计算机辅助几何设计
2023秋学期

Blossoming and Polar Forms

Bézier Splines and B-Splines Revisited

陈仁杰

中国科学技术大学

A Short Step Back

Bézier & Monomials

Matrix Form

Matrix Notation: Bézier → Monomials

$$\boldsymbol{f}(t) = \begin{pmatrix}1 & t & t^2\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -2 & 2 & 0\\ 1 & -2 & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{p}_0\\ \boldsymbol{p}_1\\ \boldsymbol{p}_2\end{pmatrix}$$ (quadratic case)

$$\boldsymbol{f}(t) = \begin{pmatrix}1 & t & t^2 & t^3\end{pmatrix}\begin{pmatrix}1 & 0 & 0 & 0\\ -3 & 3 & 0 & 0\\ 3 & -6 & 3 & 0\\ -1 & 3 & -3 & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{p}_0\\ \boldsymbol{p}_1\\ \boldsymbol{p}_2\\ \boldsymbol{p}_3\end{pmatrix}$$ (cubic case)

Format Conversion

Conversion: Compute Bézier coefficients from monomial coefficients

$$\begin{pmatrix} \boldsymbol{c}_0^{(Bez.)} \\ \boldsymbol{c}_1^{(Bez.)} \\ \boldsymbol{c}_2^{(Bez.)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{c}_0 \\ \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \end{pmatrix}$$

(quadratic case)

$$\begin{pmatrix} \boldsymbol{c}_0^{(Bez.)} \\ \boldsymbol{c}_1^{(Bez.)} \\ \boldsymbol{c}_2^{(Bez.)} \\ \boldsymbol{c}_3^{(Bez.)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \boldsymbol{c}_0 \\ \boldsymbol{c}_1 \\ \boldsymbol{c}_2 \\ \boldsymbol{c}_3 \end{pmatrix}$$

(cubic case)

Format Conversion

Conversion: quadratic to cubic

$$\begin{pmatrix} \boldsymbol{c}_0^{(3)} \\ \boldsymbol{c}_1^{(3)} \\ \boldsymbol{c}_2^{(3)} \\ \boldsymbol{c}_3^{(3)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{c}_0^{(2)} \\ \boldsymbol{c}_1^{(2)} \\ \boldsymbol{c}_2^{(2)} \\ 0 \end{pmatrix}$$

Convert to monomials and back to Bézier coefficients. (other degrees similar)

Example application: Output of TrueType fonts in Postscript.

Polar Forms & Blossoms

Idea & Definition

Affine Combinations

Definition (reminder):

- An *affine combination* of n points $\in \mathbb{R}^d$ is given by:

$$P_\alpha = \sum_{i=1}^{n} \alpha_i p_i \text{ with } \sum_{i=1}^{n} \alpha_i = 1$$

- A function f is said to be *affine* in its parameter x_i, if:

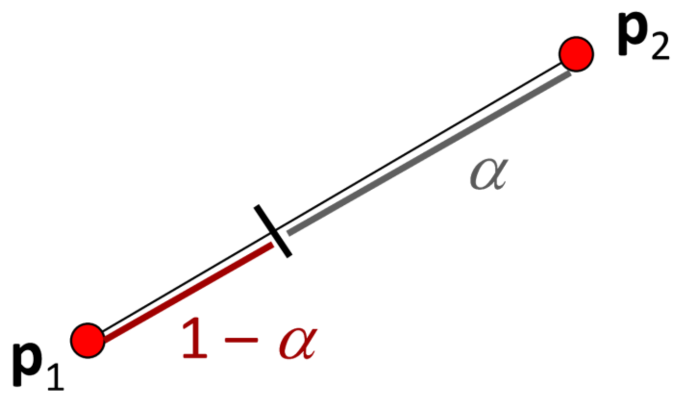
$$f\left(x_1, \dots, \sum_{i=1}^{n} \alpha_i x_i^{(k)}, \dots, x_m\right) = \sum_{i=1}^{n} \alpha_i f\left(x_1, \dots, x_i^{(k)}, \dots, x_m\right) \text{ for } \sum_{i=1}^{n} \alpha_i = 1$$

Affine Combinations

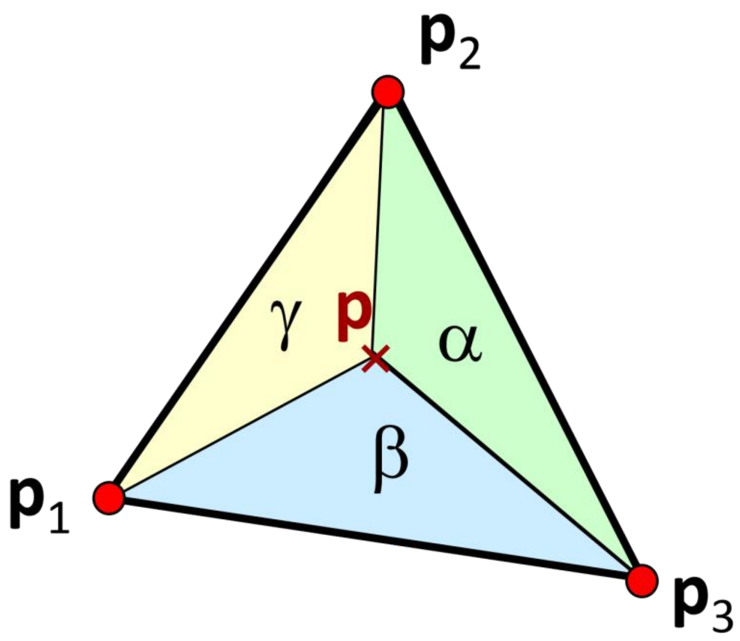
Examples:

- Linear (affine) interpolation of 2 points:

$$\boldsymbol{p}_\alpha = \alpha \boldsymbol{p}_1 + (1 - \alpha)\boldsymbol{p}_2$$



Affine Combinations



Examples:

- Barycentric combinations of 3 points
  (“barycentric coordinates”)
  $p = \alpha \boldsymbol{p}_1 + \beta \boldsymbol{p}_2 + \gamma \boldsymbol{p}_3$, **with** $\alpha + \beta + \gamma = 1$

  Properties:
  $\gamma = 1 - \alpha - \beta$
  $\alpha = \frac{\Delta(p_2,p_3,p)}{\Delta(p_1,p_2,p_3)}, \quad \beta = \frac{\Delta(p_1,p_3,p)}{\Delta(p_1,p_2,p_3)}, \quad \gamma = \frac{\Delta(p_1,p_2,p)}{\Delta(p_1,p_2,p_3)}$

  Transformation to barycentric coordinates is a linear map
  (heights in triangles)

Formalizing the Idea

- **Idea:** Express (piecewise) polynomial curves as iterated linear (affine) interpolations
- **First steps:**
  - A polynomials: $P(t) = at^3 + bt^2 + ct + d$
  - Can be written as: $P(t) = a \cdot t \cdot t \cdot t + b \cdot t \cdot t + c \cdot t + d$
  - Interpret each variable t as a separate parameter:

$$p(t_1, t_2, t_3) = a \cdot t_1 \cdot t_2 \cdot t_3 + b \cdot t_1 \cdot t_2 + c \cdot t_1 + d$$

new function

Polar Forms

Solution: Polar Forms / Blossoms

A *polar form* or *blossom* f of a polynomial F of degree d is a function in d variables:

$$F: \mathbb{R} \rightarrow \mathbb{R}$$

$$f: \mathbb{R}^d \rightarrow \mathbb{R}$$

with the following properties:

- *Diagonality*: $f(t, t, \ldots, t) = F(t)$
- *Symmetry*: $f(t_1, t_2, \ldots, t_d) = f\left(t_{\pi(1)}, t_{\pi(2)}, \ldots t_{\pi(d)}\right)$
  for all permutations of indices π
- *Multi-affine*: $\sum \alpha_k = 1$

$$\Rightarrow f\left(t_1, t_2, \ldots, \sum \alpha_k\, t_i^{(k)}, \ldots, t_d\right)$$

$$= \alpha_1 f\left(t_1, t_2, \ldots, t_i^{(1)}, \ldots, t_d\right) + \cdots + \alpha_n f\left(t_1, t_2, \ldots, t_i^{(n)}, \ldots, t_d\right)$$

Polar Forms

Rationale:

- Model polynomial as multi-affine function
  - (multi-affine property)
- Plug in a common parameter to obtain the original polynomial
  - (diagonal of the blossom)
- Symmetry property – makes the solution unique
  - There is exactly one polar form for each polynomial
  - This standardization makes different polars “compatible”, we can compare them with each other
  - We will see how this works in a few slides …

Properties

The mapping from polynomials to their corresponding polar forms is one-to-one

- For each polar form $f(t_1, t_2, \dots, t_n)$,
  a unique polynomials $F(t)$ exists

- For each polynomial $F(t)$,
  a unique polar form $f(t_1, t_2, \dots, t_n)$ exists

Properties

Polar forms of monomials:

- Degree 0: $1 \rightarrow ?$
- Degree 1:
- Degree 2:
- Degree 3:

Properties

Polar forms of monomials:

- Degree 0: $1 \rightarrow 1$
- Degree 1: $1 \rightarrow 1, t \rightarrow t$
- Degree 2: $1 \rightarrow 1,\ t \rightarrow \frac{t_1+t_2}{2},\ t^2 \rightarrow t_1 t_2$
- Degree 3: $1 \rightarrow 1,\quad t^2 \rightarrow \frac{t_1 t_2+t_2 t_3+t_1 t_3}{3}$

  $t \rightarrow \frac{t_1+t_2+t_3}{3},\ t^3 \rightarrow t_1 t_2 t_3$

Properties

Polar forms of monomials:

- Degree 0: $f = c_0$
- Degree 1: $f(t_1) = c_0 + c_1 t_1$
- Degree 2: $f(t_1, t_2) = c_0 + c_1 \frac{t_1+t_2}{2} + c_2 t_1 t_2$
- Degree 3: $f(t_1, t_2, t_3) = c_0 + c_1 \frac{t_1+t_2+t_3}{3} + c_2 \frac{t_1 t_2+t_2 t_3+t_1 t_3}{3}$

  $+c_3 t_1 t_2 t_3$

Properties

General Case:

$$f(t_1, \dots, t_n) = \sum_{i=0}^{n} c_i \binom{n}{i}^{-1} \sum_{\substack{S \subseteq \{1 \dots n\}, \\ |S| = i}} \prod_{j \in S} t_j$$

- The c_i are the monomial coefficients
- Idea: use all possible subsets of t_i to make it symmetric
- This solution is unique
- Without the symmetry property, there would be a large number of solutions

$|S|$ denotes the cardinality of the set S

Generalizations

Blossoms for polynomials curves (points as output):

- Polar form of a polynomial curve of degree d:

F: $\mathbb{R} \to \mathbb{R}^n$

f: $\mathbb{R}^d \to \mathbb{R}^n$

new

- Required Properties:
  - Diagonality: $f(t, t, \ldots, t) = F(t)$
  - Symmetry: $f(t_1, t_2, \ldots, t_d) = f\left(t_{\pi(1)}, t_{\pi(2)}, \ldots, t_{\pi(d)}\right)$
    for all permutations of indices π
  - Multi-affine:: $\sum \alpha_k = 1$

$$\Rightarrow f\left(t_1, t_2, \ldots, \sum \alpha_k\, t_i^{(k)}, \ldots, t_d\right)$$

$$= \alpha_1 f\left(t_1, t_2, \ldots, t_i^{(1)}, \ldots, t_d\right) + \cdots + \alpha_n f\left(t_1, t_2, \ldots, t_i^{(n)}, \ldots, t_d\right)$$

Generalizations

Blossoms with points as arguments:

- Polar form of degree d with points as input and output:

  $F: \quad \mathbb{R}^m \to \mathbb{R}^n$ **new**

  $f: \quad \mathbb{R}^{d \times m} \to \mathbb{R}^n$

- Required Properties:
  - Diagonality: $f(\boldsymbol{t}, \boldsymbol{t}, \dots, \boldsymbol{t}) = F(\boldsymbol{t})$
  - Symmetry: $f(\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_d) = f\left(\boldsymbol{t}_{\pi(1)}, \boldsymbol{t}_{\pi(2)}, \dots, \boldsymbol{t}_{\pi(d)}\right)$

    for all permutations of indices π
  - Multi-affine:: $\sum \alpha_k = 1$

$$\Rightarrow f\left(\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \sum \alpha_k \, \boldsymbol{t}_i^{(k)}, \dots, \boldsymbol{t}_d\right)$$

$$= \alpha_1 f\left(\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_i^{(1)}, \dots, \boldsymbol{t}_d\right) + \dots + \alpha_n f\left(\boldsymbol{t}_1, \boldsymbol{t}_2, \dots, \boldsymbol{t}_i^{(n)}, \dots, \boldsymbol{t}_d\right)$$

Generalizations

Vector arguments

- We will have to distinguish between *points* and *vectors* (differences of points)
- Use "hat" notation $\hat{v} = p - q$ to denote vectors
- Also defined in the one dimensional case (vectors in $\mathbb{R}$)
- One vector: $\hat{1} = 1 - 0$, $\hat{\mathbf{1}} = [1, \ldots 1]^{\mathrm{T}} - \mathbf{0}$
- Define shorthand notation (recursive) to define vectors in polar form:

$$f(\underbrace{t_1, \ldots, t_{n-k}}_{n-k}, \underbrace{\hat{v}_1, \ldots, \hat{v}_k}_{k}) := f(\underbrace{t_1, \ldots, t_{n-k}}_{n-k}, p_1, \underbrace{\hat{v}_2, \ldots, \hat{v}_k}_{k-1}) - f(\underbrace{t_1, \ldots, t_{n-k}}_{n-k}, q_1, \underbrace{\hat{v}_2, \ldots, \hat{v}_k}_{k-1})$$

Properties

Derivatives of blossoms:

$$f(t_1, \dots, t_n) = \sum_{i=0}^{n} c_i \binom{n}{i}^{-1} \sum_{\substack{S\subseteq\{1\dots n\},\\ |S|=i}} \prod_{j\in S} t_j$$

- The c_i are related to the derivatives at $t = 0$
- Hence: $c_k = \frac{1}{k!}\frac{d^k}{dt^k}F(0) = \binom{n}{k} f(\underbrace{0, \dots, 0}_{n-k}, \underbrace{\hat{1}, \dots, \hat{1}}_{k})$
- In general: $\frac{d^k}{dt^k}F(t) = \frac{n!}{(n-k)!} f(\underbrace{t, \dots, t}_{n-k}, \underbrace{\hat{1}, \dots, \hat{1}}_{k})$

Example

$$f(t_1, t_2, t_3) = c_0 + c_1 \frac{t_1 + t_2 + t_3}{3} + c_1 \frac{t_1 t_2 + t_1 t_3 + t_2 t_3}{3} + c_3 t_1 t_2 t_3$$

$$f'(t) = \frac{3!}{2!}\left[\left(c_0 + \frac{1 + t + t}{3} + c_1 \frac{1t + tt + 1t}{3} + c_3 1tt\right) - \left(c_0 + \frac{0 + t + t}{3} + c_1 \frac{tt}{3}\right)\right]$$

$$= 3\left(c_1 \frac{1}{3} + c_2 \frac{2t}{3} + c_3 tt\right)$$

$$= 3c_3 t^2 + 2c_2 t + c_1$$

Continuity Condition

Theorem: continuity condition for polynomials

The following statements are equivalent:

1. F **and** G **are** C^k**-continuous at** t
2. $\forall t_1, \dots, t_k: f(t, \dots, t, t_1, \dots, t_k) = g(t, \dots, t, t_1, \dots, t_k)$
3. $f(t, \dots, t, \underbrace{\hat{1}, \dots, \hat{1}}_{k\text{-times}}) = g(t, \dots, t, \underbrace{\hat{1}, \dots, \hat{1}}_{k\text{-times}})$

$2 \Leftrightarrow 3$:

$$\begin{aligned} f(t, .., t, t_1) &= f(t, \dots, t, t_1 - 0) \\ &= t_1\big(f(t, \dots, t, 1) - f(t, \dots, t, 0)\big) \\ &= t_1 f\big(t, \dots, t, \hat{1}\big) \end{aligned}$$

Continuity Condition

Examples:

- $\forall t_1, t_2, t_3: f(t_1, t_2, t_3) = g(t_1, t_2, t_3) \Rightarrow$ same curve
- $\forall t_1, t_2: \quad f(t_1, t_2, t) = g(t_1, t_2, t) \Rightarrow C^2$ at t
- $\forall t_1: \quad f(t_1, t, t) = g(t_1, t, t) \Rightarrow C^1$ at t
- $f(t, t, t) = g(t, t, t) \Rightarrow C^0$ at t

Raising the Degree

Raising the degree of a blossom:

- Can we directly construct a polar form with degree elevated by one from a lower degree one, without changing the polynomial?
- [Other than transforming to monomials, adding $\mathbf{0} \cdot t^{d+1}$, and transforming back?]

Solution:

- Given: $f(t_1, \dots, t_d)$
- We obtain: $f^{(+1)}(t_1, \dots, t_{d+1}) = \frac{1}{d+1} \sum_{i=1}^{d+1} f(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{d+1})$

leave out t_i

Raising the Degree

Proof:

$$\forall t: f^{(+1)}(t, \dots, t) = \frac{1}{d+1} \sum_{i=1}^{d+1} f(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{d+1}) \Big|_{t_1 = \cdots = t_{d+1} = t}$$

$$= \frac{1}{d+1} \sum_{i=1}^{d+1} f(t, \dots, t)$$

$$= f(t, \dots, t)$$

$$\Rightarrow F^{(+1)}(t) = F(t)$$

Polars and Control Points

Interpretation (Examples):

- Multi-variate function: $f(t_1, t_2, t_3)$
  - Describes a point depending on three parameters
  - Where $f(t_1, t_2, t_3)$ moves for changing (t_1, t_2, t_3) (think of storing monomial coefficients inside)
- Polynomial value: $f(1.5, 1.5, 1.5)$
  - One value of the polynomial curve: $F(1.5)$
- Off-curve points: $f(1, 2, 3)$
  - Evaluate points not necessarily on the polynomial curve
  - Question: what meaning do various off-curve points have?
  - We will use off-curve points as control points

Polars and Control Points

Interpretation (Examples):

- Specifying: $f(t_1, t_2, t_3)$
  - Assume, f is not known yet
  - We want to determine a polar (i.e. a polynomial)
- On-curve points:
  $\{f(0,0,0) = \boldsymbol{x}_0, f(1,1,1) = \boldsymbol{x}_1, f(2,2,2) = \boldsymbol{x}_2, f(3,3,3) = \boldsymbol{x}_3\}$
  - Degree d polynomial has $d + 1$ degrees of freedom
  - We know already how to do polynomial interpolation
- Off-curve points:
  $\{f(1, 2, 3) = \boldsymbol{x}_{123}, f(2,3,4) = \boldsymbol{x}_{234}\}$
  - We can also use off-curve points to specify the polar/polynomial
  - This is the main motivation for the whole formalism

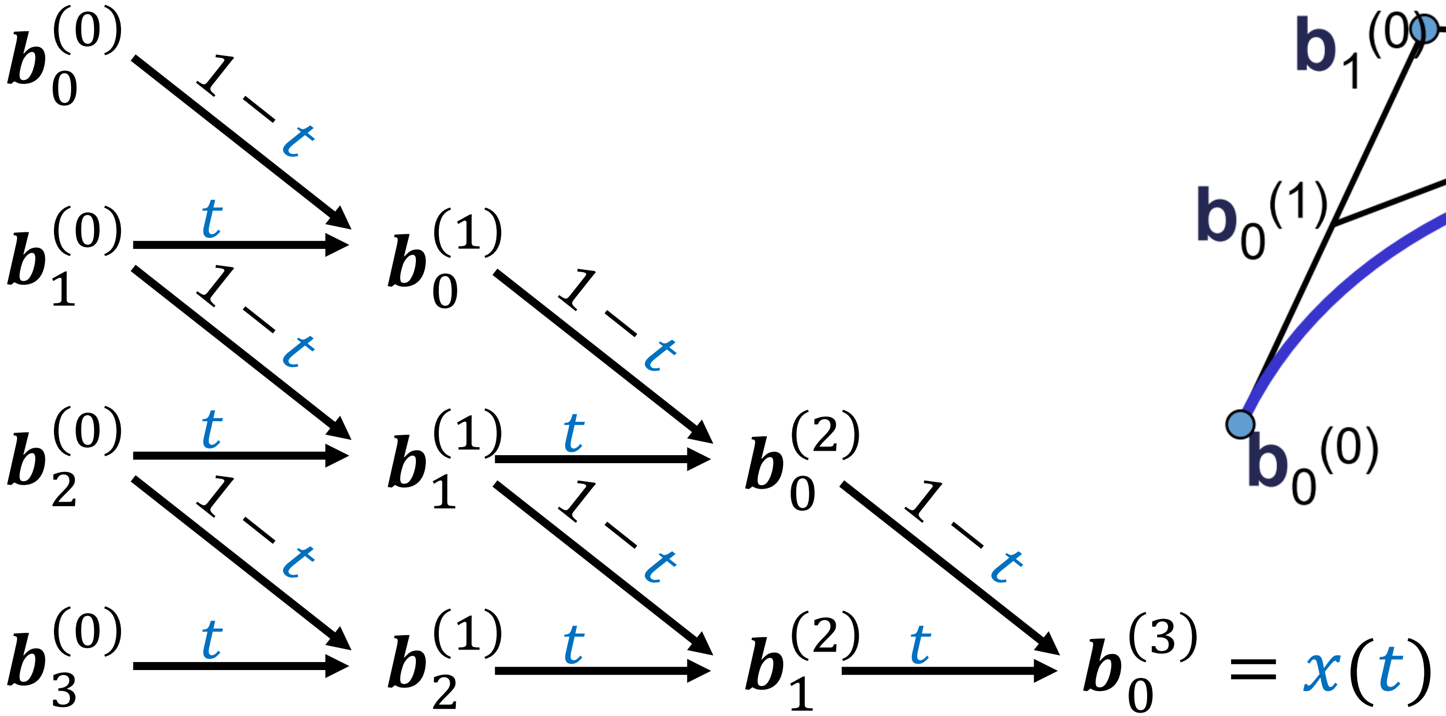
Polar Forms & Blossoms

Bézier Splines

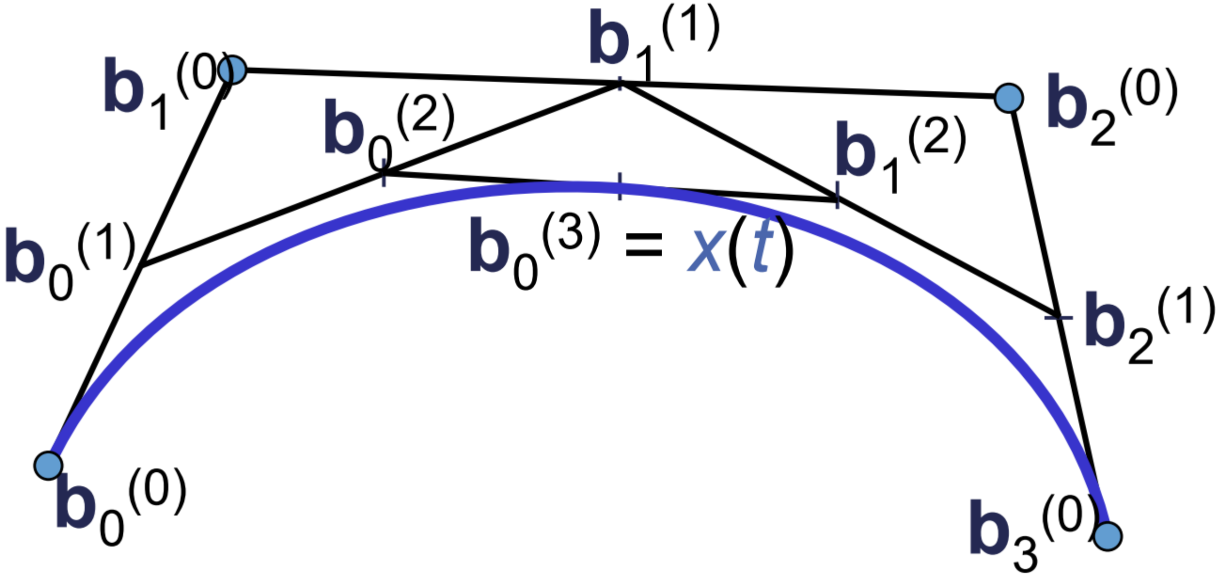
De Casteljau algorithm (from earlier)

Repeated convex combination of control points

$$\boldsymbol{b}_i^{(r)} = (1-t)\boldsymbol{b}_i^{(r-1)} + t\boldsymbol{b}_{i+1}^{(r-1)}$$



de Casteljau scheme

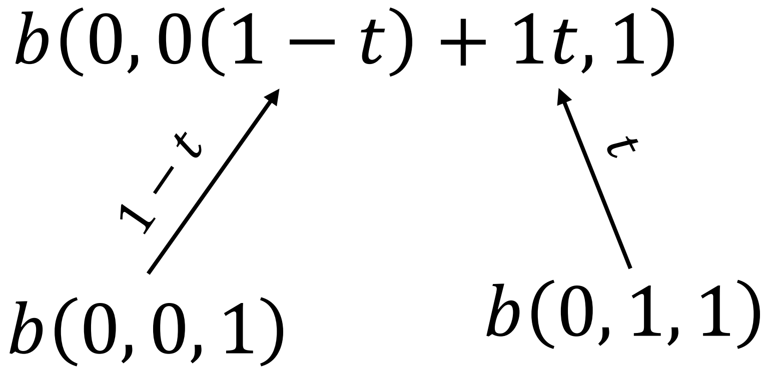


De Casteljau Algorithm

The de Casteljau algorithm is simple to state with blossoms:

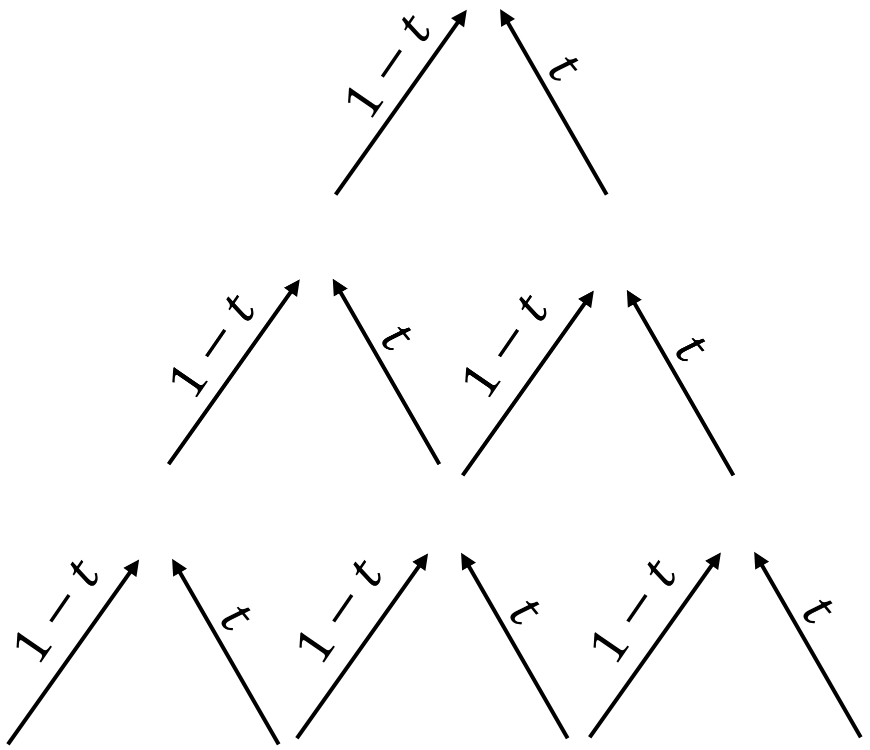
- We just have to exchange the labels
- Then use the multi-affinity property in order to compute intermediate points
- With this view, we can easily show that the de Casteljau algorithm is equivalent to the formulation based on Bernstein polynomials

Key observation



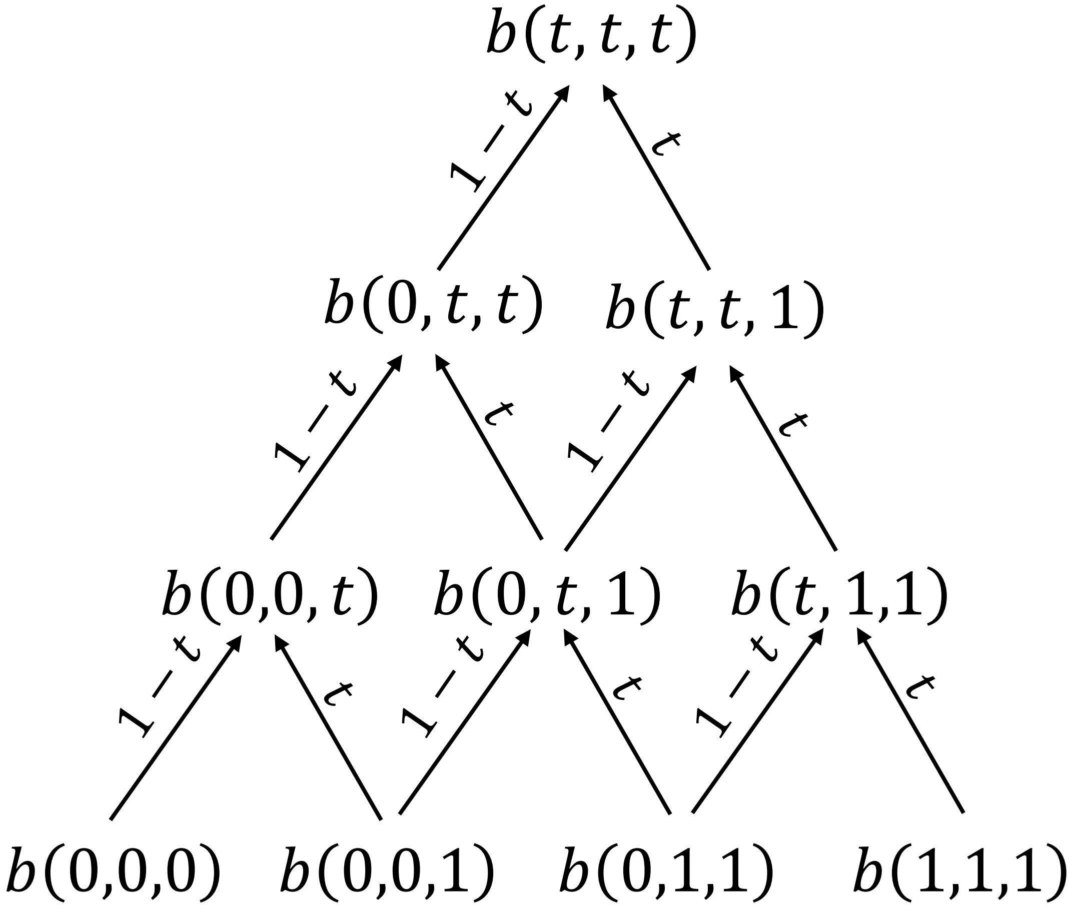
De Castljau Algorithms for Bézier Curves

$$(1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t) t^2 p_2 + t^3 p_3$$

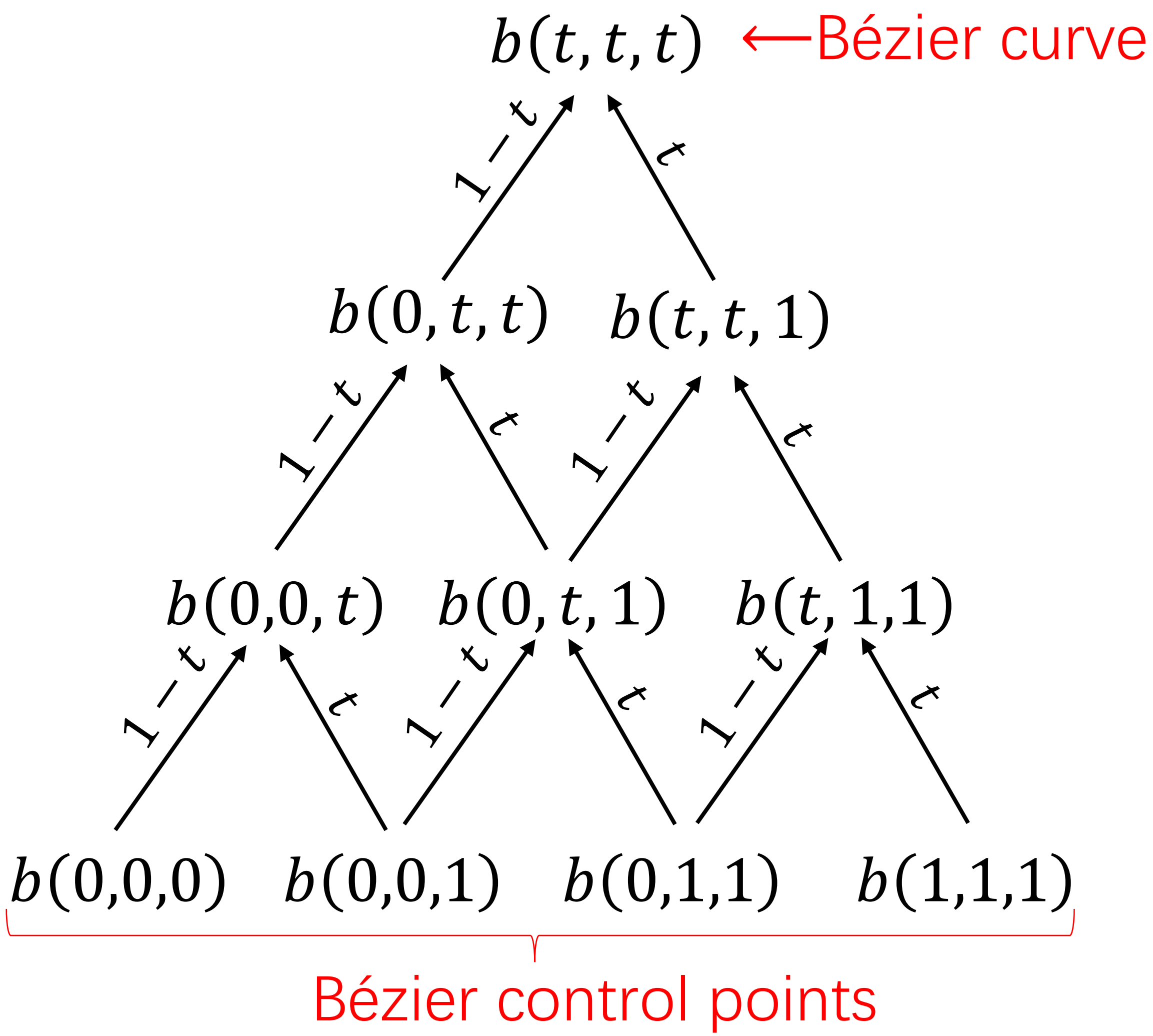


p_0 p_1 p_2 p_3

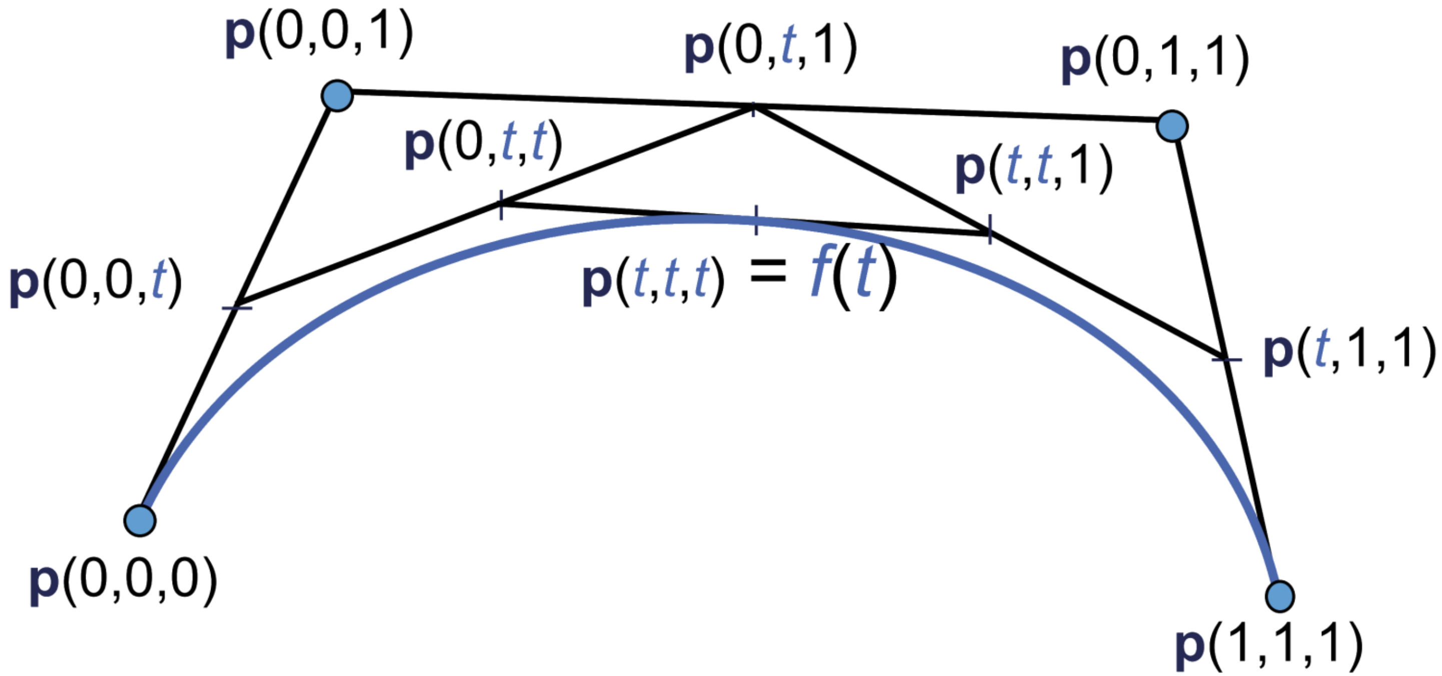
De Castljau Algorithms for Bézier Curves



De Castljau Algorithms for Bézier Curves



De Casteljau (Polar forms)



Bézier control points: $\boldsymbol{p}(0,0,0)$, $\boldsymbol{p}(0,0,1)$, $\boldsymbol{p}(0,1,1)$, $\boldsymbol{p}(1,1,1)$

Analysis

Transforming a polar to the Bernstein basis:

$$f(t, \dots, t) = (1 - t)f(t, \dots, t, 0) + tf(t, \dots, t, 1)$$

Analysis

Transforming a polar to the Bernstein basis:

$$\begin{aligned} f(t,\dots,t) &= (1-t)f(t,\dots,t,0) + tf(t,\dots,t,1) \\ &= (1-t)[(1-t)f(t,\dots,t,0,0) + tf(t,\dots,0,1)] + t[(1-t)f(t,\dots,t,1,0) + tf(t,\dots,t,1,1)] \end{aligned}$$

Analysis

Transforming a polar to the Bernstein basis:

$$
\begin{aligned}
f(t,\dots,t) &= (1-t)f(t,\dots,t,0) + tf(t,\dots,t,1) \\
&= (1-t)[(1-t)f(t,\dots,t,0,0) + tf(t,\dots,0,1)] + t[(1-t)f(t,\dots,t,1,0) + tf(t,\dots,t,1,1)] \\
&= (1-t)^2 f(t,\dots,t,0,0) + 2t(1-t)tf(t,\dots,0,1) + t^2 f(t,\dots,t,1,1)
\end{aligned}
$$

Analysis

Transforming a polar to the Bernstein basis:

$$\begin{aligned}
f(t,\dots,t) &= (1-t)f(t,\dots,t,0) + tf(t,\dots,t,1) \\
&= (1-t)[(1-t)f(t,\dots,t,0,0) + tf(t,\dots,0,1)] + t[(1-t)f(t,\dots,t,1,0) + tf(t,\dots,t,1,1)] \\
&= (1-t)^2 f(t,\dots,t,0,0) + 2t(1-t)tf(t,\dots,0,1) + t^2 f(t,\dots,t,1,1) \\
&= \cdots \\
&= \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} f(\underbrace{0,\dots,0}_{n-i},\underbrace{1,..,1}_{i})
\end{aligned}$$

Analysis

De Castlejau Algorithm: Perform this in reverse order

- Bézier points: $$p_i^{(0)}(t) = f(\underbrace{0, \dots, 0}_{d-i}, \underbrace{1, \dots, 1}_{i})$$
- Intermediate points: $$p_i^{(j)}(t) = f(\underbrace{0, \dots, 0}_{d-i-j}, \underbrace{1, \dots, 1}_{i}, \underbrace{t, \dots, t}_{j})$$
- Recursive computation: $$\begin{aligned} p_i^{(j)}(t) &= f(\underbrace{0, \dots, 0}_{d-i-j}, \underbrace{t, \dots, t}_{j}, \underbrace{1, \dots, 1}_{i}) \\ &= (1-t) f(\underbrace{0, \dots, 0}_{d-i-j+1}, \underbrace{t, \dots, t}_{j-1}, \underbrace{1, \dots, 1}_{i}) + t f(\underbrace{0, \dots, 0}_{d-i-j}, \underbrace{t, \dots, t}_{j-1}, \underbrace{1, \dots, 1}_{i+1}) \\ &= (1-t) p_i^{(j-1)}(t) + t p_{i+1}^{(j-1)}(t) \end{aligned}$$

Consequence: Bernstein / de Casteljau lead to the same result

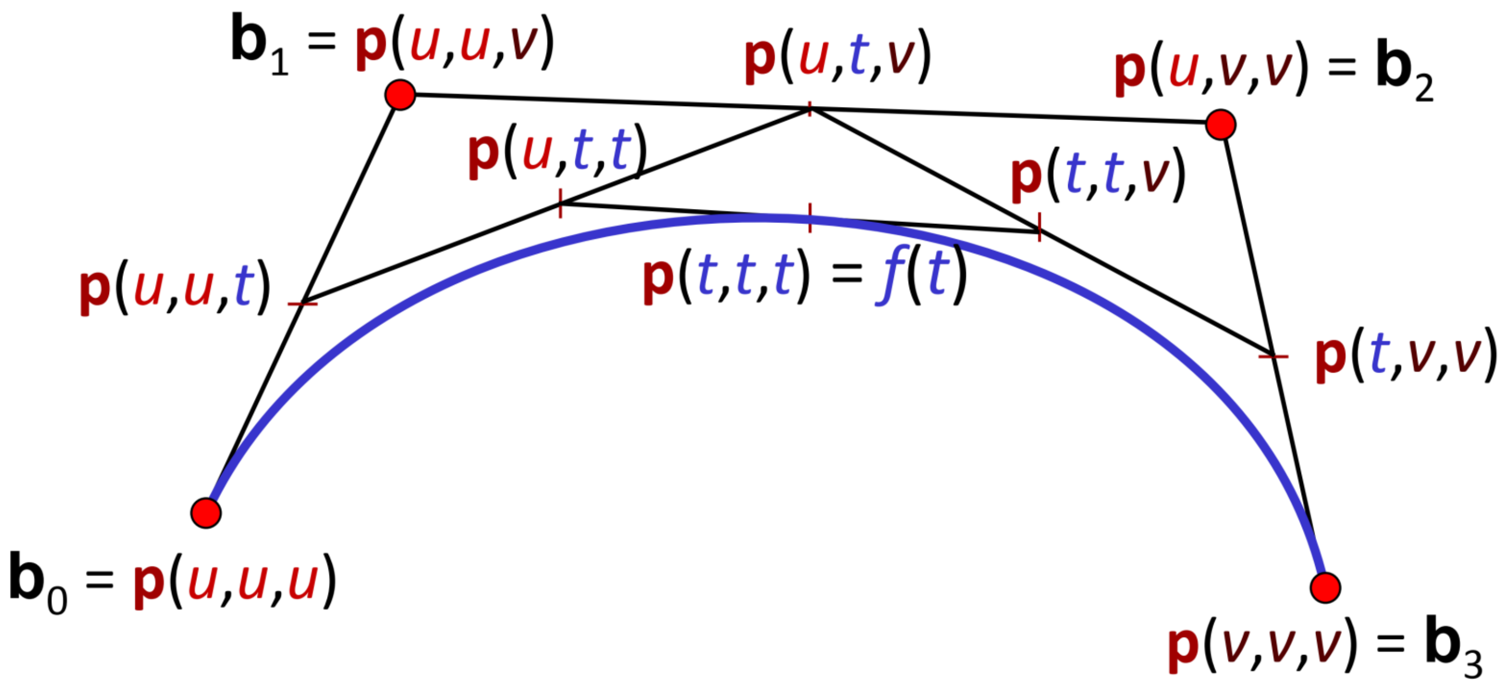
Generalized Parameter Intervals

- Let $f(t)$ be a Bézier curve of degree d over the domain $t \in [u, v]$
- Let $\boldsymbol{p}$ be the polar form of f
- Then the Bézier points of f are given in polar form as:
  - $\boldsymbol{b}_i = p(u, \ldots, u, v, \ldots, v)$

Example for a cubic Bézier curve:

$\boldsymbol{p}(u, u, u), \boldsymbol{p}(u, u, v), \boldsymbol{p}(u, v, v), \boldsymbol{p}(v, v, v)$

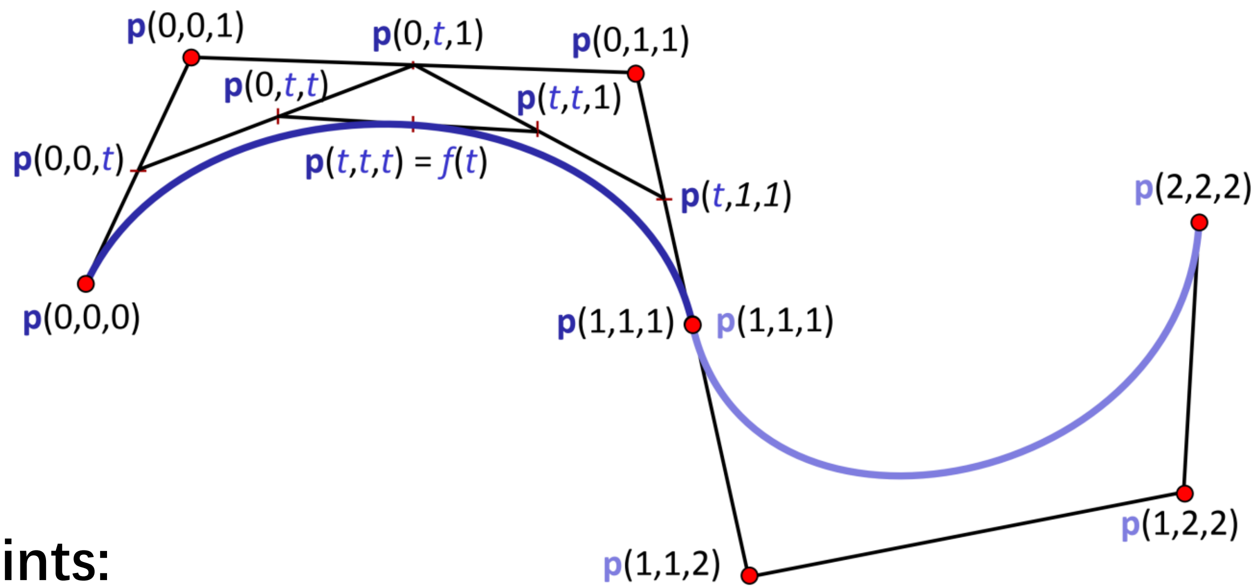
Generalized Parameter Intervals



Example for a cubic Bézier curve:

$\boldsymbol{p}(u, u, u), \boldsymbol{p}(u, u, v), \boldsymbol{p}(u, v, v), \boldsymbol{p}(v, v, v)$

Multiple Segments



Bézier Control points:

$\boldsymbol{p}(0,0,0), \boldsymbol{p}(0,0,1), \boldsymbol{p}(0,1,1), \boldsymbol{p}(1,1,1) = \boldsymbol{p}(1,1,1), \boldsymbol{p}(1,1,2), \boldsymbol{p}(1,2,2), \boldsymbol{p}(2,2,2)$

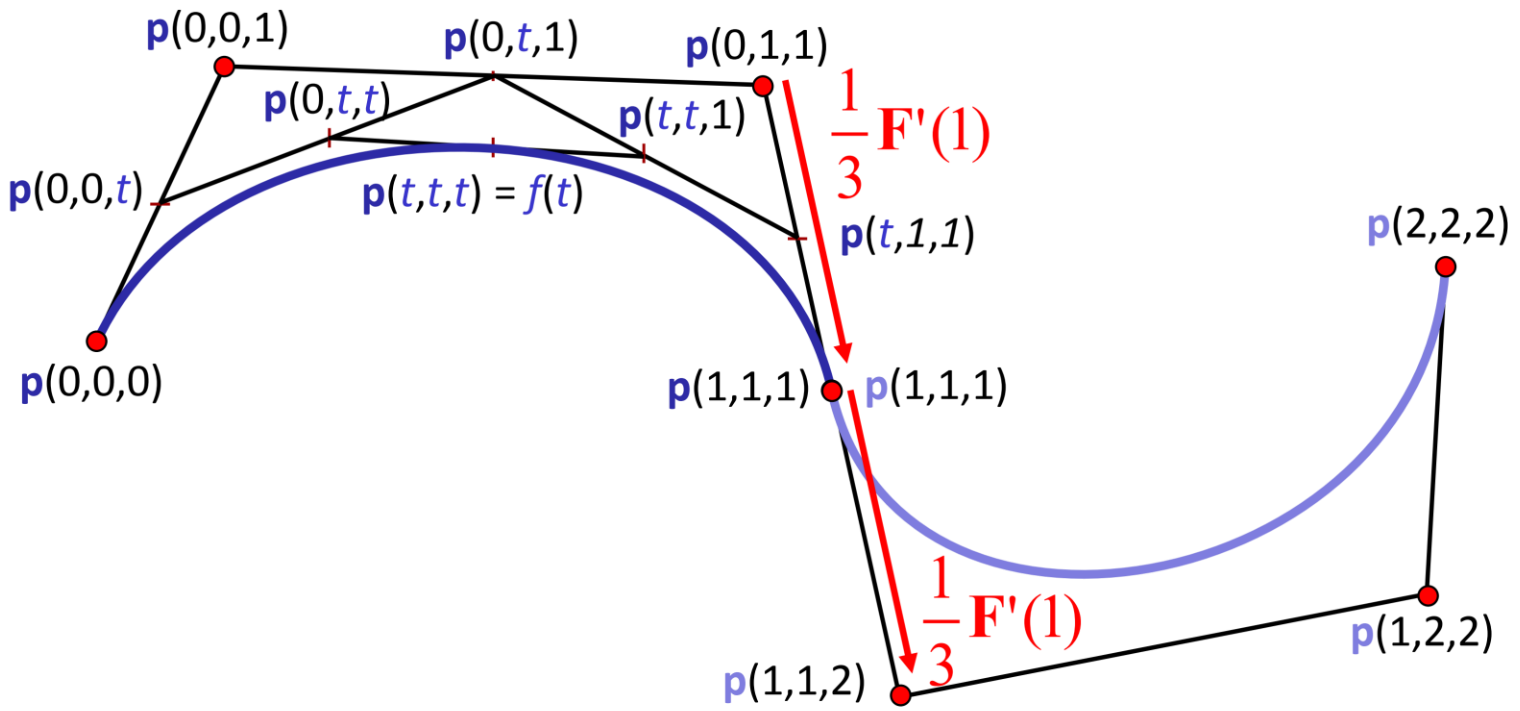
Two Curve Segments:

$\{\boldsymbol{p}(0,0,0), \boldsymbol{p}(0,0,1), \boldsymbol{p}(0,1,1), \boldsymbol{p}(1,1,1)\}, \ \{\boldsymbol{p}(1,1,1), \boldsymbol{p}(1,1,2), \boldsymbol{p}(1,2,2), \boldsymbol{p}(2,2,2)\}$

Remark: no intersection between different segments

(e.g.: combination of $p(0,1,1)$ and $p(2,1,1)$ is not defined)

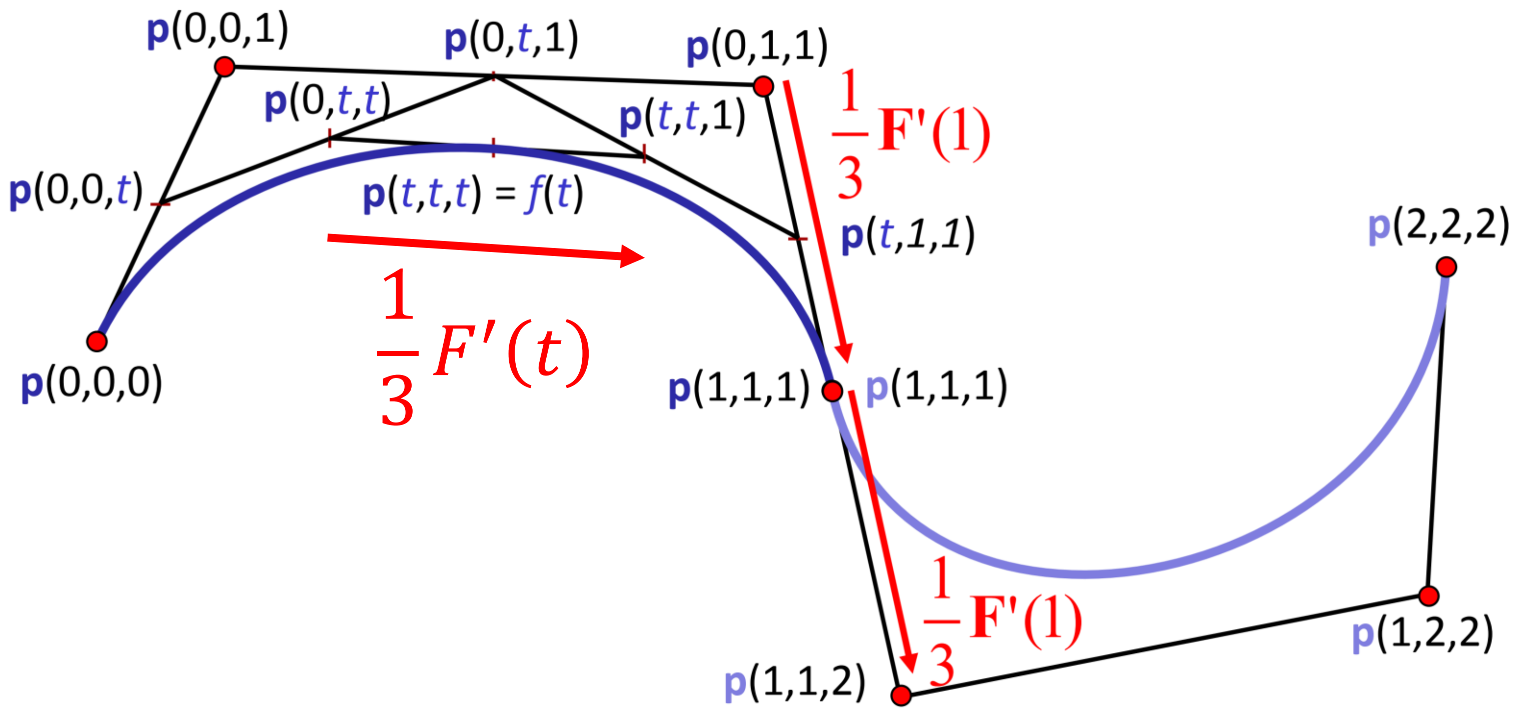
More Observations



Derivatives:

- $\frac{d}{dt}F(t) = df\left(t, \dots, t, \hat{1}\right) = d\left(f(t, \dots, t, 1) - f(t, \dots, t, 0)\right)$ (degree d)
- C^1 continuity condition follows

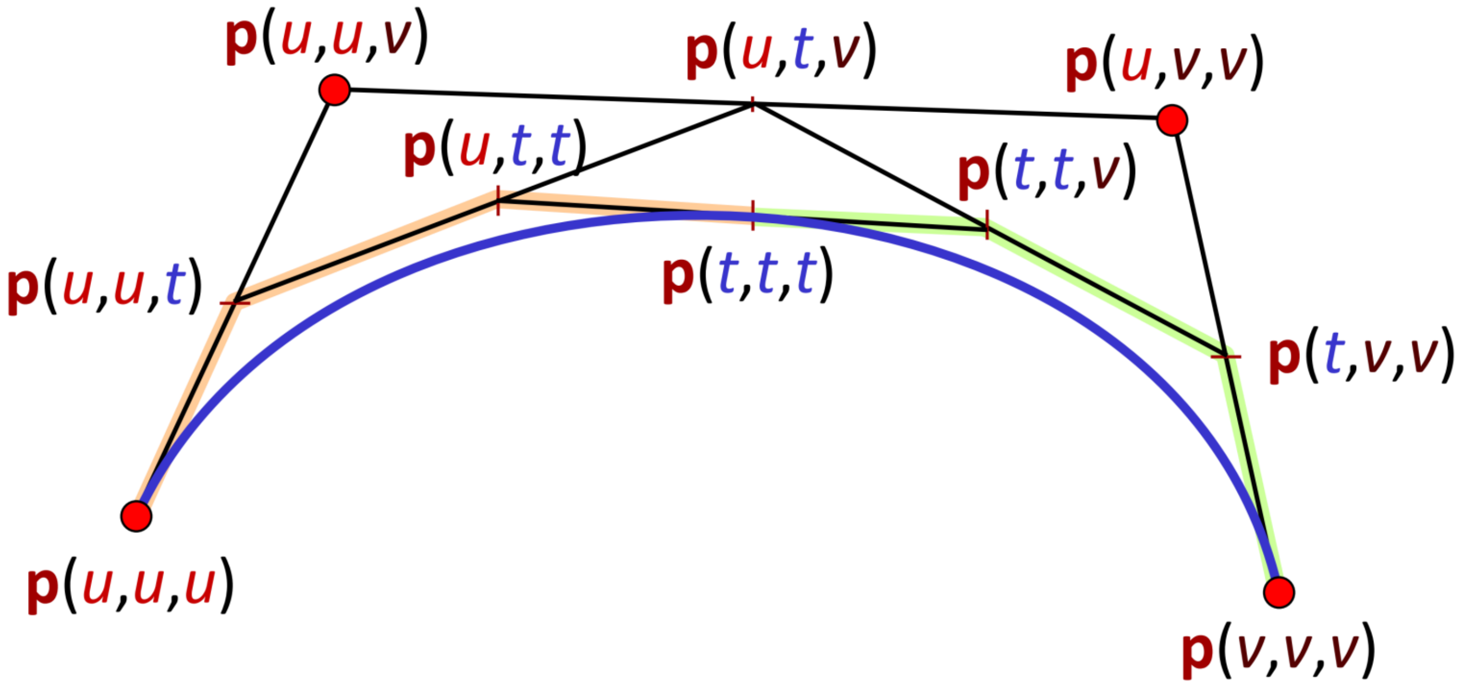
More Observations



Derivatives:

- de Casteljau algorithm computes tangent vectors at any points as a byproduct
- Proportional to last line segment that is bisected

More Observations



Subdivision:

- After each de Casteljau step, we obtain two new control polygons left and right of $f(t)$ describing the same curve.
- We can divide a segment into two
- Recursive subdivision can be used for rendering

Observations

Remark: The de Casteljau algorithm for computing

- Derivatives
  - At end points
  - At inner points t
- Subdivisions

Hold for Bézier curves of arbitrary degree $d \geq 1$

(General degree derivatives: $F'(t)/d$)

More Bézier Curve Properties…

General degree elevation

- Increase the degree of a Bézier curve segment by one
- What are the new control points?

Polar forms:

- Old curve: $b(t_1, \dots, t_d)$
- New curve: $b^{(+1)}(t_1, \dots, t_{d+1}) = \frac{1}{d+1} \sum_{i=1}^{d+1} b(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{d+1})$

leave out t_i

Degree Elevation

$$\boldsymbol{b}^{(+1)}(0,\dots,0) = \frac{1}{d+1}\sum_{i=1}^{d+1} \boldsymbol{b}(0,\dots,0) = \boldsymbol{b}(0,\dots,0)$$

$$\boldsymbol{b}^{(+1)}(1,\dots,0) = \frac{1}{d+1}\boldsymbol{b}(0,\dots,0) + \frac{d}{d+1}\boldsymbol{b}(1,0,\dots,0)$$

$$\boldsymbol{b}^{(+1)}(1,1,0,\dots,0) = \frac{2}{d+1}\boldsymbol{b}(0,\dots,0) + \frac{d-1}{d+1}\boldsymbol{b}(1,1,0,\dots,0)$$

$$\boldsymbol{b}^{(+1)}(1,1,1,\dots,1,0) = \frac{d}{d+1}\boldsymbol{b}(1,\dots,1,0) + \frac{1}{d+1}\boldsymbol{b}(1,\dots,1)$$

$$\boldsymbol{b}^{(+1)}(1,\dots,1) = \boldsymbol{b}(1,\dots,1)$$

Degree Elevation

Result: new control points

$$\boldsymbol{p}_i^{(+1)} = \frac{i}{n+1}\boldsymbol{p}_{i-1} + \left(1 - \frac{i}{n+1}\right)\boldsymbol{p}_i,\ \ i = 0, \dots, n+1$$ (zero points if out of range)

Repeated degree elevation:

$$\boldsymbol{p}_i^{(+k)} = \sum_{i=1}^{d+1} \boldsymbol{p}_j \frac{\binom{d}{j}\binom{k}{i-j}}{\binom{d+k}{j}}$$ (proof by induction)

Repeating degree elevation let the control point converge to the Bézier curve in the limit

Change of basis, the easy way

- Given: Polynomial $p(t)$ of degree n in monomial form
- Wanted: coefficients of the same Bézier curve

Change of basis, the easy way

- Given: Polynomial $p(t)$ of degree n in monomial form
- Wanted: coefficients of the same Bézier curve
- Solution:

$$p(t) \rightarrow b(t_1, \dots, t_n)$$

coefficients: $b(\underbrace{0, \dots, 0}_{n+1-k}, \underbrace{1, \dots 1}_{k})$

- This is a direct implication of de Casteljau in polar form

Example

- Example: Bézier coefficients of $p(t) = 1 + 2t + 3t^2 - t^3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5/3 \\ 10/3 \\ 5 \end{pmatrix}$$

Example

- Using polar forms

$$b(t_0, t_1, t_2) = 1 + 2\frac{t_0 + t_1 + t_2}{3} + 3\frac{t_0 t_1 + t_1 t_2 + t_0 t_2}{3} - t_0 t_1 t_2$$

$$b(0,0,0) = 1, \qquad b(0,0,1) = \frac{5}{3}, \qquad b(0,1,1) = \frac{10}{3}, \qquad b(1,1,1) = 5$$

Polar Forms & Blossoms

B-Splines

B-Spline Curves in Polar Form

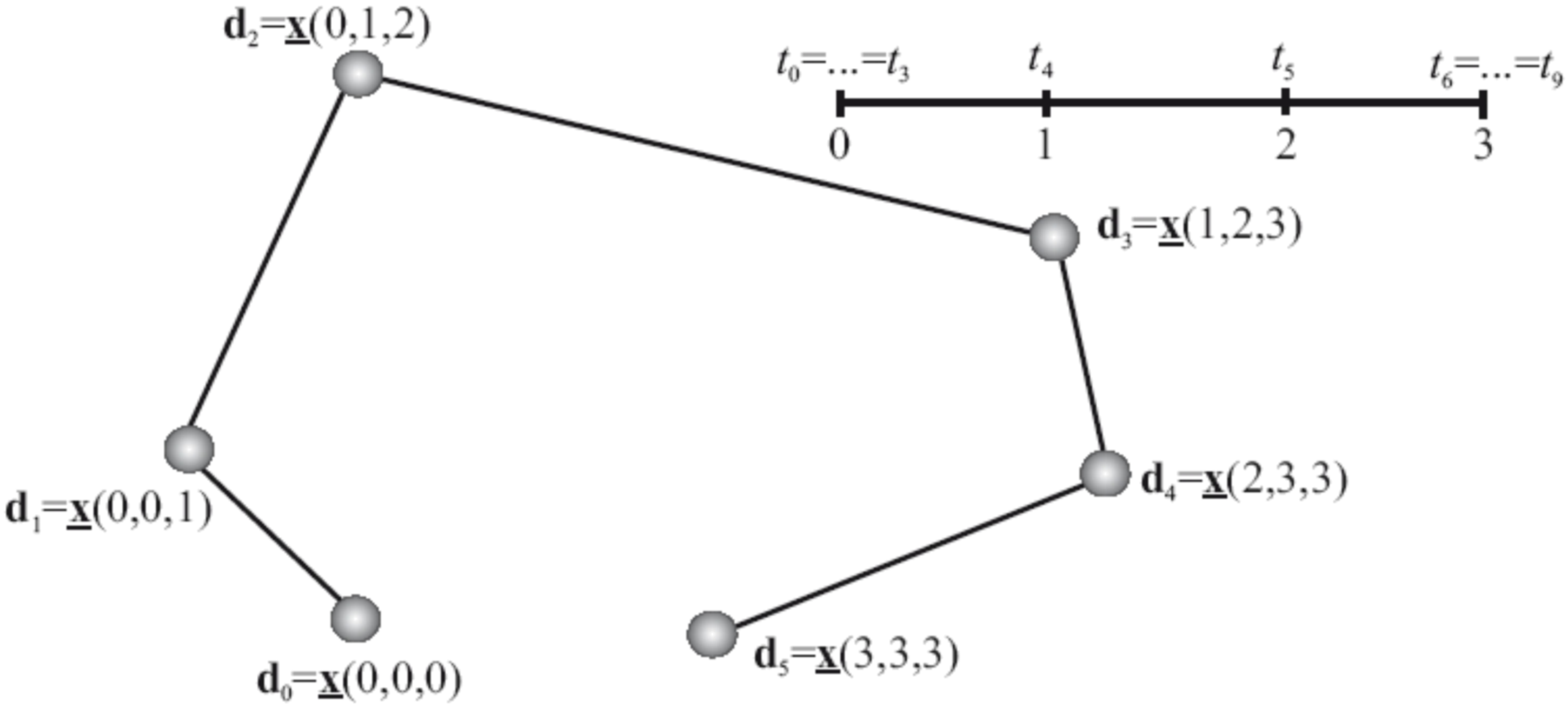
An unique description in polar form exists for piecewise polynomial curves as well

- Given: B-Spline curve $\boldsymbol{x}$ of order k
  - with knot vector $T = (t_0, \dots, t_{n+k})$
  - and de Boor points $\boldsymbol{d}_0, \dots, \boldsymbol{d}_n$
- Let $\underline{\boldsymbol{x}}$ be the polar form of $\boldsymbol{x}$
- Then the de Boor points of $\boldsymbol{x}$ are given as:
  - $\boldsymbol{d}_i = \underline{\boldsymbol{x}}(t_{i+1}, \dots, t_{i+k-1})$

we just use consecutive knot values as blossom arguments

B-Spline Curves in Polar form

- **Example:** $k = 4, n = 5$



De Boor Algorithm in Polar Form

de Boor algorithm in polar form

- To Define the curve value at t, we look for the relevant part of the de Boor polygon, i.e.,

  → the (partial) knot sequence

$$r_{k-1} \leq \cdots \leq r_1 < s_1 \leq \cdots \leq s_{k-1}$$

with $r_1 \leq t < s_1$

De Boor Algorithm in Polar Form

de Boor algorithm in polar form

- Then the intermediate points of the de Boor algorithm result are

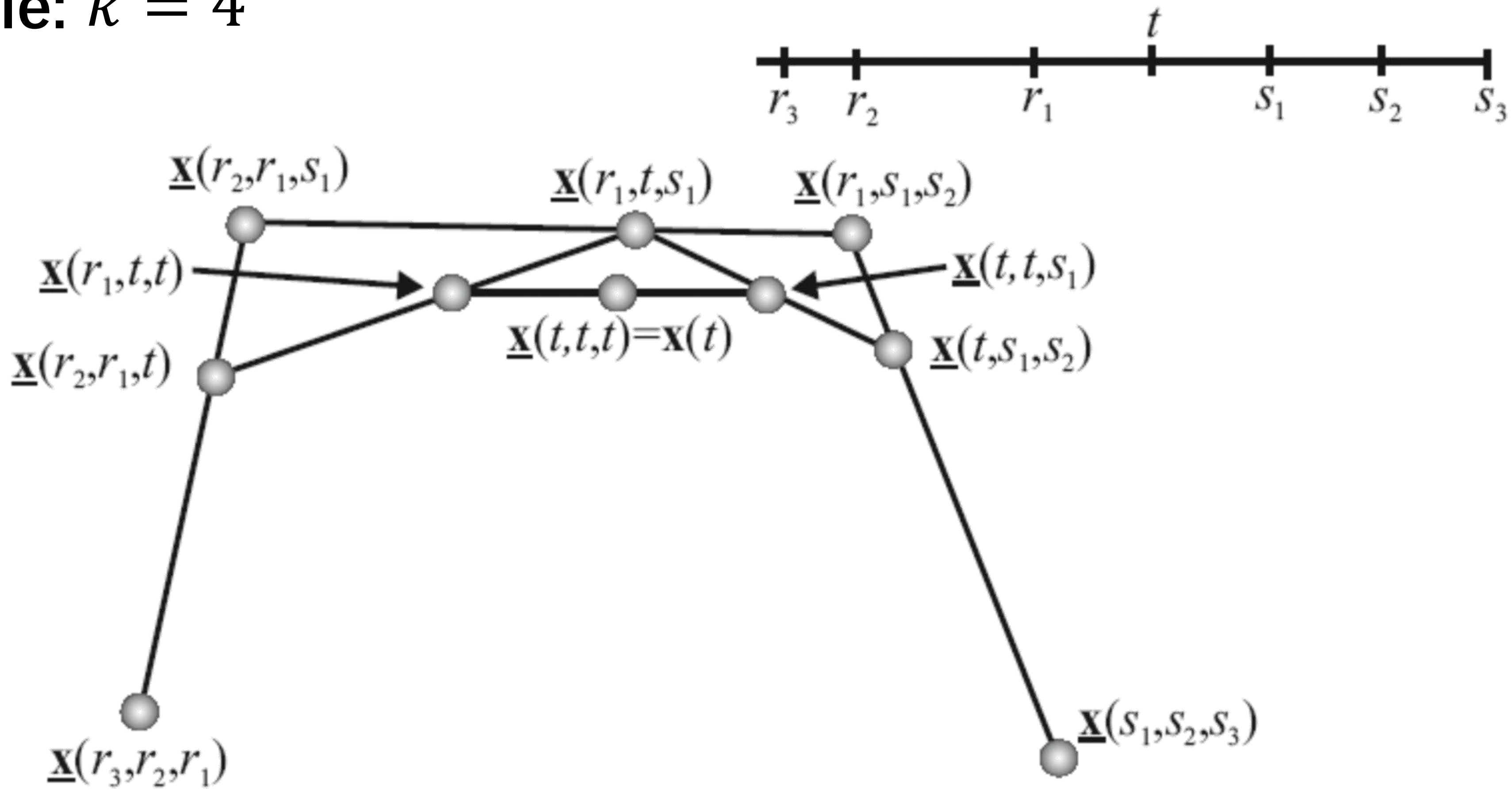
$$d_j^l(t) = \underline{x}\left(r_1, \ldots, r_{k-1-l-j}, \underbrace{t, \ldots, t}_{l}, s_1, \ldots, s_j\right)$$

and the desired curve point is

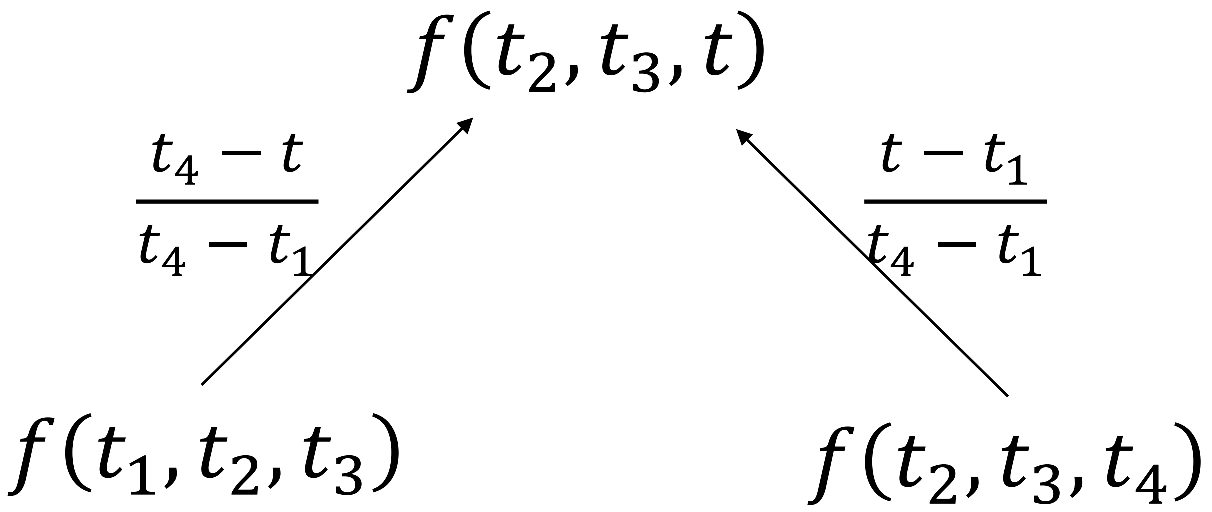
$$x(t) = \underline{x}(\underbrace{t, \ldots, t}_{k-1})$$

De Boor Algorithm in Polar form

Example: $k = 4$

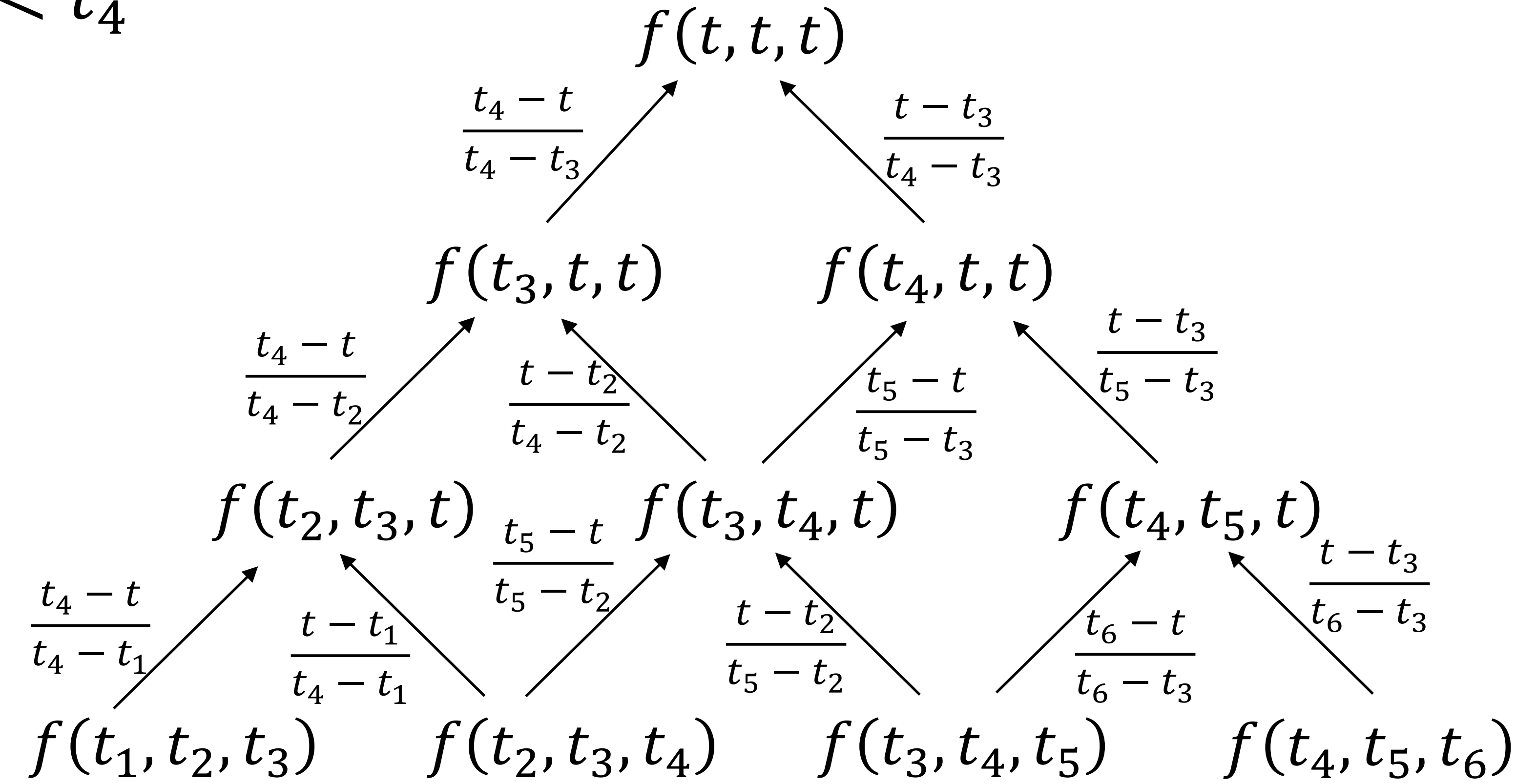


Key observation



De Boor Alg. In Polar form

- For $t_3 \le t < t_4$



De Boor Algorithm in Polar Form

for $2 \le t < 3$

$$\frac{3-t}{3}\,\mathbf{p}(0,1,2) + \left[1-\frac{3-t}{3}\right]\mathbf{p}(3,1,2)$$

$$\frac{4-t}{3}\,\mathbf{p}(1,2,3) + \left[1-\frac{4-t}{3}\right]\mathbf{p}(4,2,3)$$

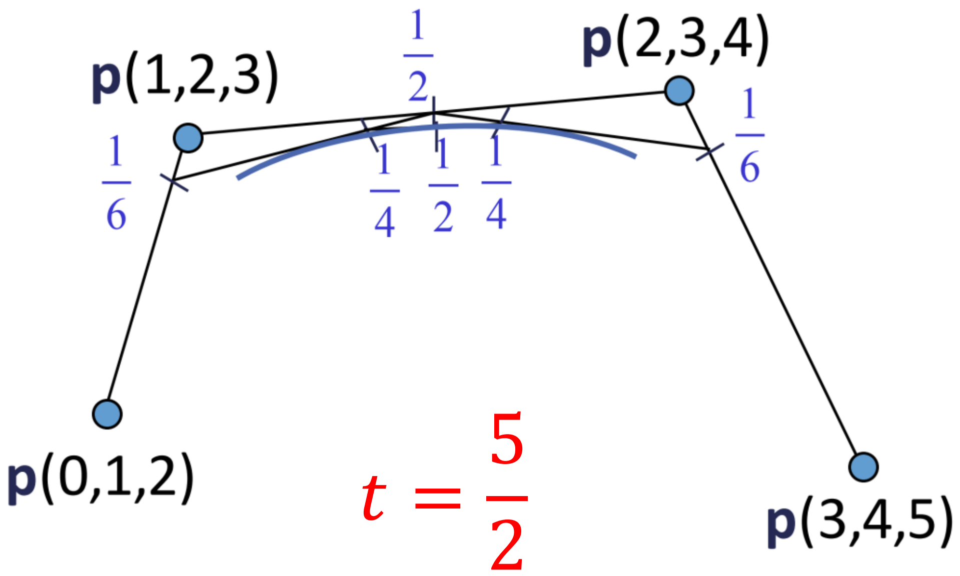
$$\frac{5-t}{3}\,\mathbf{p}(2,3,4) + \left[1-\frac{5-t}{3}\right]\mathbf{p}(5,3,4)$$

$$\frac{3-t}{2}\,\mathbf{p}(t,1,2) + \left[1-\frac{3-t}{2}\right]\mathbf{p}(t,3,2)$$

$$\frac{4-t}{2}\,\mathbf{p}(t,3,2) + \left[1-\frac{4-t}{2}\right]\mathbf{p}(t,3,4)$$

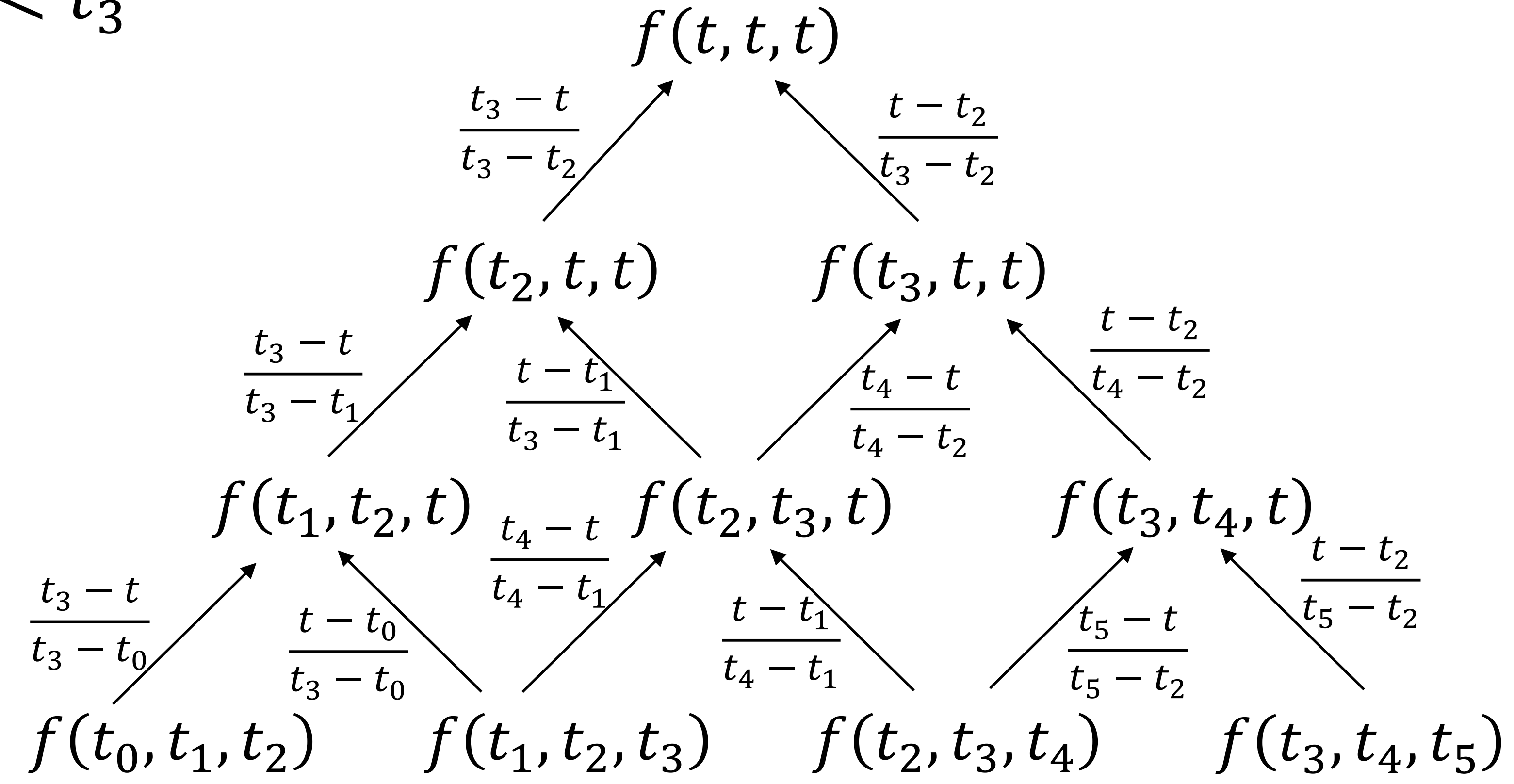
$$\frac{3-t}{1}\,\mathbf{p}(t,t,2) + \left[1-\frac{3-t}{1}\right]\mathbf{p}(t,t,3)$$

$$\mathbf{p}(t,t,t)$$



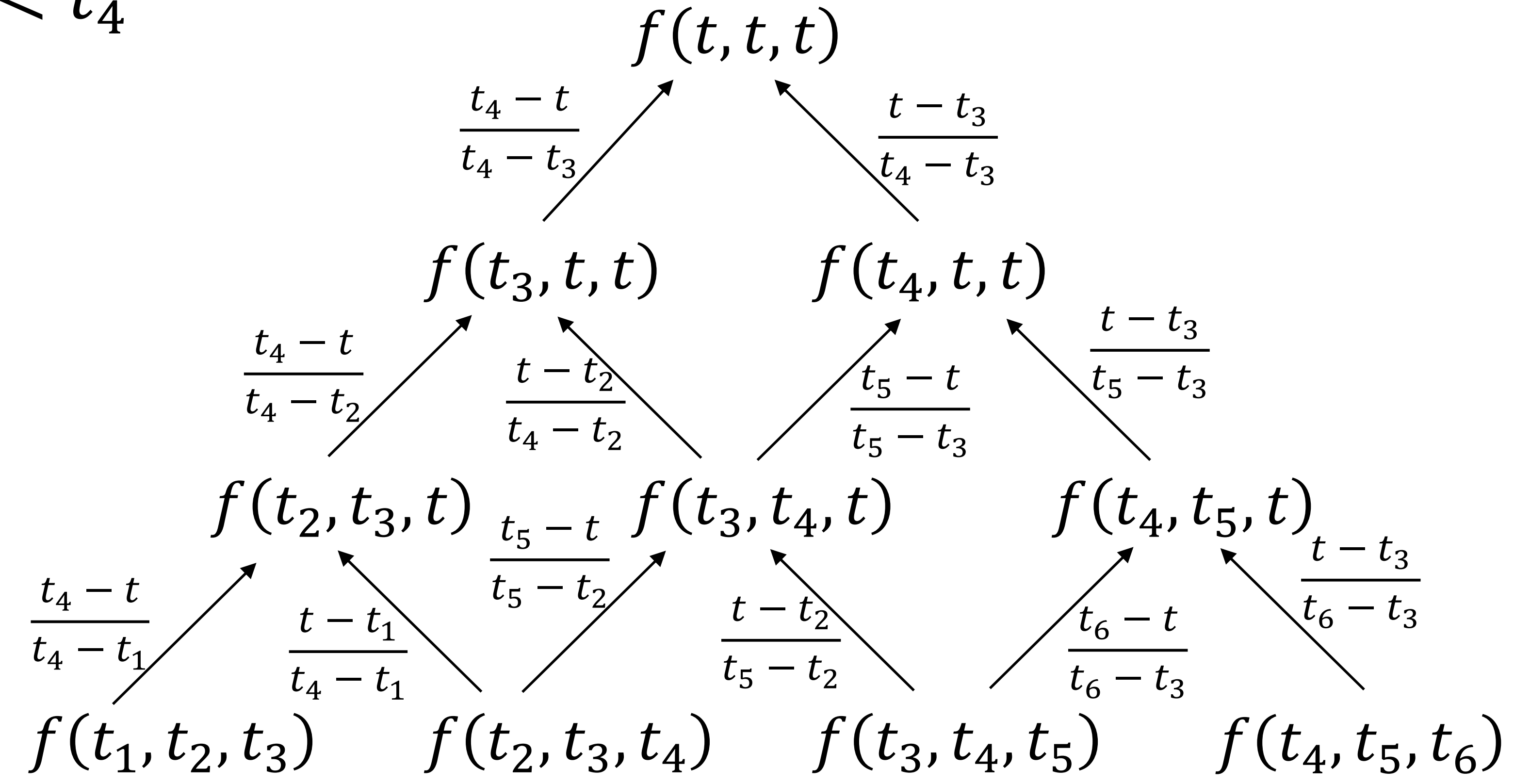
B-Splines in Polar form

- For $t_2 \leq t < t_3$

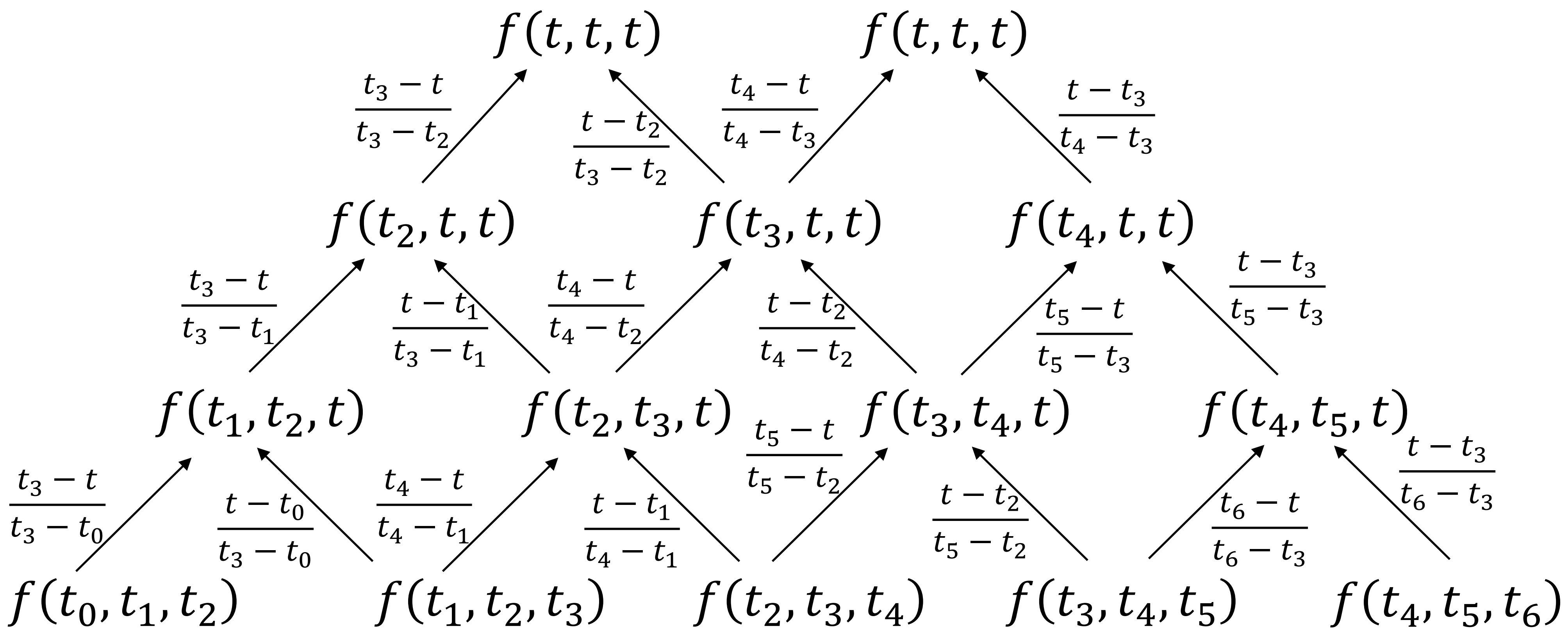


B-Splines in Polar form

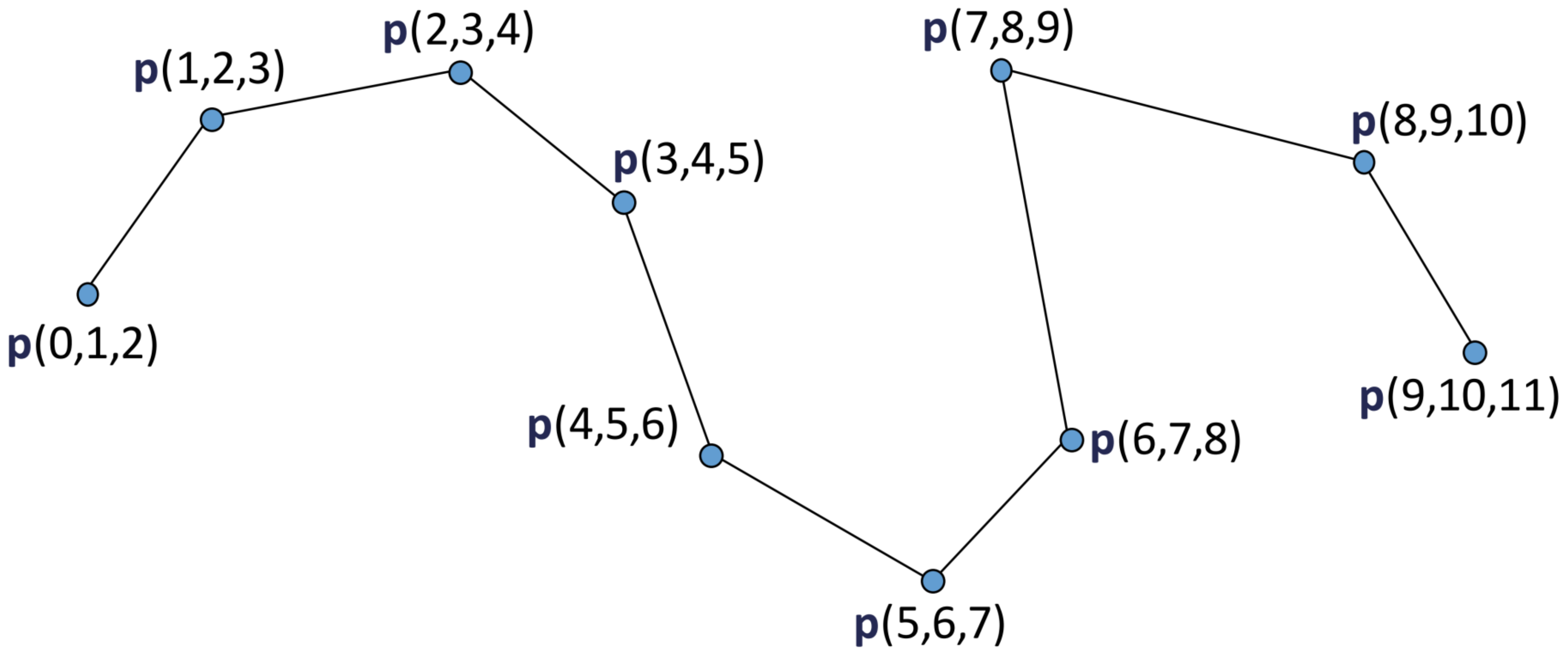
- For $t_3 \leq t < t_4$



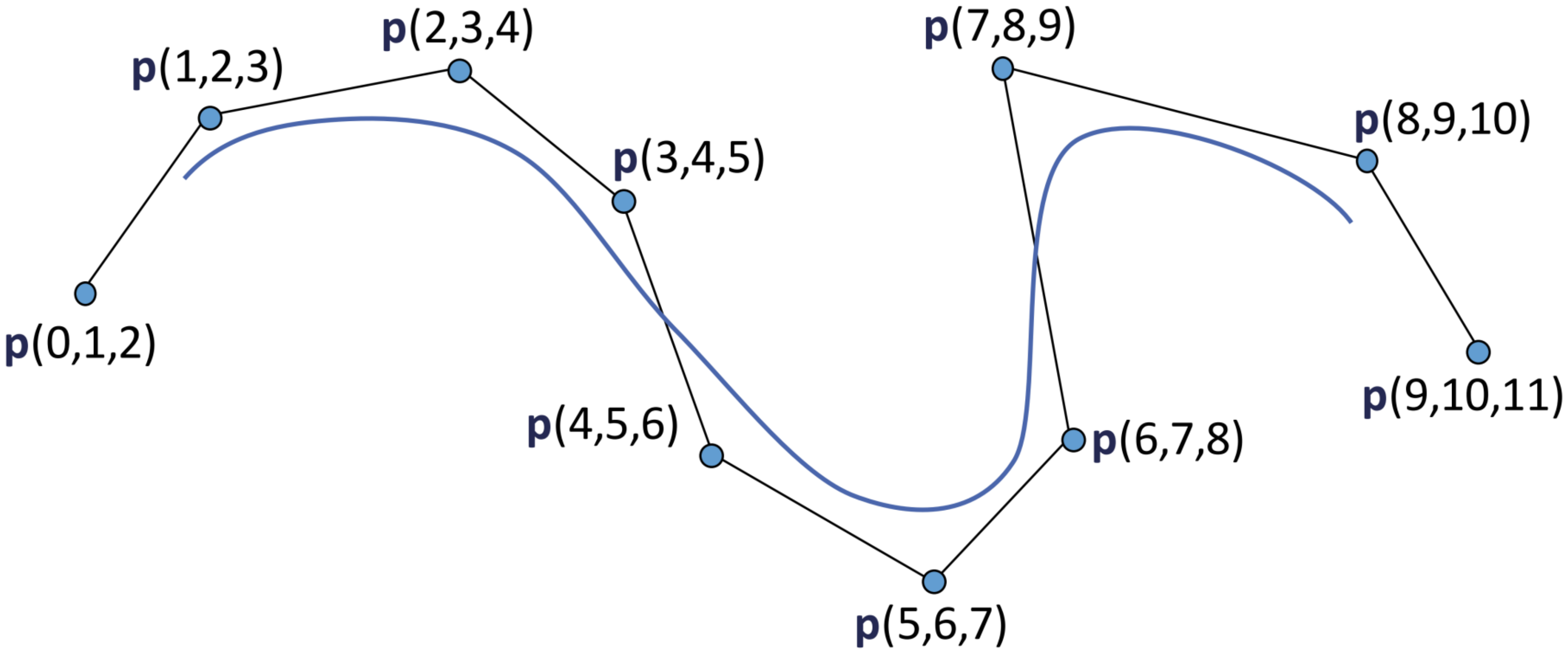
B-Splines in Polar form



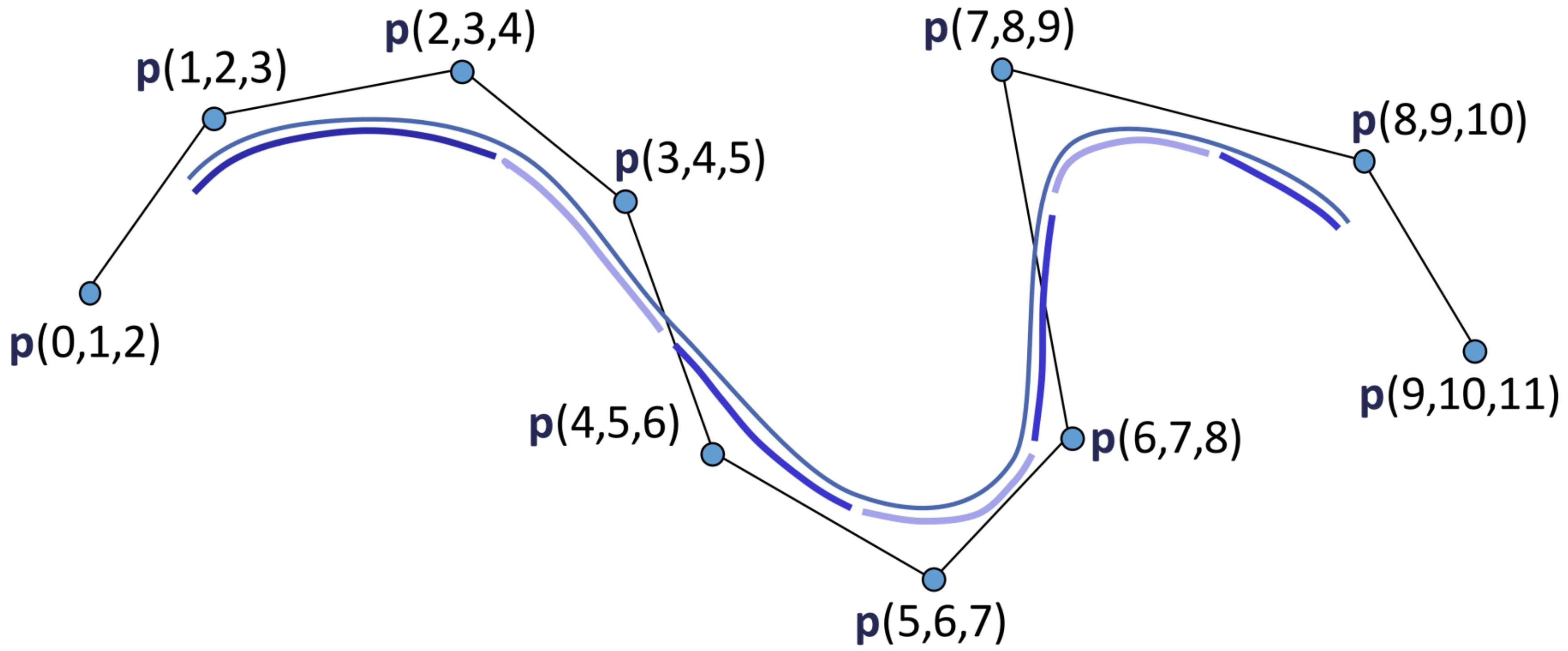
Example: General Case



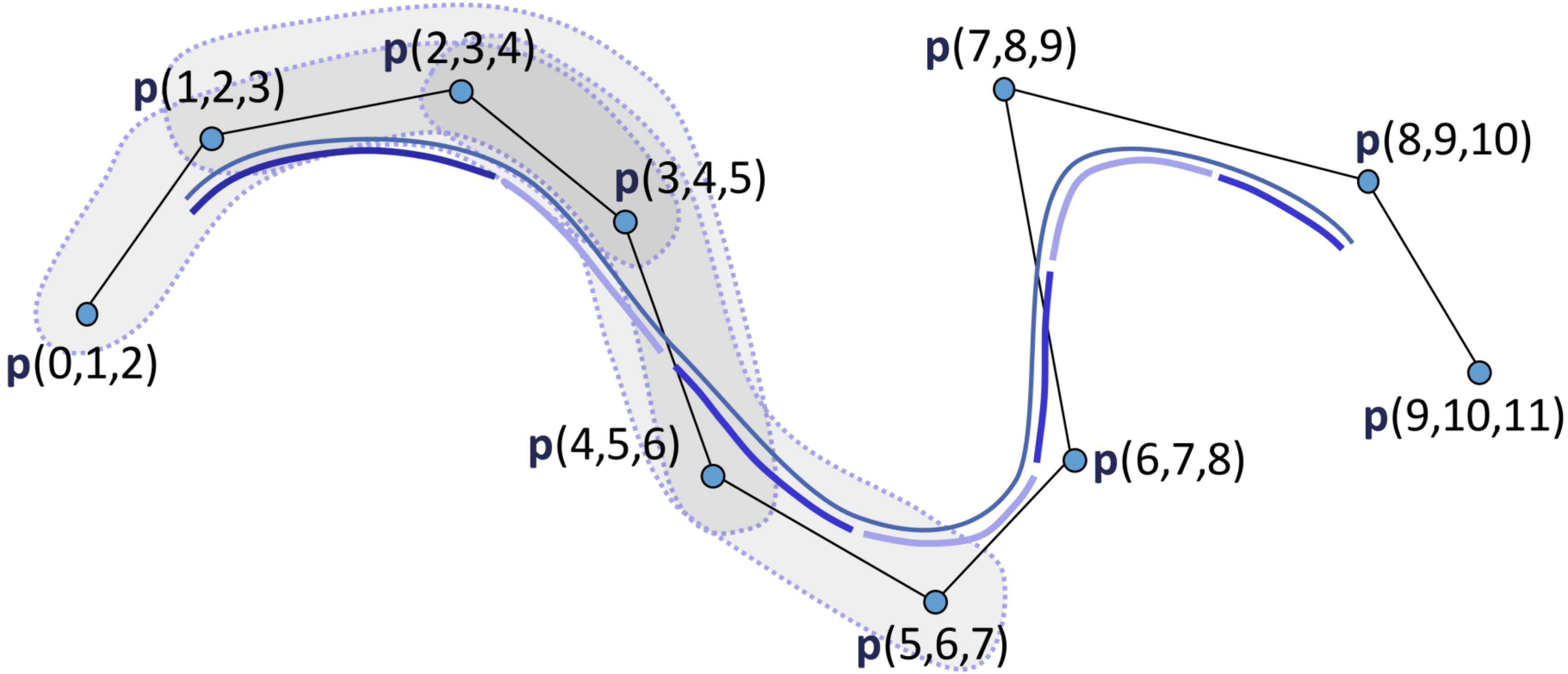
Example: General Case



Example: General Case



Example: General Case



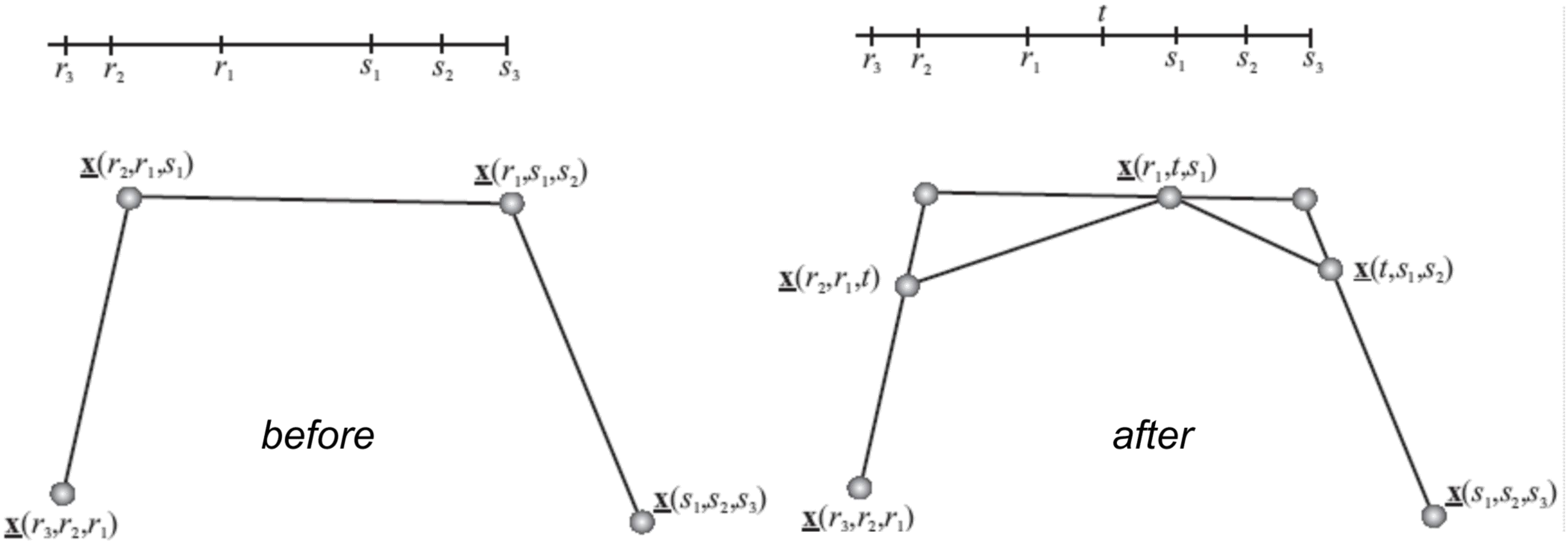
Knot Insertion

Knot insertion

- Increases the number of curve segments, but not the polynomial degree
- Insertion at t: First step of the de Boor algorithm!

Knot Insertion

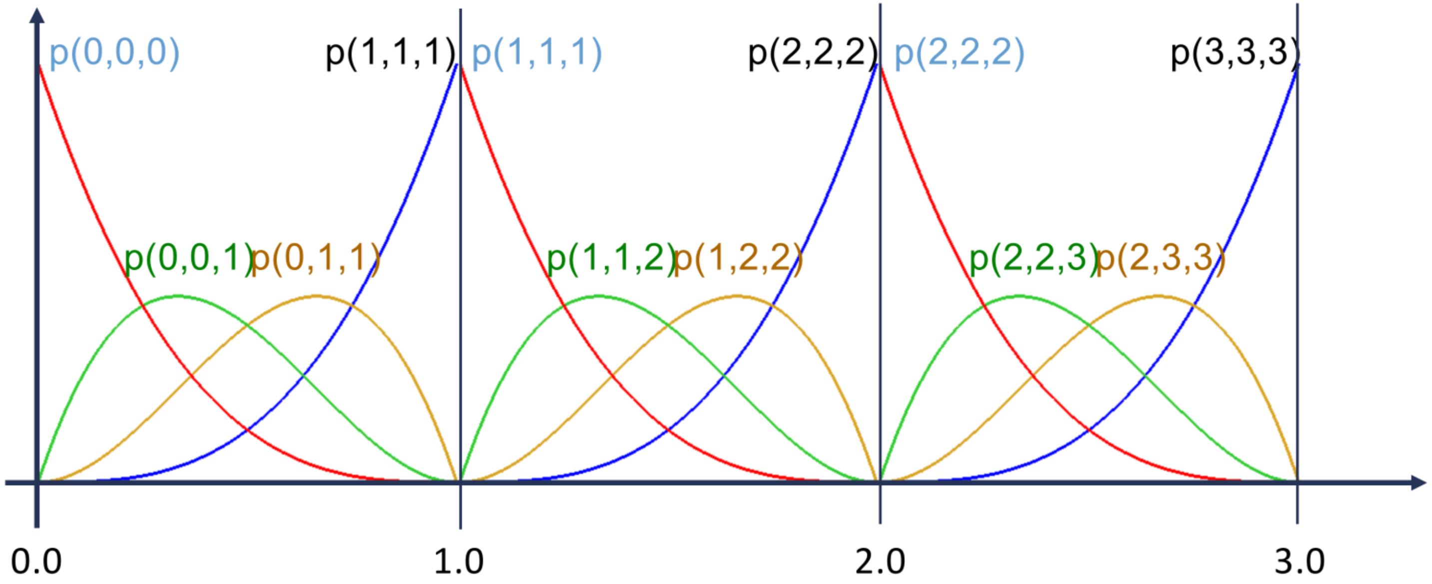
- **Insertion of knots** Example



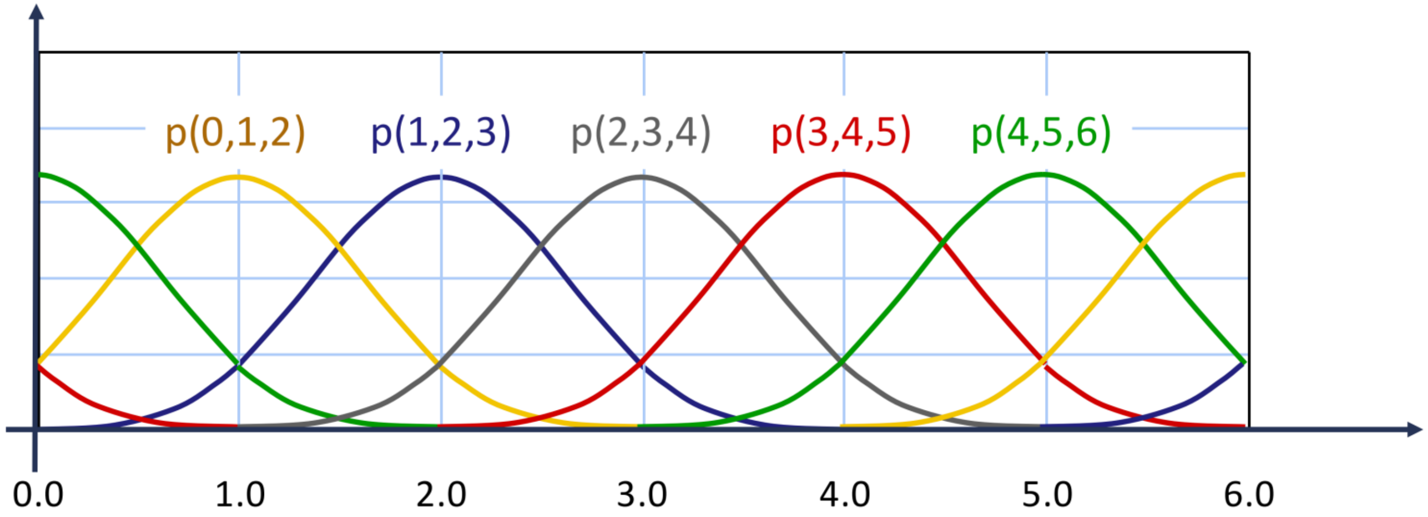
Polar Forms & Blossoms

Illustrations

Structure

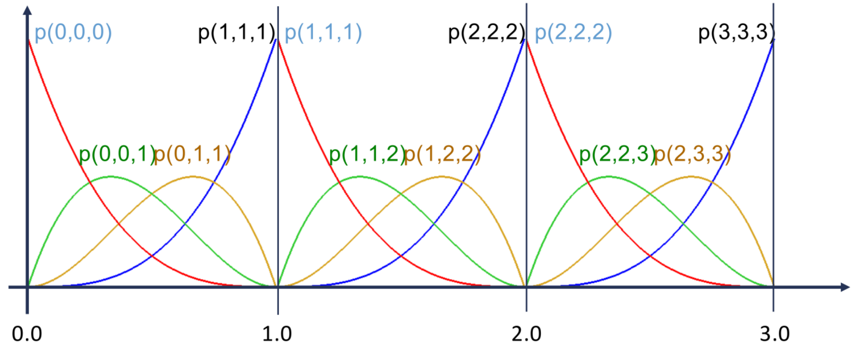


Bézier Spline

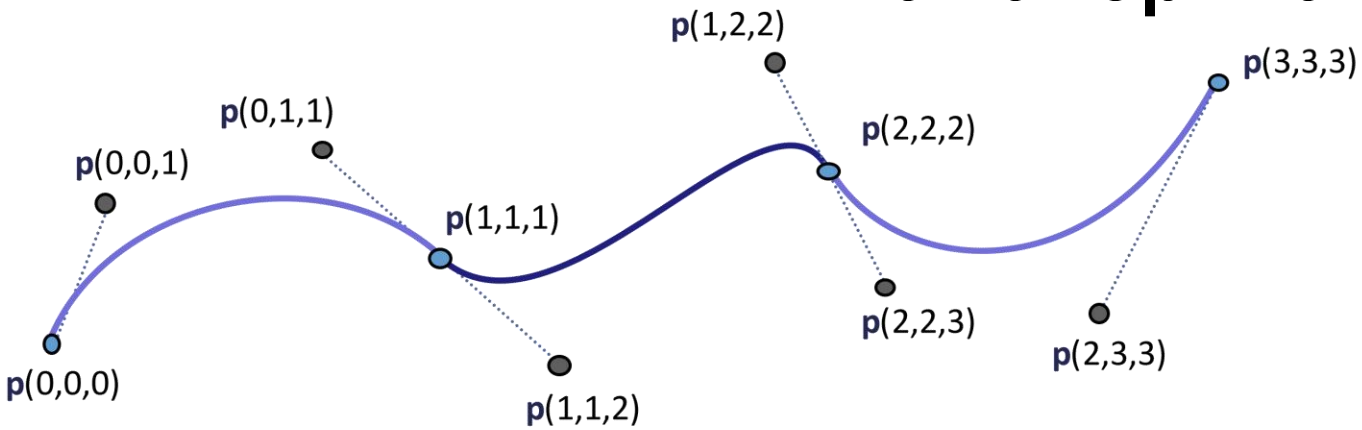


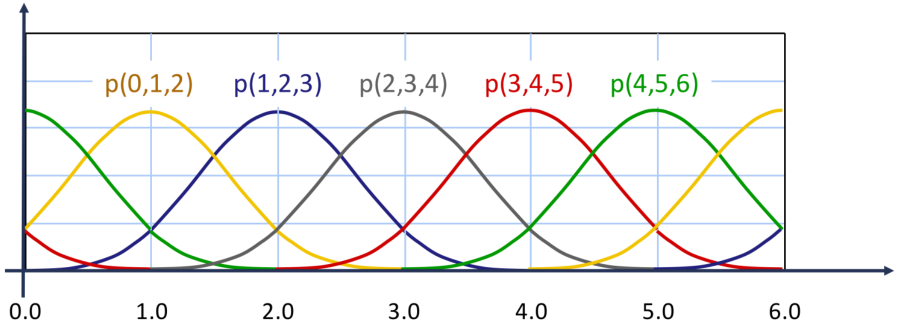
B-Spline

Structure



Bézier Spline





B-Spline