_2, \dots, v_n\}$
- Output: Orthogonal set $\{b_1, b_2, \dots, b_n\}$
  - Set $b_1 = \frac{v_1}{\|v_1\|}$
  - For $k = 2, \dots, n$
    - $\widetilde{b_k} = v_k - \sum_{i=1}^{k-1} \langle v_k, b_i \rangle \ b_i$
    - $b_k = \frac{\widetilde{b_k}}{\|\widetilde{b_k}\|}$



$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$

Planar Curves

The Frenet Frame of an arc-length parametrized planar curve

Tangent vector

$$e_1(s) = c'(s)$$

Normal vector

$$e_2(s) = R^{90^\circ} e_1(s)$$

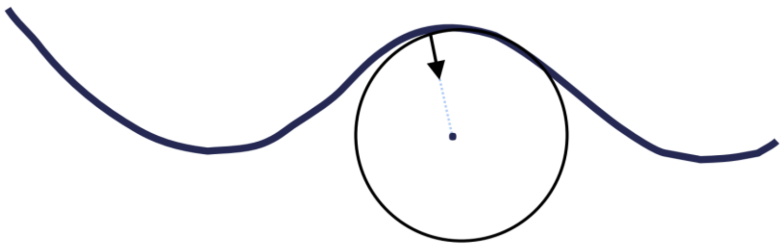
Frame equation

$$\begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \end{pmatrix}$$

Signed Curvature

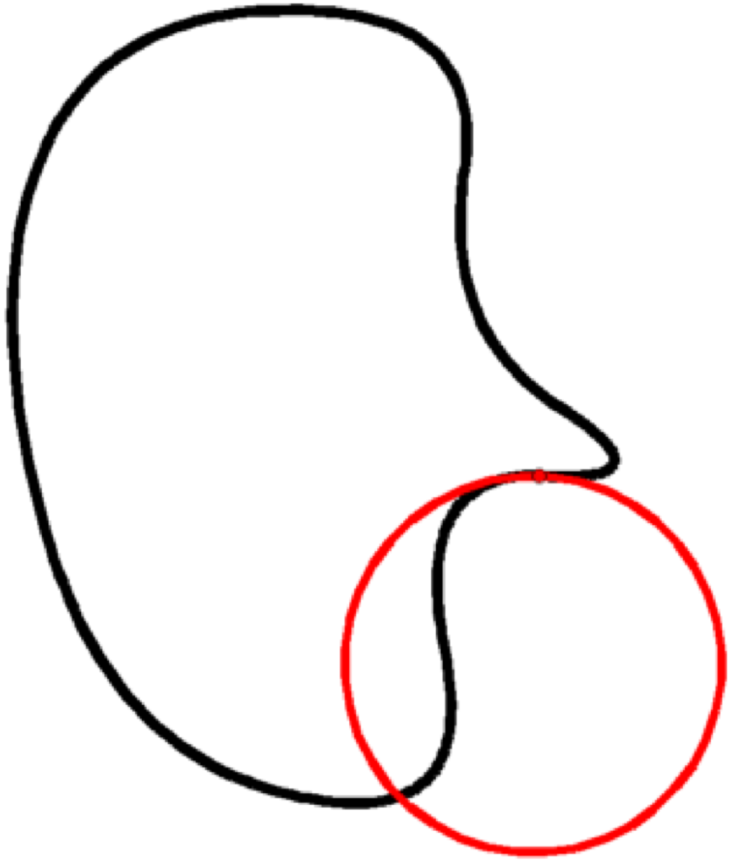
$\kappa(s) = \langle e_1'(s), e_2(s) \rangle$ is called the *signed curvature* of the curve

Osculating circle



Osculating circle

- Radius: $^1/_\kappa$
- Center: $c(s) + \frac{1}{\kappa} e_2(s)$



Properties

- Rigid motions
  - Rigid motion: $x \to Ax + b$ with orthogonal A (in other words: affine maps that preserve distances)
    - Orientation preserving (no mirroring) if $|A| = +1$
    - Mirroring leads to $|A| = -1$
- Invariance under rigid motions for planar curves
  - Curvature is invariant under rigid motion
    - Absolute value is invariant
    - Signed value is invariant for orientation preserving rigid motion
- Rigidity of planar curves
  - Two Frenet curves with identical signed curvature function differ only by an orientation preserving rigid motion

Fundamental Theorem

Fundamental theorem for planar curves

- Let $\kappa: (a, b) \mapsto \mathbb{R}$ be a smooth function. For some $s_0 \in (a, b)$, suppose we are given a point p_0 and two orthonormal vectors t_0 and n_0. Then there exists a unique Frenet curve $c: (a, b) \mapsto \mathbb{R}^2$ such that
  - $c(s_0) = p_0$
  - $e_1(s_0) = t_0$
  - $e_2(s_0) = n_0$
  - The curvature of c equals the given function κ
- In other words: for every <u>smooth</u> function there is a unique (up to rigid motion) curve that has this function as its curvature

Arc-length Derivative

- Arc-length parameterization
  - Finding an arc-length parameterization for a parameterized curve is usually difficult
  - Still one can compute the Frenet frame and its derivatives. For this we define the so called arc-length derivative
- Arc-length derivative
  - For a parameterized curve $c: [a, b] \mapsto \mathbb{R}^n$, we define the *arc-length derivative* of any differentiable function $f: [a, b] \mapsto \mathbb{R}$ as

$$f'(s) = \frac{1}{\|c'(t)\|} f'(t)$$

Compute the signed curvature

- Computing the Frenet frame
  - For $c: [a, b] \mapsto \mathbb{R}^2$, the Frenet frame at $c(t)$ can be computed as (using arc length derivative)

$$e_1(t) = c'(s) = \frac{c'(t)}{\|c'(t)\|}$$
$$e_2(t) = R^{90^\circ} e_1(t)$$

- Computing the signed curvature
  - The signed curvature is given by

$$\kappa(t) = \langle e_1'(t), e_2(t) \rangle = \frac{\left\langle c''(t), R^{90^\circ} c'(t) \right\rangle}{\|c'(t)\|^3}$$

Space Curves

- Frenet frame of arc-length parametrized space curves
  - Frenet frame of a Frenet curve in $\mathbb{R}^3$
    - Tangent vector

$$e_1(s) = c'(s)$$

    - Normal vector

$$e_2(s) = \frac{1}{\|c''(t)\|} c''(t)$$

    - Binormal vector

$$e_3(s) = e_1(s) \times e_2(s)$$

Frenet Frame of Space Curves

- Frenet–Serret equations

$$\begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}$$

- The signed curvature still is $\kappa(s) = \langle e_1'(s), e_2(s) \rangle$

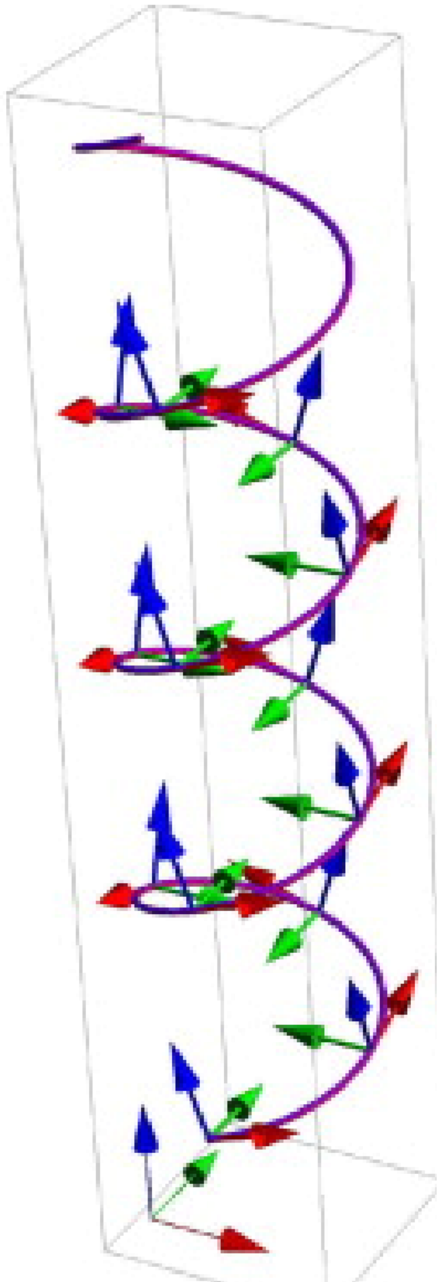
$$\tau(t) = \frac{(c' \times c'') \cdot c'''}{\|c' \times c''\|^2}$$

Frenet Frame of Space Curves

- Frenet–Serret equations

$$\begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ e_3(s) \end{pmatrix}$$

- The torsion $\tau(s) = \langle e_2'(s), e_3(s) \rangle$ measures how the curve bends out of the plane spanned by e_1 and e_2



Frenet Frame of Space Curves

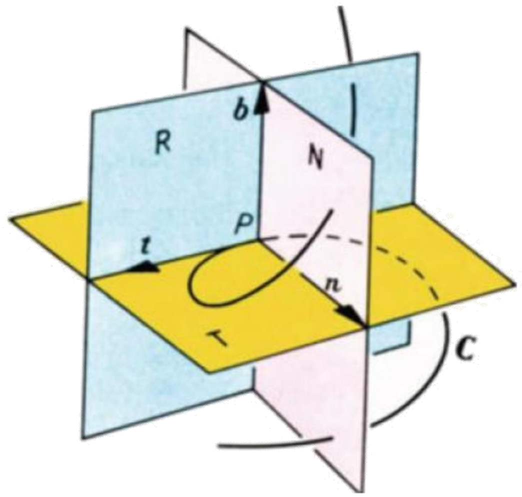
- Frenet equations for curves in $\mathbb{R}^n$

$$\begin{pmatrix} e_1(s) \\ e_2(s) \\ \dots \\ e_n(s) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa_1(s) & 0 & \dots & 0 \\ -\kappa_1(s) & 0 & \kappa_2(s) & \dots & 0 \\ 0 & -\kappa_2(s) & 0 & \dots & \\ & & & \dots & \kappa_{n-1}(s) \\ 0 & \dots & & -\kappa_{n-1}(s) & 0 \end{pmatrix} \begin{pmatrix} e_1(s) \\ e_2(s) \\ \dots \\ e_n(s) \end{pmatrix}$$

- The function $\kappa_i(s)$ are called the i^{th} Frenet curvatures

Summary of relations

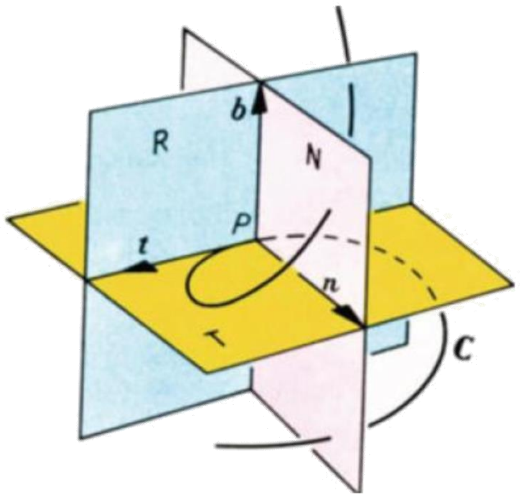
- **For regular curves:**
  - The tangent $t = \frac{c'}{\|c'\|}$, the normal plane $(p - p_0) \cdot t = 0$
  - The binormal $b = \frac{c' \times c''}{\|c' \times c''\|}$, the osculating plane $(p - p_0) \cdot b = 0$
  - The principal normal $n = b \times t$, the rectifying plane $(p - p_0) \cdot n = 0$
  - The curvature $\kappa(t) = \frac{c' \times c''}{\|c'\|^3}$
  - The torsion $\tau(t) = \frac{(c' \times c'') \cdot c'''}{\|c' \times c''\|^2}$



Summary of relations

For an <u>arc-length parameterized</u> (unit speed) curves $c(s)$:

- The tangent $t = c'$
- The binormal $b = t \times n$
- The principal normal $n = \frac{t'}{\|t'\|} = \frac{c''}{\|c''\|}$
- The curvature $\kappa(t) = \|t'\| = \|c''\|$
- The signed curvature $\kappa(s) = t' = c''$
- The torsion $\tau(t) = -b' \cdot n$



Special case: planar curves

- For a regular planar curve $c(t) = \big(x(t), y(t)\big)$, it is defined as

$$\kappa(t) = \frac{|x'y'' - x''y'|}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$

- Sometimes we talk about signed curvature, and then curvature can be allowed to be signed (negative, zero, or positive)

$$\kappa(t) = \frac{x'y'' - x''y'}{\left(x'^2 + y'^2\right)^{\frac{3}{2}}}$$