计算机辅助几何设计
2023秋学期

Rational Spline Curves

Projective Geometry · Rational Bézier Curves · NURBS

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Some Projective Geometry

Projective Geometry

- **A very short overview of projective geometry**
  - The computer graphics perspective
  - Formal definition

Homogeneous Coordinates

- **Problem**
  - Linear maps (matrix multiplication in $\mathbb{R}^d$) can represent ⋯
    - Rotations
    - Scaling
    - Shearing
    - Orthogonal projection
  - ⋯but not:
    - Translations
    - Perspective projections
  - This is a problem in computer graphics:
    - We would like to represent compound operations in a single closed representation

Translations

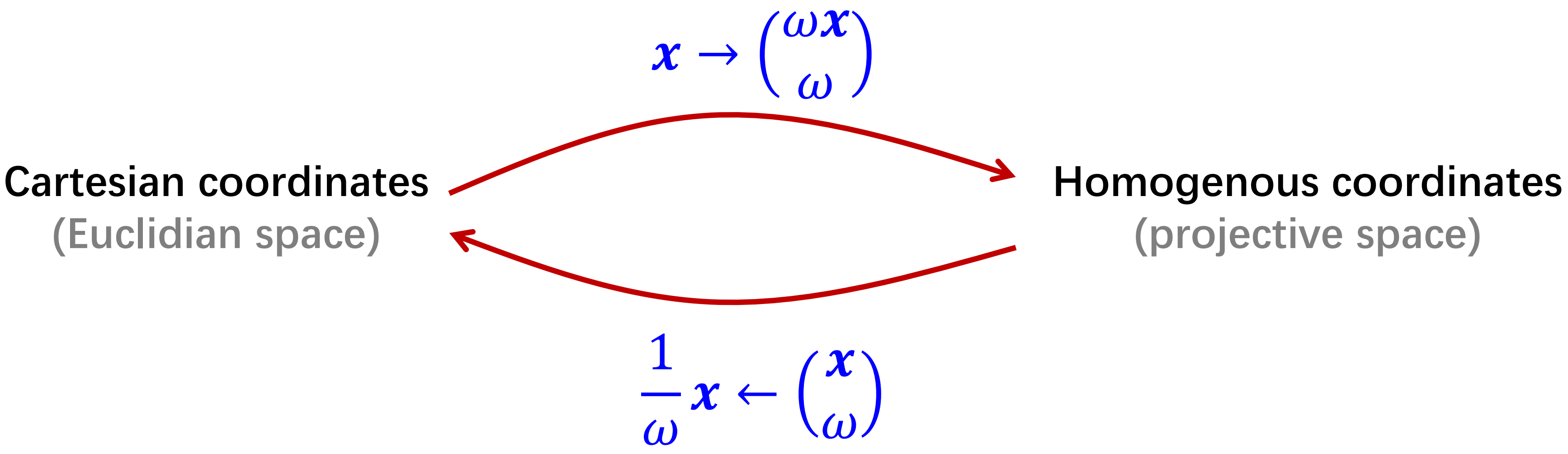
- **“Quick Hack” #1: Translations**
  - Linear maps cannot represent translations:
    - Every linear map maps the zero vector to zero $M0 = 0$
    - Thus, non-trivial translations are non-linear
  - Solution:
    - Add one dimension to each vector
    - Fill in a one
    - Now we can do translations by adding multiples of the one:

$$Mx = \begin{pmatrix} r_{11} & r_{21} & t_x \\ r_{12} & r_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \\ 1 \end{pmatrix}$$

Normalization

Problem: what if the last entry is not 1?

- It's not a bug, it's a feature…
- If the last component is not 1, divide everything by it before using the result



Notation

Notation:

- The extra component is called the *homogenous component* of the vector.
- It is usually denoted by ω:
  - 2D case:

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \omega x \\ \omega y \\ \omega \end{pmatrix}$$

  - 3D case:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} \omega x \\ \omega y \\ \omega z \\ \omega \end{pmatrix}$$

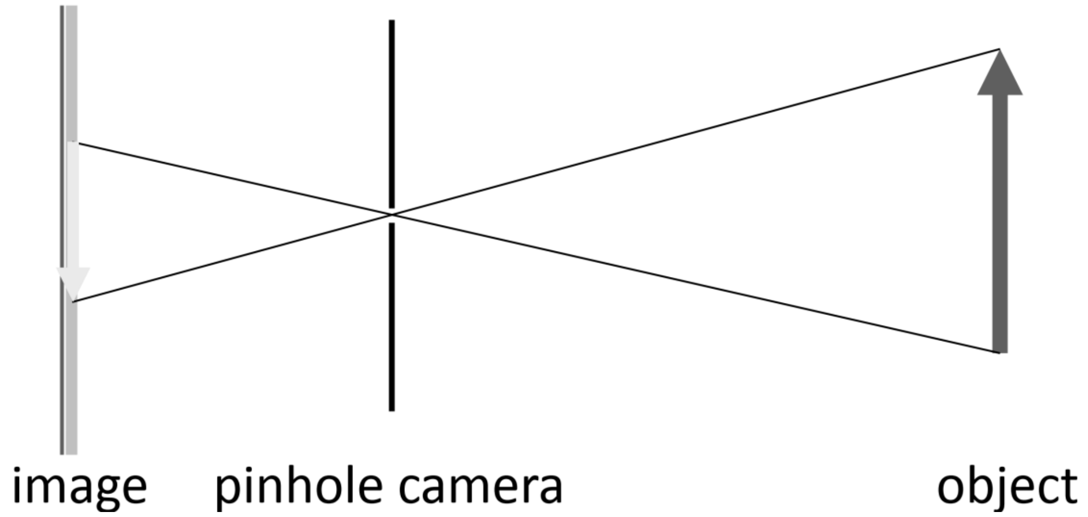
  - Ceneral case:

$$\boldsymbol{x} \rightarrow \begin{pmatrix} \omega \boldsymbol{x} \\ \omega \end{pmatrix}$$

Perspective Projections

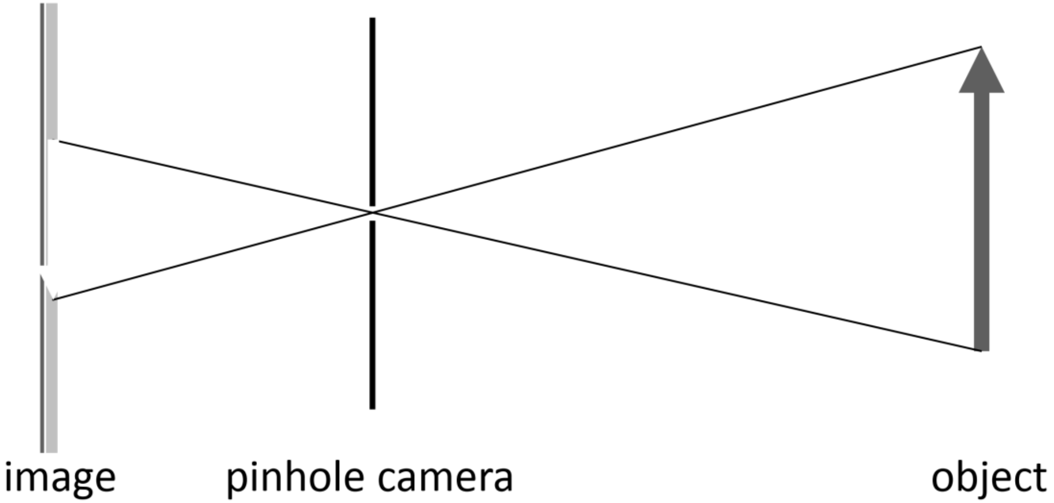
New Feature: Perspective projections

- Very useful for 3D computer graphics
- Perspective projection (central projection)
  - involves divisions
  - can be packed into homogeneous component

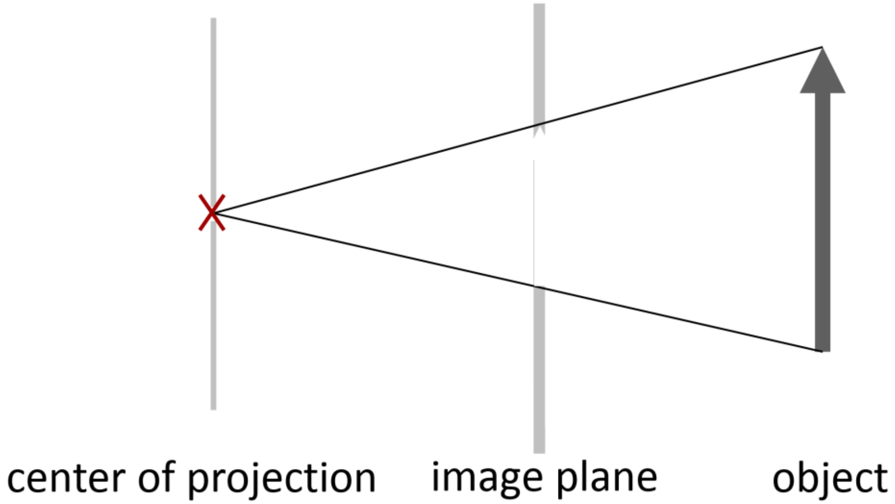


Perspective Projection

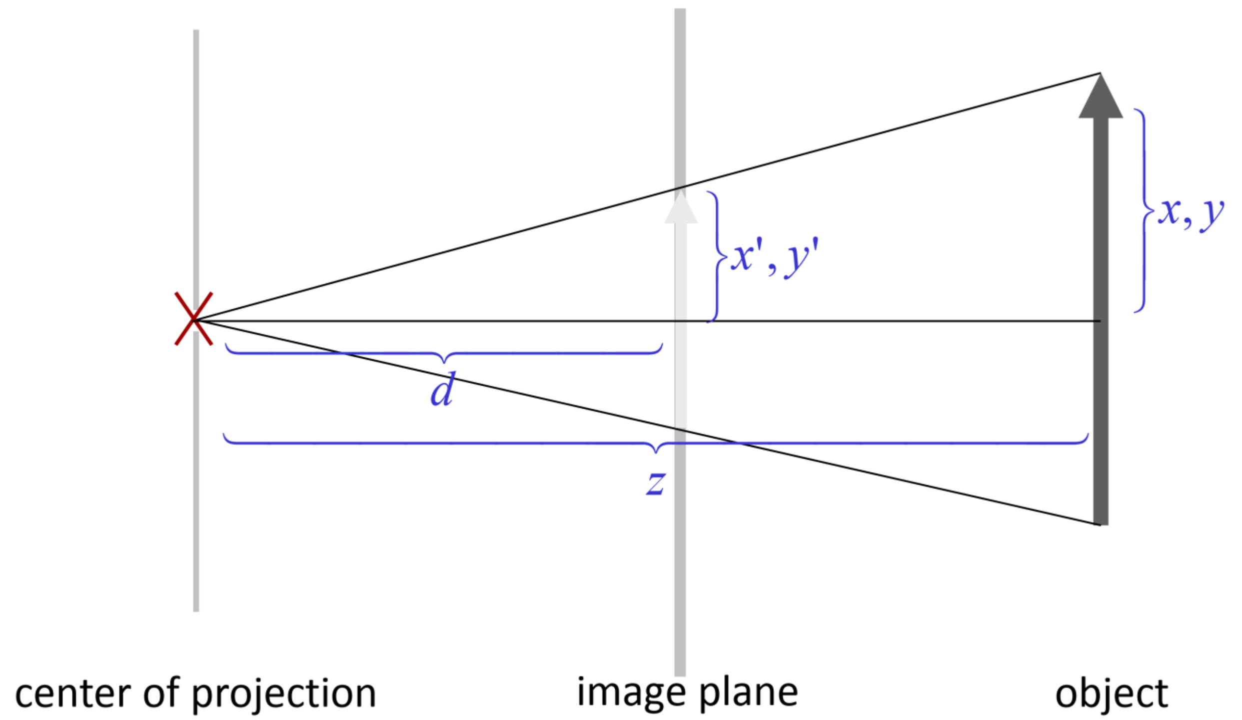
Physical camera:



Virtual camera:



Perspective Projection



Perspective projection: $x' = d\frac{x}{z}, \; y' = d\frac{y}{z}$

Homogenous Transformation

- **Projection as linear transformation in homogenous coordinates:**
  - Trick: Put the denominator into the ω component

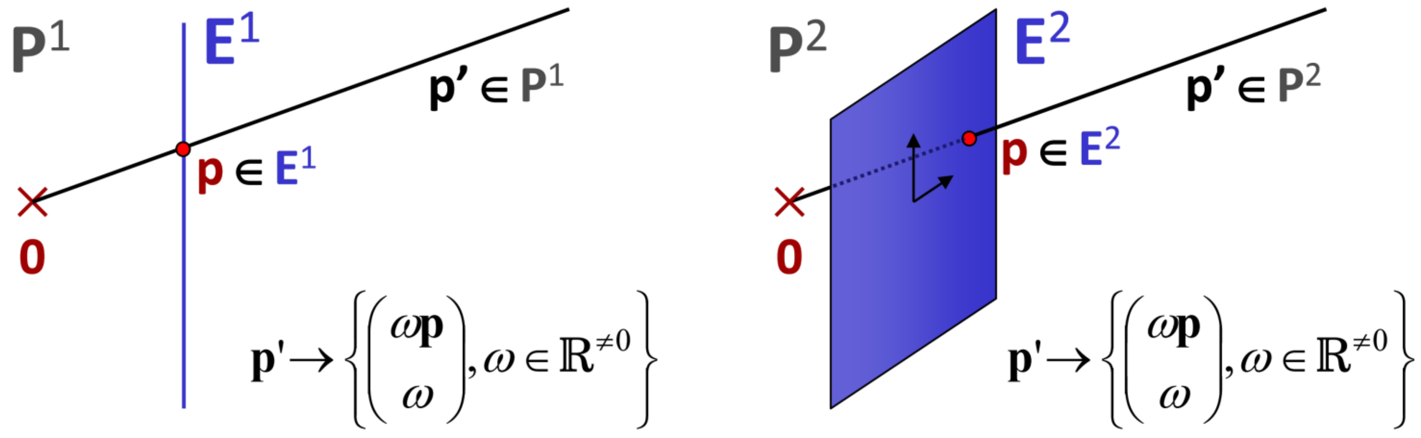
$$x' = d\frac{x}{z}, \quad y' = d\frac{y}{z}$$

$$\begin{pmatrix} x' \\ y' \\ z' \\ \omega' \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Formal Definition

Projective Space P^d

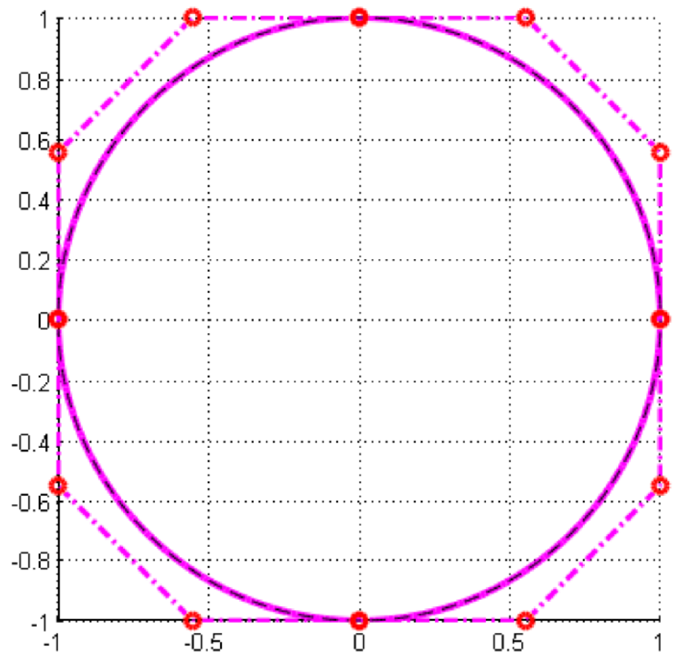
- Embed Euclidian space E^d
  - Into $d+1$ dimensional Euclidian space at $\omega = 1$
  - Additional dimension usually named ω
- Identify all points on lines through the origin
  - *Representing* the same Euclidian point



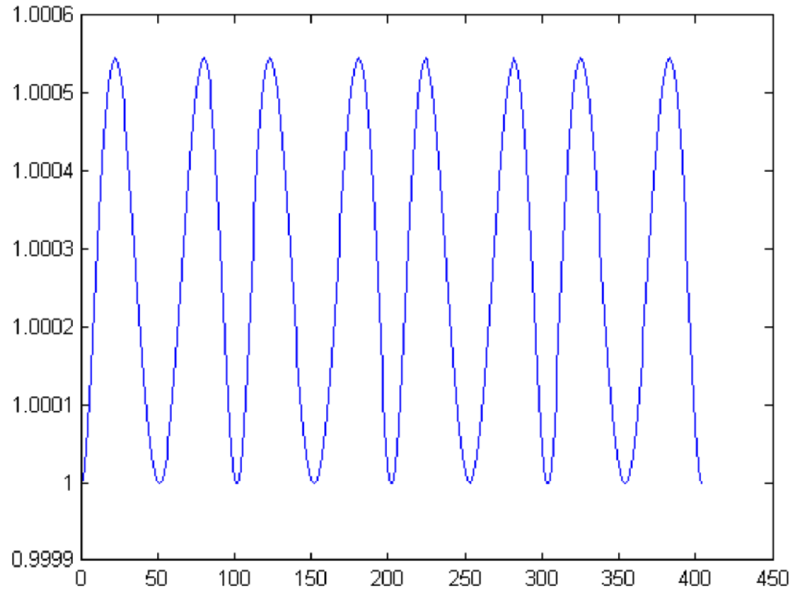
Question

Can we represent a circle arc using a Bézier curve?

Approximation of Circle using Cubic Bézier



Evaluation of $\left(x^2 + y^2\right)$ for points on the Bézier curve



Rational Curves

- **Motivation**
  - Bézier and B-spline curves **cannot** represent conic sections (circles, hyperbolas, etc.)
  - But we require those for some tasks
- **Goal**
  - Uniform and easily manageable description of polynomial curves and conic sections
- **Idea**
  - Control points are equipped with weights…but not any weights!

Planetarium of the St. Louis Science Center

Tycho Brahe Planetarium, Copenhagen

Quadrics and Conics

Modeling Wish List

We want to model:

- Circles (surfaces: Spheres)
- Ellipses (surfaces: Ellipsoids)
- And segments of those
- Surfaces: Objects with circular cross section
  - Cylinders
  - Cones
  - Surfaces of revolution (lathing)

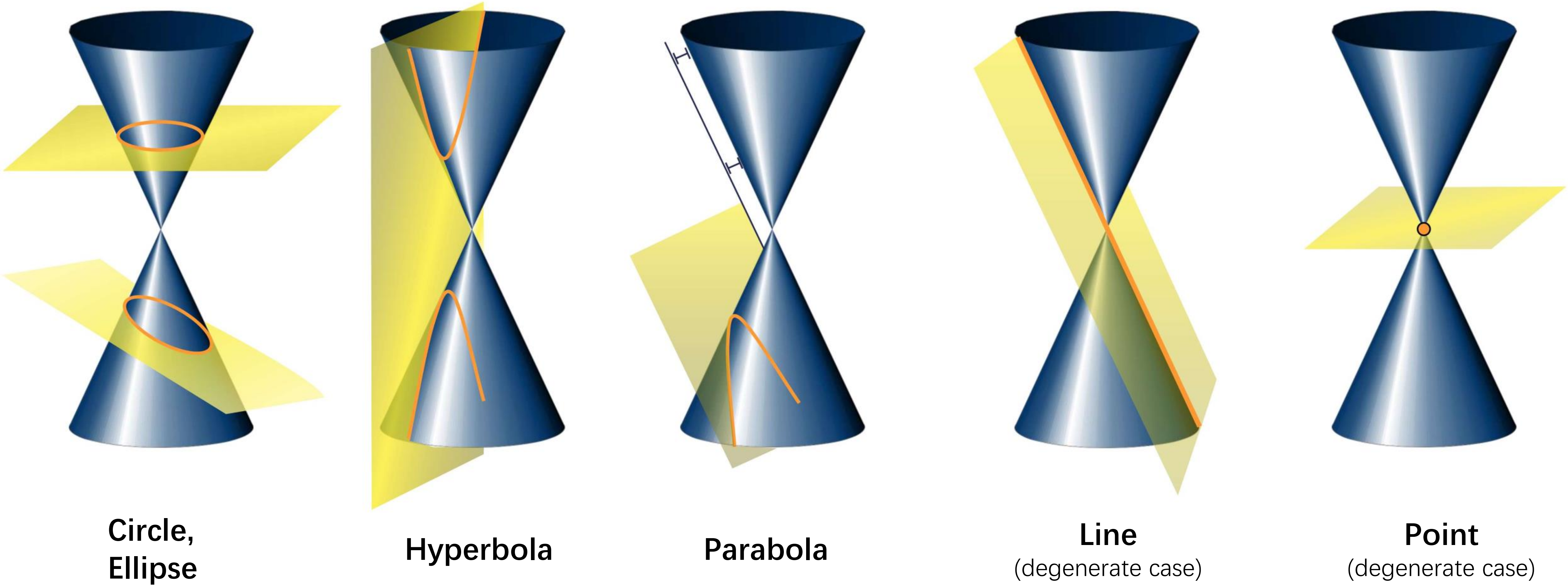
These objects cannot be represented exactly by piecewise polynomials (they are only approximated)

Conical Sections

Classic description of such objects:

- Conical sections (conics)
- Intersections of a cone and a plane
- Resulting Objects:
  - Circles
  - Ellipses
  - Hyperbolas
  - Parabolas
  - Points
  - Lines

Conic Sections



Implicit Form

Implicit quadrics:

- Conic sections can be expressed as zero set of a quadratic function:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$\Leftrightarrow \boldsymbol{x}^T \begin{bmatrix} a & {}^{b}/_{2} \\ {}^{b}/_{2} & c \end{bmatrix} \boldsymbol{x} + [d \quad e]\boldsymbol{x} + f = 0$$

- Easy to see why:

  Implicit eq. for a cone: $Ax^2 + by^2 = z^2$

  Explicit eq. for a plane: $z = Dx + Ey + F$

  Conical Section: $Ax^2 + By^2 = (Dx + Ey + F)^2$

Quadrics & Conics

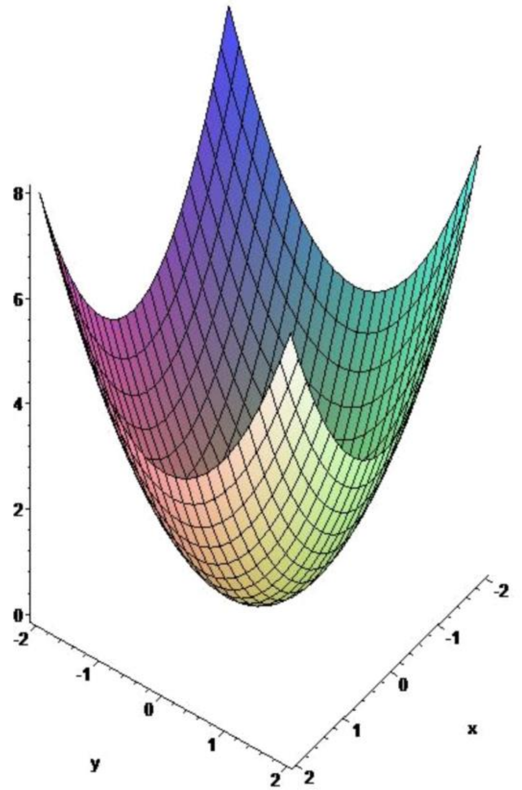
Quadrics:

- Zero sets of quadratic functions (any dimension) are called *quadrics*:

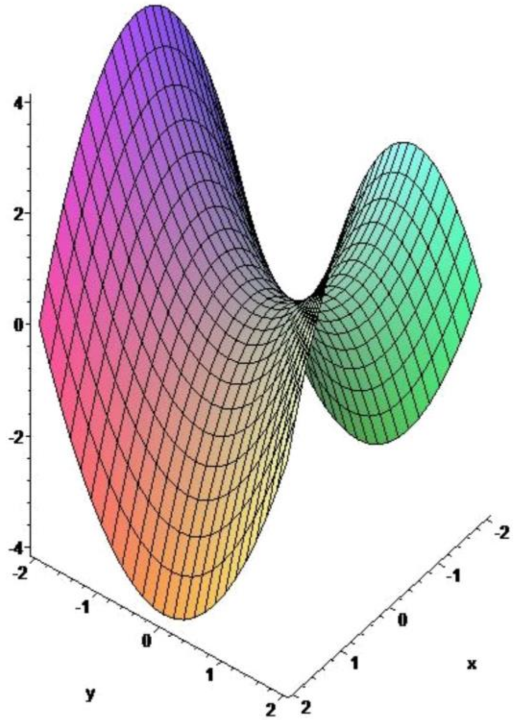
$$\{\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{x}^T \boldsymbol{M} \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + \boldsymbol{c} = \boldsymbol{0}\}$$

- *Conics* are the special case for $d = 2$

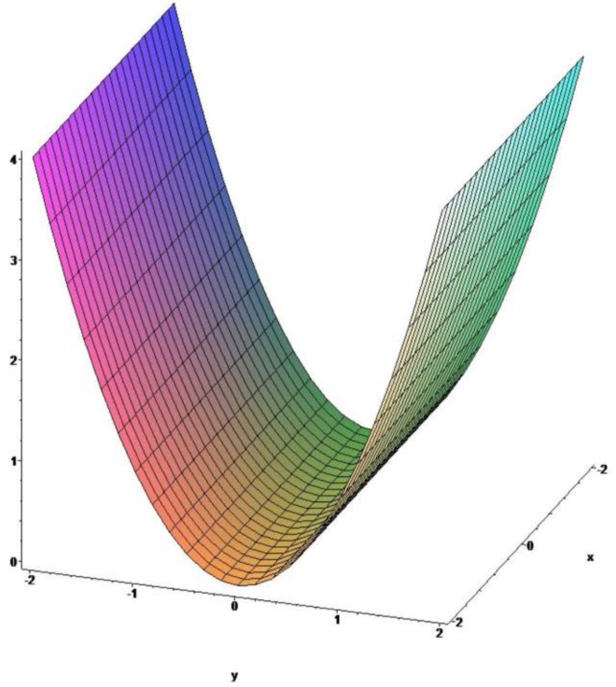
Shapes of Quadratic Polynomials



$\lambda_1 = 1, \lambda_2 = 1$



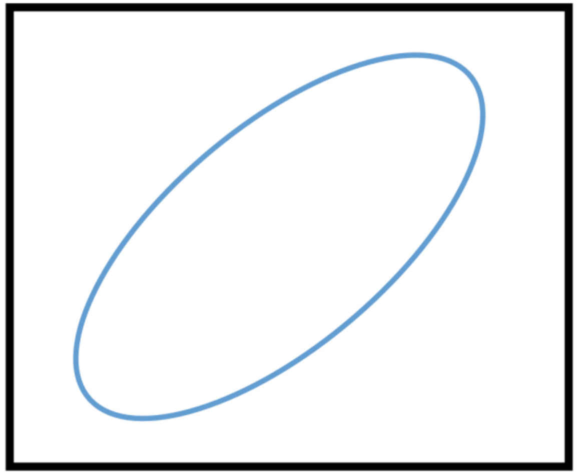
$\lambda_1 = 1, \lambda_2 = -1$



$\lambda_1 = 1, \lambda_2 = 0$

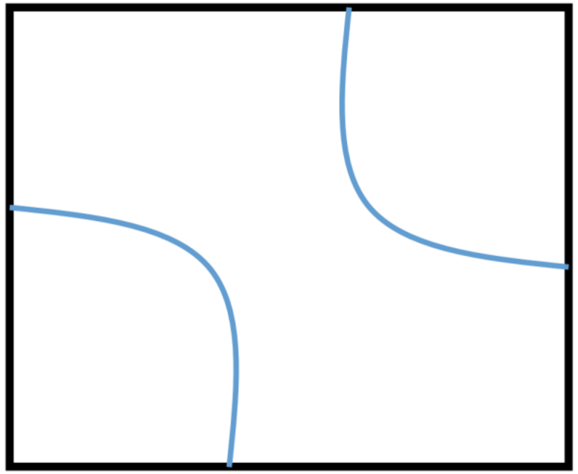
The Iso-Lines: Quadrics

Elliptic



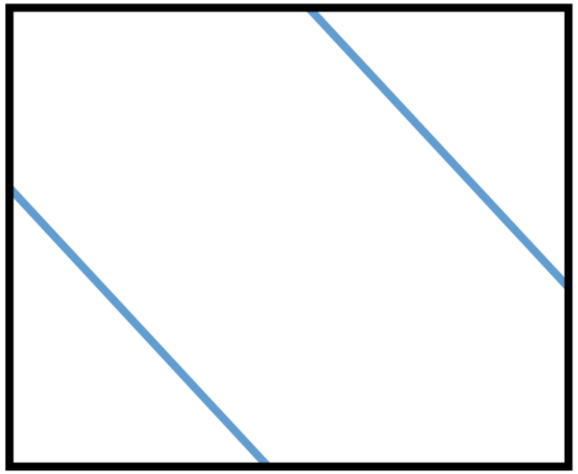
$\lambda_1 > 0, \lambda_2 > 0$

hyperbolic



$\lambda_1 < 0, \lambda_2 > 0$

degenerate case



$\lambda_1 = 0, \lambda_2 > 0$

Characterization

Determining the type of Conic from the implicit form:

- Implicit function: quadratic polynomial

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

$$\Leftrightarrow \boldsymbol{x}^T \underbrace{\begin{bmatrix} a & {}^{b}/_{2} \\ {}^{b}/_{2} & c \end{bmatrix}}_{M} \boldsymbol{x} + \begin{bmatrix} d & e \end{bmatrix} \boldsymbol{x} + f = 0$$

- Eigenvalues of M

$$\lambda_{1,2} = \frac{a + c}{2} \pm \frac{1}{2}\sqrt{(a - c)^2 + b^2}$$

Cases

Implicit function:
$ax^2 + bxy + cy^2 + dx + ey + f = 0$

We obtain the following cases:

- Ellipse: $b^2 < 4ac$
  - Circle: $b = 0, a = c$
  - Otherwise: general ellipse
- Parabola: $b^2 = 4ac$ (border case)
- Hyperbola: $b^2 > 4ac$

Cases

Implicit function:
$ax^2 + bxy + cy^2 + dx + ey + f = 0$

Explanation:

$$
\begin{aligned}
b^2 = 4ac \Rightarrow \lambda_{1,2} &= \frac{a+c}{2} \pm \frac{1}{2}\sqrt{(a-c)^2 + 4ac} \\
&= \frac{a+c}{2} \pm \frac{1}{2}\sqrt{a^2 - 2ac + c^2 + 4ac} \\
&= \frac{a+c}{2} \pm \frac{1}{2}\sqrt{a^2 + 2ac + c^2} \\
&= \frac{a+c}{2} \pm \frac{1}{2}\sqrt{(a+c)^2} \\
&= \frac{a+c}{2} \pm \frac{a+c}{2} \\
&= \{0, a+c\}
\end{aligned}
$$

Polynomial Curves & Conics

We want to represent conics with parametric curves:

- How can we represent (pieces) of conics as parametric curves?
- How can we generalize our framework of piecewise polynomial curves to include conical sections?

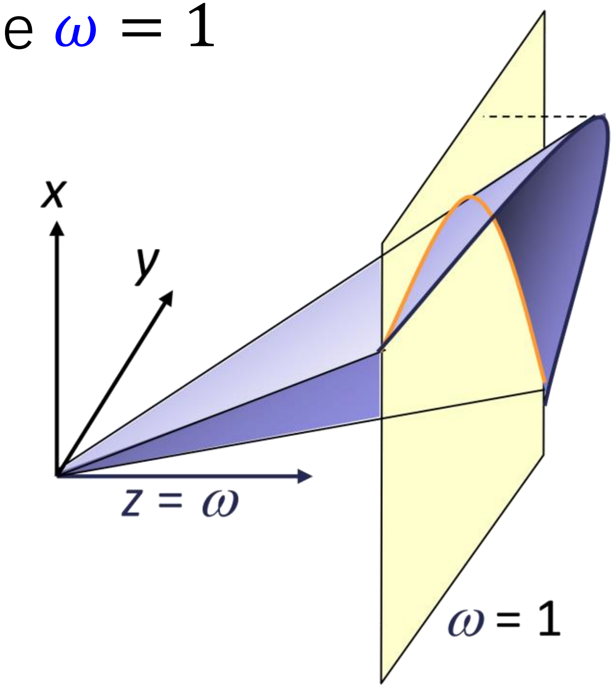
Projections of Parabolas:

- We will look at a certain class of parametric functions – projections of parabolas
- This class turns out to be general enough
- And can be expressed easily with the tools we know.

Projections of Parabolas

Definition: Projection of a Parabola

- We start with a quadratic space curve
- Interpret the z-coordinate as homogenous component ω
- Project the curve on the plane $\omega = 1$



Projected Parabola

Formal Definition:

- Quadratic polynomial curve in three space
- Project by dividing by the third coordinates

$$\boldsymbol{f}^{(hom)}(t) = \boldsymbol{p}_0 + t\boldsymbol{p}_1 + t^2\boldsymbol{p}_2 = \begin{pmatrix} \boldsymbol{p}_0.x \\ \boldsymbol{p}_0.y \\ \boldsymbol{p}_0.\omega \end{pmatrix} + t\begin{pmatrix} \boldsymbol{p}_1.x \\ \boldsymbol{p}_1.y \\ \boldsymbol{p}_1.\omega \end{pmatrix} + t^2\begin{pmatrix} \boldsymbol{p}_2.x \\ \boldsymbol{p}_2.y \\ \boldsymbol{p}_2.\omega \end{pmatrix}$$

$$\boldsymbol{f}^{(eucl)}(t) = \frac{\begin{pmatrix} \boldsymbol{p}_0.x \\ \boldsymbol{p}_0.y \end{pmatrix} + t\begin{pmatrix} \boldsymbol{p}_1.x \\ \boldsymbol{p}_1.y \end{pmatrix} + t^2\begin{pmatrix} \boldsymbol{p}_2.x \\ \boldsymbol{p}_2.y \end{pmatrix}}{\boldsymbol{p}_0.\omega + t\boldsymbol{p}_1.\omega + t^2\boldsymbol{p}_2.\omega}$$

Parameterizing Conics

Conics can be parameterized using projected parabolas:

- We show that we can represent (piecewise):
  - Points and lines (obvious ✓)
  - A unit parabola
  - A unit circle
  - A unit hyperbola
- General cases (ellipses etc.) can be obtained by affine mappings of the control points (which leads to affine maps of the curve)

Parameterizing Parabolas

Parabolas as rational parametric curves:

$$\boldsymbol{f}^{(eucl)}(t) = \frac{\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\end{pmatrix} + t^2\begin{pmatrix}0\\1\end{pmatrix}}{1 + 0t + 0t^2} \qquad \begin{pmatrix} x(t) = t \\ y(t) = t^2 \end{pmatrix}$$ ✓ (obvious)

Circle

Let's try to find a rational parameterization of a (**piece of a**) unit circle:

$$\boldsymbol{f}^{(eucl)}(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

Circle

Let's try to find a rational parameterization of a (**piece of a**) unit circle:

$$\boldsymbol{f}^{(eucl)}(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\cos\varphi = \frac{1-\tan^2\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}, \quad \sin\varphi = \frac{2\tan\frac{\varphi}{2}}{1+\tan^2\frac{\varphi}{2}}$$ (tangent half-angle formula)

$$t := \tan\frac{\varphi}{2} \Rightarrow \boldsymbol{f}^{(eucl)}(\varphi) = \begin{pmatrix} \frac{1-t^2}{1+t^2} \\ \frac{2t}{1+t^2} \end{pmatrix}$$

Circle

Let's try to find a rational parameterization of a (**piece of a**) unit circle:

$$\boldsymbol{f}^{(eucl)}(\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \frac{1-t^2}{1+t^2} \\ \frac{2t}{1+t^2} \end{pmatrix} \text{ with } t := \tan\frac{\varphi}{2}$$

$$\Rightarrow \boldsymbol{f}^{(hom)}(t) = \begin{pmatrix} 1-t^2 \\ 2t \\ 1+t^2 \end{pmatrix}$$

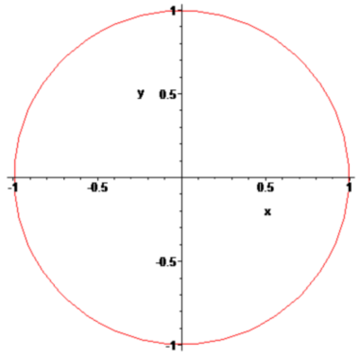
parameterization for $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow$ we need at least three segments to parametrize a full circle

Hyperbolas

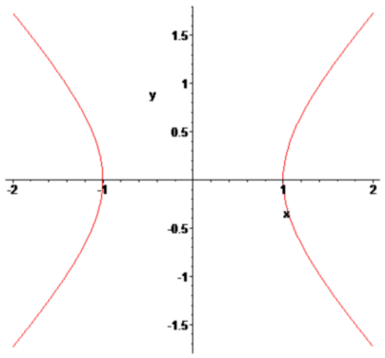
Unit Circle: $x^2 + y^2 = 1$

$$\Rightarrow x(t) = \frac{1 - t^2}{1 + t^2}, \qquad y(t) = \frac{2t}{1 + t^2} \ (t \in \mathbb{R})$$



Unit Hyperbola: $x^2 - y^2 = 1$

$$\Rightarrow x(t) = \frac{1 + t^2}{1 - t^2}, \qquad y(t) = \frac{2t}{1 - t^2}, (t \in [0, 1])$$



Rational Bézier Curves

Rational Bézier Curves

Rational Bézier curves in $\mathbb{R}^n$ of degree d:

- Form a Bézier curve of degree d in $n+1$ dimensional space
- Interpret last coordinates as homogenous component
- Euclidean coordinates are obtained by projection

$$\boldsymbol{f}^{(hom)}(t) = \sum_{i=0}^{n} B_i^{(d)}(t)\boldsymbol{p}_i\,, \quad \boldsymbol{p}_i \in \mathbb{R}^{n+1}$$

$$\boldsymbol{f}^{(eucl)}(t) = \frac{\sum_{i=0}^{n} B_i^{(d)}(t)\begin{pmatrix} p_i^{(1)} \\ \dots \\ p_i^{(n)} \end{pmatrix}}{\sum_{i=0}^{n} B_i^{(d)}(t)p_i^{(n+1)}}$$

More Convenient Notation

The curve can be written in “weighted points” form:

$$\boldsymbol{f}^{(eucl)}(t) = \frac{\sum_{i=0}^{n} B_i^{(d)}(t)\omega_i \begin{pmatrix} p_i^{(1)} \\ \dots \\ p_i^{(n)} \end{pmatrix}}{\sum_{i=0}^{n} B_i^{(d)}(t)\omega_{\boldsymbol{i}}}$$

Interpretation:

- Points are weighted by weights ω_i
- Normalized by interpolated weights in the denominator
- Large weights → more influence of that point

Properties

What about affine invariance, convex hull prop.?

$$\boldsymbol{f}^{(eucl)}(t) = \sum_{i=0}^{n} \boldsymbol{p}_i \frac{B_i^{(d)}(t)\omega_i}{\sum_{j=0}^{n} B_j^{(d)}(t)\omega_i} = \sum_{i=0}^{n} q_i(t)\boldsymbol{p}_i \quad \text{with } \sum_{i=0}^{n} q_i(t) = 1$$

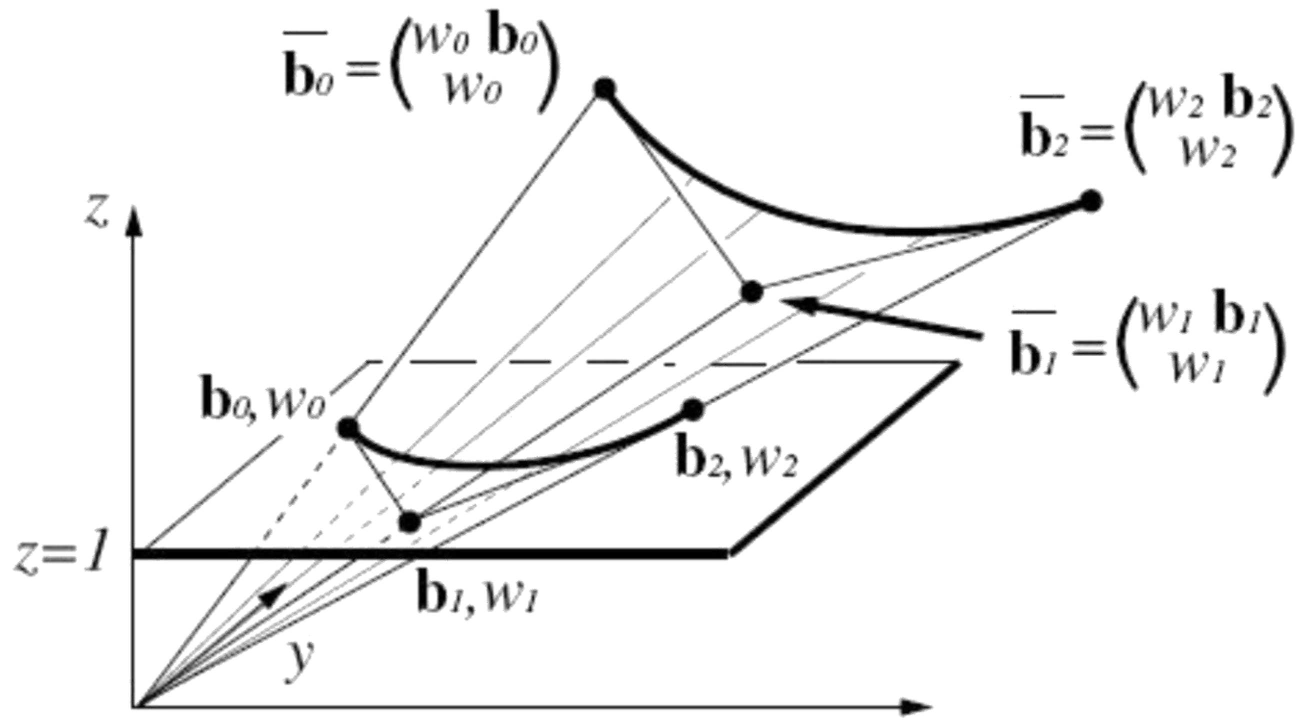
Consequences:

- Affine invariance still holds
- For strictly positive weights:
  - Convex hull property still holds
  - This is not a big restriction (potential singularities otherwise)
- Projective invariance (projective maps, hom. coord's)

Rational Bézier Curves

Geometric interpretation of rational Bézier curves:

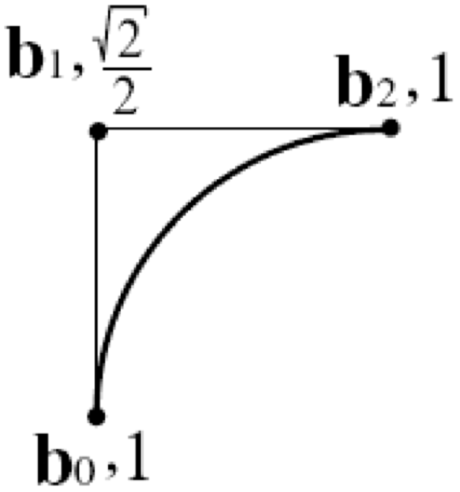
- Rational Bézier curves are obtained by central projection of "normal" Bézier curves



Rational Bézier Curves

Examples:

- $\omega_i = 1 \; (i = 0, \ldots, n)$: "normal" Bézier curves
- Generally:
  - Each conic section can be described as rational Bézier curve of degree two
  - Each rational Bézier curve of degree two is a conic section
- Example: Circular arc



Rational de Casteljau Algorithm

Evaluation with de Casteljau Algorithm

- Three variants:
  - Compute in $n+1$ dimensional space, then project
  - Compute numerator and denominator separately, then divide
  - Divide in each intermediate step ("rational de Casteljau")
- Non-rational de Casteljau algorithm:

$$\boldsymbol{b}_i^{(r)}(t) = (1-t)\boldsymbol{b}_i^{(r-1)}(t) + t\boldsymbol{b}_{i+1}^{(r-1)}(t)$$

- Rational de Casteljau algorithm

$$\boldsymbol{b}_i^{(r)}(t) = (1-t)\frac{\omega_i^{(r-1)}(t)}{\omega_i^{(r)}(t)}\boldsymbol{b}_i^{(r-1)}(t) + t\,\frac{\omega_{i+1}^{(r-1)}(t)}{\omega_i^{(r)}(t)}\boldsymbol{b}_{i+1}^{(r-1)}(t)$$

with $\omega_i^{(r)}(t) = (1-t)\omega_i^{(r-1)}(t) + t\omega_{i+1}^{(r-1)}(t)$

Rational de Casteljau Algorithm

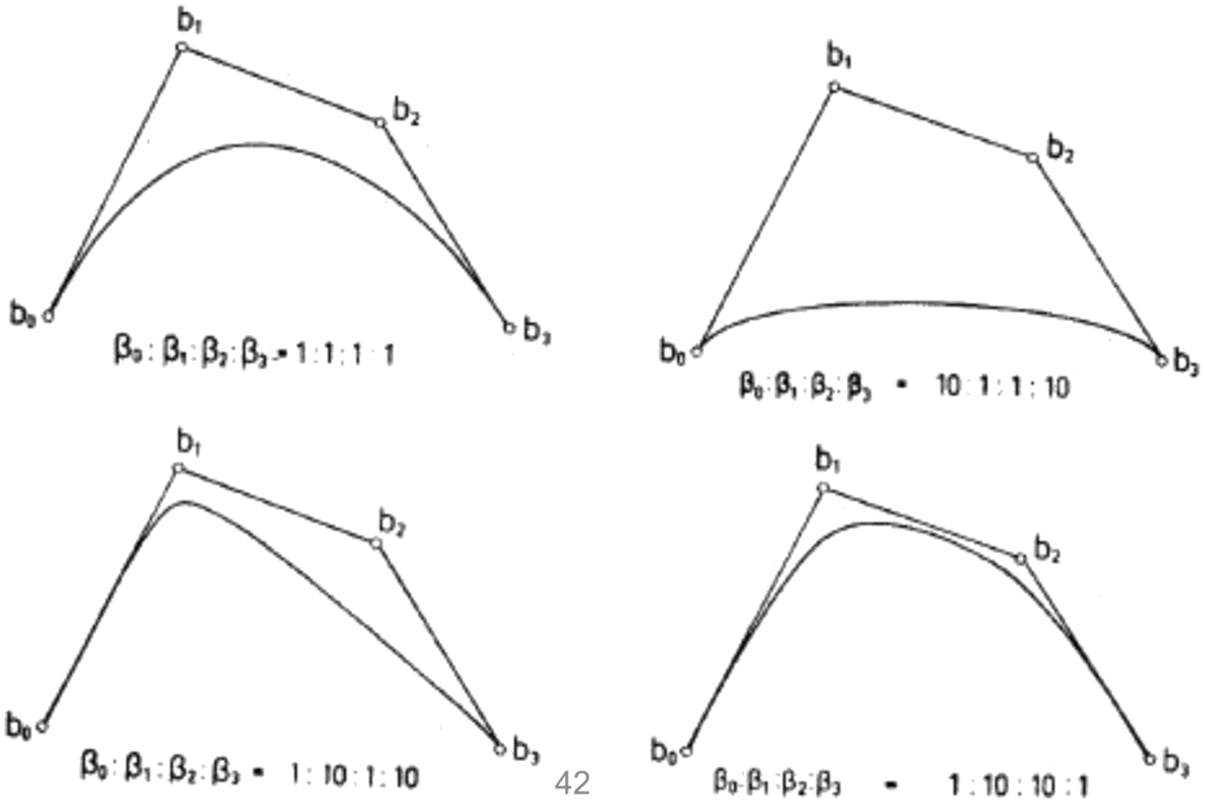
Advantages:

- More intuitive (repeated weighted linear interpolation of points and weights)
- Numerically more stable (only convex combinations for the standard case of positive weights, $t \in [0,1]$)

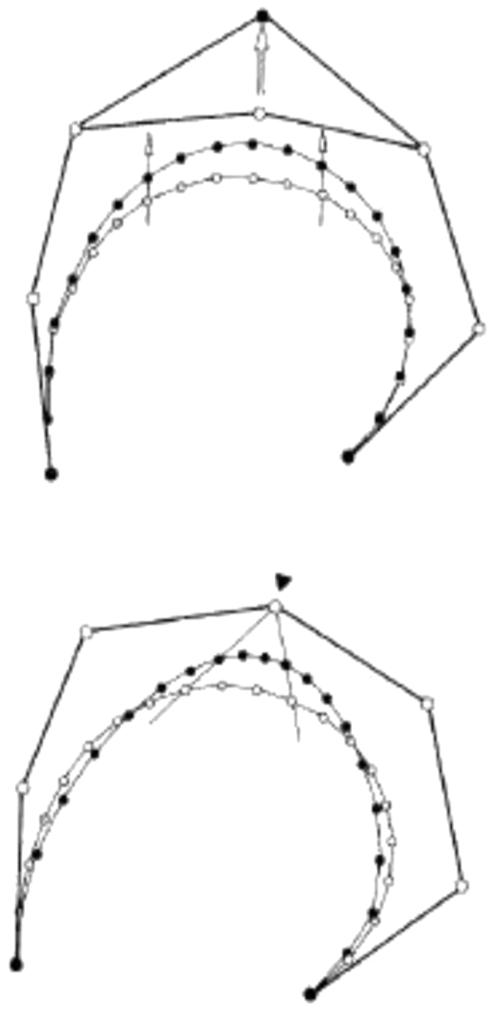
Influence of the Weights

Influence of the weights on the curve shape:

- Increasing ω_i moves the curve towards the Bézier point b_i
- Examples:



Influence of the Weights



Moving a control point

↑

Not the same!

↓

Increasing the weight of a control point

Quadratic Bézier Curves

- Quadratic curves:
  - Necessary and sufficient to represent conics
  - Therefore, we will examine them closer ⋯
- Quadratic rational Bézier curve:

$$\boldsymbol{f}^{(eucl)}(t) = \frac{B_0^{(2)}(t)\omega_0\boldsymbol{p}_0 + B_1^{(2)}(t)\omega_1\boldsymbol{p}_1 + B_2^{(2)}(t)\omega_2\boldsymbol{p}_2}{B_0^{(2)}(t)\omega_0 + B_1^{(2)}(t)\omega_1 + B_2^{(2)}(t)\omega_2}, \qquad \boldsymbol{p}_i \in \mathbb{R}^n, \omega_i \in \mathbb{R}$$

Standard Form (or Normal Form)

How many degrees of freedom are in the weights?

- Quadratic rational Bézier curve:

$$f^{(eucl)}(t) = \frac{B_0^{(2)}(t)\omega_0 \boldsymbol{p}_0 + B_1^{(2)}(t)\omega_1 \boldsymbol{p}_1 + B_2^{(2)}(t)\omega_2 \boldsymbol{p}_2}{B_0^{(2)}(t)\omega_0 + B_1^{(2)}(t)\omega_1 + B_2^{(2)}(t)\omega_2}$$

If one of the weights is $\neq \mathbf{0}$ (which must be the case), we can divide numerator and denominator by this weight and thus remove one degree of freedom. *No impact on the curve.*

If we are only interested in the *shape of the curve*, we can remove one more degree of freedom by a *reparameterization* ⋯ *No impact on* ***shape*** *of the curve*

Standard Form

How many degrees of freedom are in the weights?

- Concerning the shape of the curve, the parameterization does not matter
- We have

$$f^{(eucl)}(t) = \frac{(1-t)^2\omega_0\boldsymbol{p}_0 + 2t(1-t)\omega_1\boldsymbol{p}_1 + t^2\omega_2\boldsymbol{p}_2}{(1-t)^2\omega_0 + 2t(1-t)\omega_1 + t^2\omega_2}$$

- We set: (with α to be determined later)

$$t \leftarrow \frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}, \quad \text{i.e.,}\ (1-t) \leftarrow \frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}$$

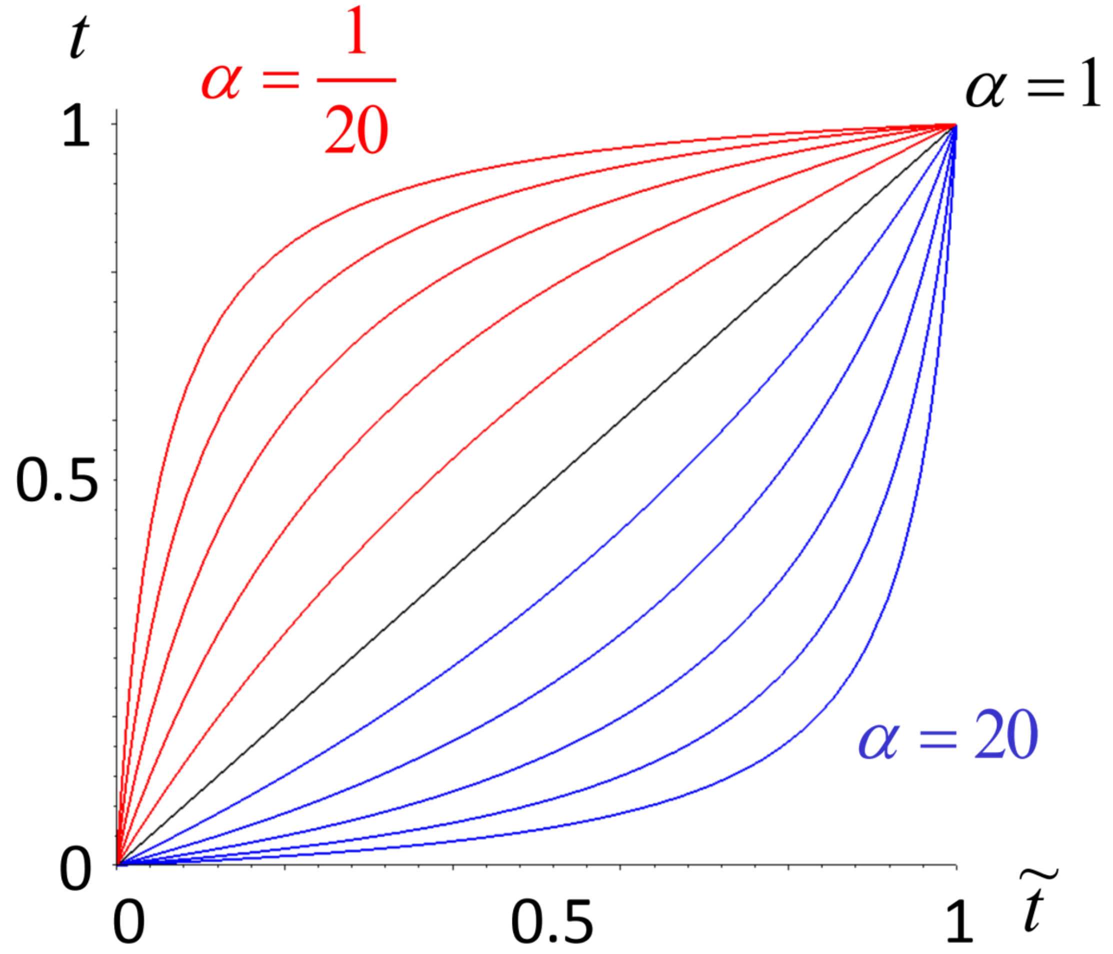
Remark: Why this reparameterization?

Reparameterization:

$$t \leftarrow \frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}$$

Properties:

- $0 \rightarrow 0, 1 \rightarrow 1,$ monotonic in between
- Shape determined by parameter α



Standard Form

$t \leftarrow \frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}$, i.e., $(1-t) \leftarrow \frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}$

Standard Form

$$t \leftarrow \frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}, \text{ i.e., } (1-t) \leftarrow \frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}$$

$$\boldsymbol{f}^{(eucl)}(t) = \frac{\left(\frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}\right)^2 \omega_0 \boldsymbol{p}_0 + 2\left(\frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}\right)\left(\frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}\right)\omega_1 \boldsymbol{p}_1 + \left(\frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}\right)^2 \omega_2 \boldsymbol{p}_2}{\left(\frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}\right)^2 \omega_0 + 2\left(\frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}\right)\left(\frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}\right)\omega_1 + \left(\frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}\right)^2 \omega_2}$$

$$= \frac{\alpha^2(1-\tilde{t})^2\omega_0\boldsymbol{p}_0 + 2\alpha(1-\tilde{t})\tilde{t}\omega_1\boldsymbol{p}_1 + \tilde{t}^2\omega_2\boldsymbol{p}_2}{\alpha^2(1-\tilde{t})^2\omega_0 + 2\alpha(1-\tilde{t})\tilde{t}\omega_1 + \tilde{t}^2\omega_2}$$

Standard Form

$$t \leftarrow \frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}, \text{ i.e., } (1-t) \leftarrow \frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}$$

$$\boldsymbol{f}^{(eucl)}(t) = \frac{\left(\frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}\right)^2 \omega_0 \boldsymbol{p}_0 + 2\left(\frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}\right)\left(\frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}\right)\omega_1 \boldsymbol{p}_1 + \left(\frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}\right)^2 \omega_2 \boldsymbol{p}_2}{\left(\frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}\right)^2 \omega_0 + 2\left(\frac{\alpha(1-\tilde{t})}{\alpha(1-\tilde{t})+\tilde{t}}\right)\left(\frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}\right)\omega_1 + \left(\frac{\tilde{t}}{\alpha(1-\tilde{t})+\tilde{t}}\right)^2 \omega_2}$$

$$= \frac{\alpha^2(1-\tilde{t})^2\omega_0\boldsymbol{p}_0 + 2\alpha(1-\tilde{t})\tilde{t}\omega_1\boldsymbol{p}_1 + \tilde{t}^2\omega_2\boldsymbol{p}_2}{\alpha^2(1-\tilde{t})^2\omega_0 + 2\alpha(1-\tilde{t})\tilde{t}\omega_1 + \tilde{t}^2\omega_2}$$

$$= \frac{\alpha^2 B_0^{(2)}(\tilde{t})\omega_0\boldsymbol{p}_0 + \alpha B_1^{(2)}(\tilde{t})\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_2}{\alpha^2 B_0^{(2)}(\tilde{t})\omega_0 + \alpha B_1^{(2)}(\tilde{t})\omega_1 + B_2^{(2)}(\tilde{t})\omega_2}$$

Standard Form

$$\boldsymbol{f}^{(eucl)}(t) = \frac{\alpha^2 B_0^{(2)}(\tilde{t})\omega_0\boldsymbol{p}_0 + \alpha B_1^{(2)}(\tilde{t})\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_2}{\alpha^2 B_0^{(2)}(\tilde{t})\omega_0 + \alpha B_1^{(2)}(\tilde{t})\omega_1 + B_2^{(2)}(\tilde{t})\omega_2}$$

let $\alpha = \sqrt{\frac{\omega_2}{\omega_0}}$ (assume $0 \le \frac{\omega_2}{\omega_0} < \infty$)

Standard Form

$$\boldsymbol{f}^{(eucl)}(t) = \frac{\alpha^2 B_0^{(2)}(\tilde{t})\omega_0\boldsymbol{p}_0 + \alpha B_1^{(2)}(\tilde{t})\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_2}{\alpha^2 B_0^{(2)}(\tilde{t})\omega_0 + \alpha B_1^{(2)}(\tilde{t})\omega_1 + B_2^{(2)}(\tilde{t})\omega_2}$$

let $\alpha = \sqrt{\frac{\omega_2}{\omega_0}}$ (assume $0 \leq \frac{\omega_2}{\omega_0} < \infty$)

$$\boldsymbol{f}^{(eucl)}(t) = \frac{B_0^{(2)}(\tilde{t})\left(\sqrt{\frac{\omega_2}{\omega_0}}\right)^2 \omega_0\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t})\left(\sqrt{\frac{\omega_2}{\omega_0}}\right)^2 \omega_0 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\omega_1 + B_2^{(2)}(\tilde{t})\omega_2}$$

$$= \frac{B_0^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t})\omega_2 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\omega_1 + B_2^{(2)}(\tilde{t})\omega_2}$$

Standard Form

$$\boldsymbol{f}^{(eucl)}(t) = \frac{B_0^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t})\omega_2 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\omega_1 + B_2^{(2)}(\tilde{t})\omega_2}$$

Standard Form

$$\boldsymbol{f}^{(eucl)}(t) = \frac{B_0^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\omega_2\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t})\omega_2 + B_1^{(2)}(\tilde{t})\sqrt{\frac{\omega_2}{\omega_0}}\omega_1 + B_2^{(2)}(\tilde{t})\omega_2}$$

$$= \frac{B_0^{(2)}(\tilde{t})\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\sqrt{\frac{1}{\omega_0\omega_2}}\omega_1\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t}) + B_1^{(2)}(\tilde{t})\sqrt{\frac{1}{\omega_0\omega_2}}\omega_1 + B_2^{(2)}(\tilde{t})}$$

$$= \frac{B_0^{(2)}(\tilde{t})\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\omega\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t}) + B_1^{(2)}(\tilde{t})\omega + B_2^{(2)}(\tilde{t})}$$

with $\omega := \sqrt{\frac{1}{\omega_0\omega_2}}\omega_1$

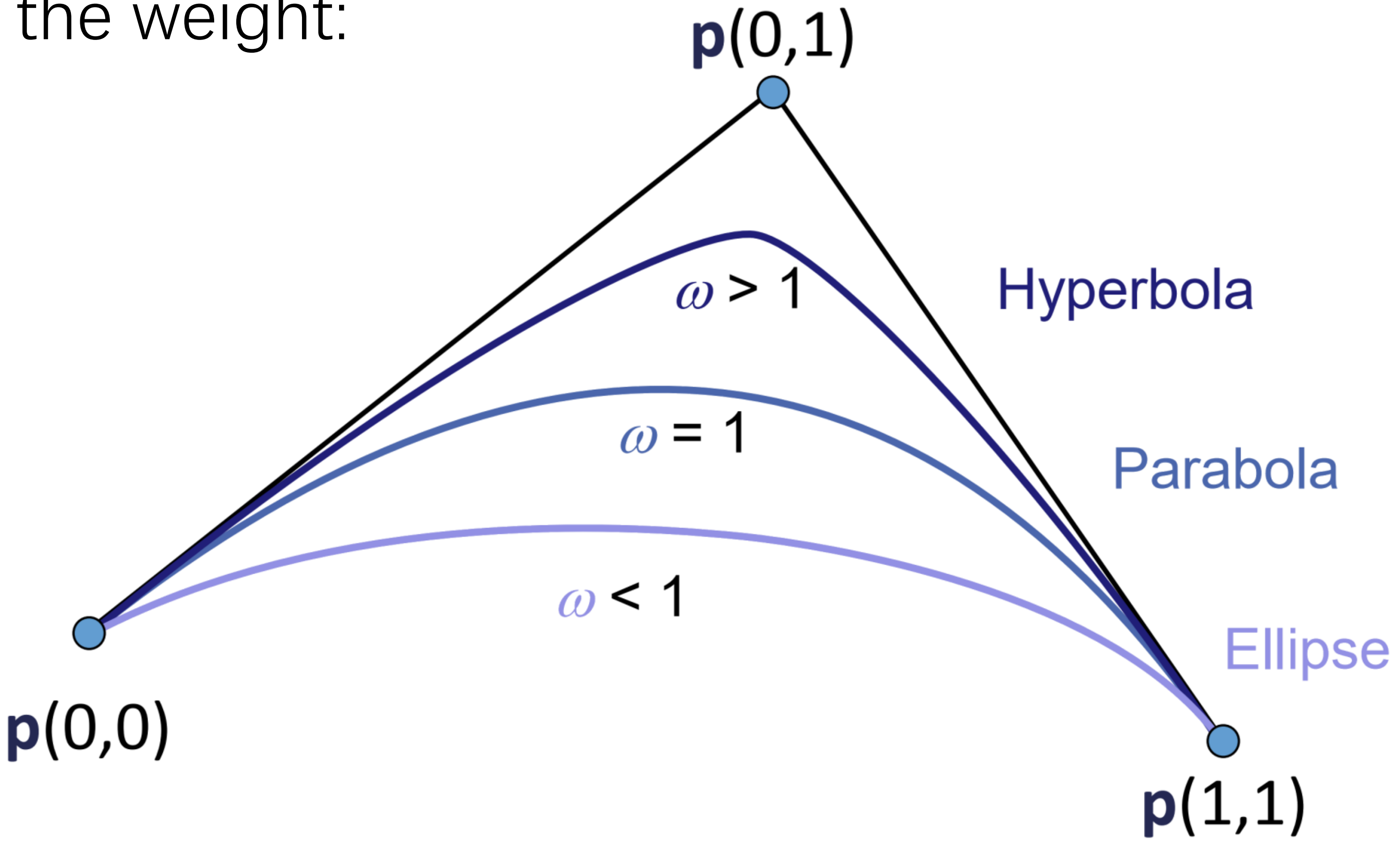
Standard Form

Consequence:

- It is sufficient to specify the weight of the inner point
- We can w.l.o.g. set $\omega_0 = \omega_2 = 1$, $\omega_1 = \omega$
- This form of a quadratic Bézier curve is called the *standard form* or the *normal form*
- Choices:
  - $\omega < 1$: ellipse segment
  - $\omega = 1$: parabola segment (non-rational curve)
  - $\omega > 1$: hyperbola segment

Illustration

- Changing the weight:



Conversion to Implicit Form

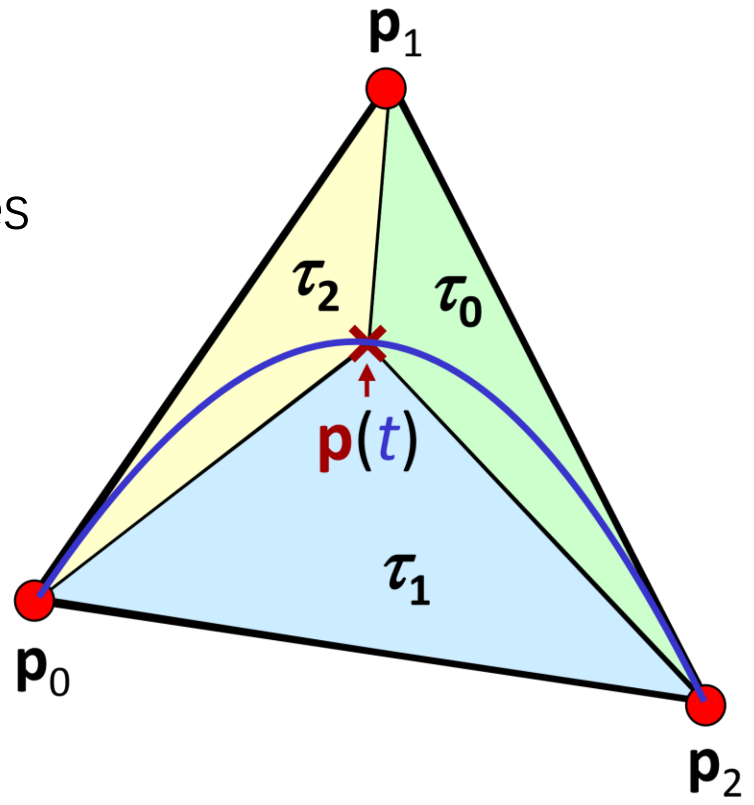
Convert parametric to implicit form

- In order to show the shape condition
- For distance computation / inside-outside tests

Express curve in barycentric coordinates

- Curve can be expressed in barycentric coordinates (linear transform)

$$\boldsymbol{f}(t) = \tau_0(t)\boldsymbol{p}_0 + \tau_1(t)\boldsymbol{p}_1 + \tau_2(t)\boldsymbol{p}_2$$



Conversion to Implicit Form

Compare the coefficients

$$\boldsymbol{f}(t) = \tau_0(t)\boldsymbol{p}_0 + \tau_1(t)\boldsymbol{p}_1 + \tau_2(t)\boldsymbol{p}_2$$

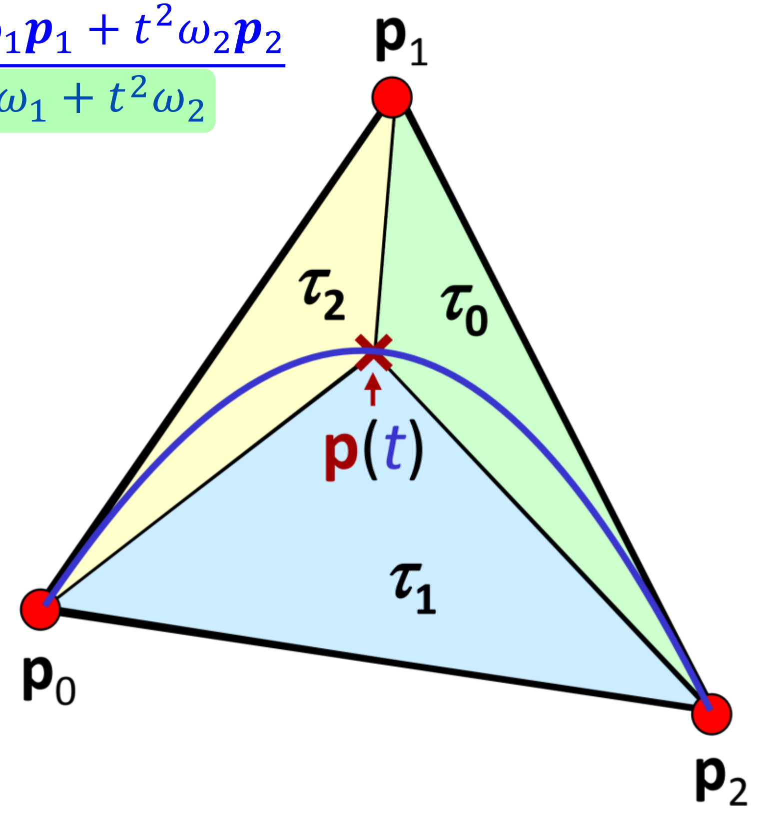
$$\boldsymbol{f}^{(eucl)}(t) = \frac{(1-t)^2\omega_0\boldsymbol{p}_0 + 2t(1-t)\omega_1\boldsymbol{p}_1 + t^2\omega_2\boldsymbol{p}_2}{(1-t)^2\omega_0 + 2t(1-t)\omega_1 + t^2\omega_2}$$

$$\tau_0 = \frac{\omega_0(1-t)^2}{\omega(t)}$$

$$\tau_1 = \frac{2\omega_1 t(1-t)}{\omega(t)}$$

$$\tau_2 = \frac{\omega_2 t^2}{\omega(t)}$$

$$\omega(t) = (1-t)^2\,\omega_0 + 2t(1-t)\omega_1 + t^2\omega_2$$



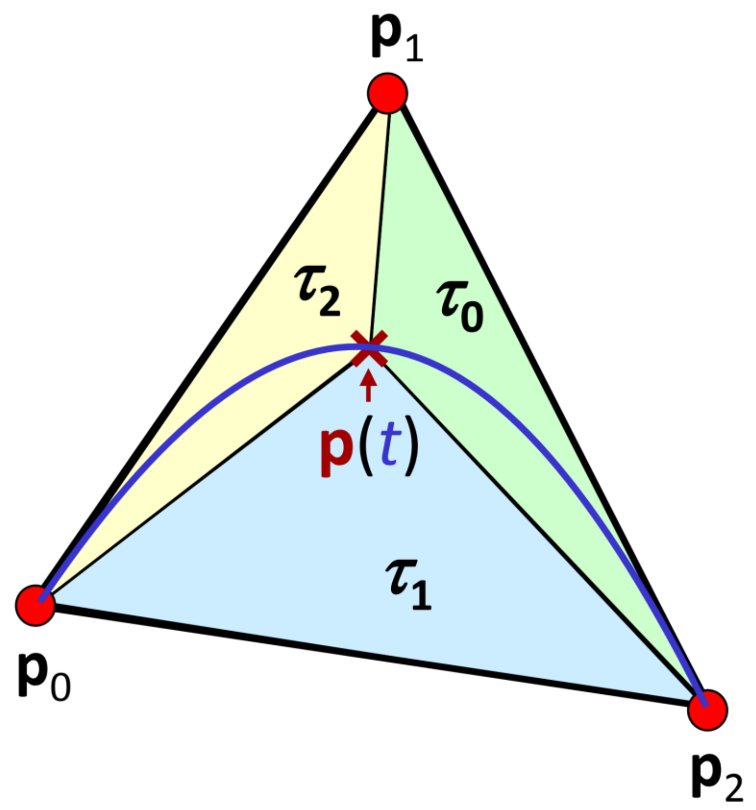
Conversion to Implicit Form

Solving for t, $1-t$

$$\tau_0 = \frac{\omega_0(1-t)^2}{\omega(t)} \Rightarrow 1-t = \sqrt{\frac{\tau_0(t)\omega(t)}{\omega_0}}$$

$$\tau_1 = \frac{2\omega_1 t(1-t)}{\omega(t)}$$

$$\tau_2 = \frac{\omega_2 t^2}{\omega(t)} \Rightarrow t = \sqrt{\frac{\tau_2(t)\omega(t)}{\omega_2}}$$



Conversion to Implicit Form

Solving for t, $1-t$

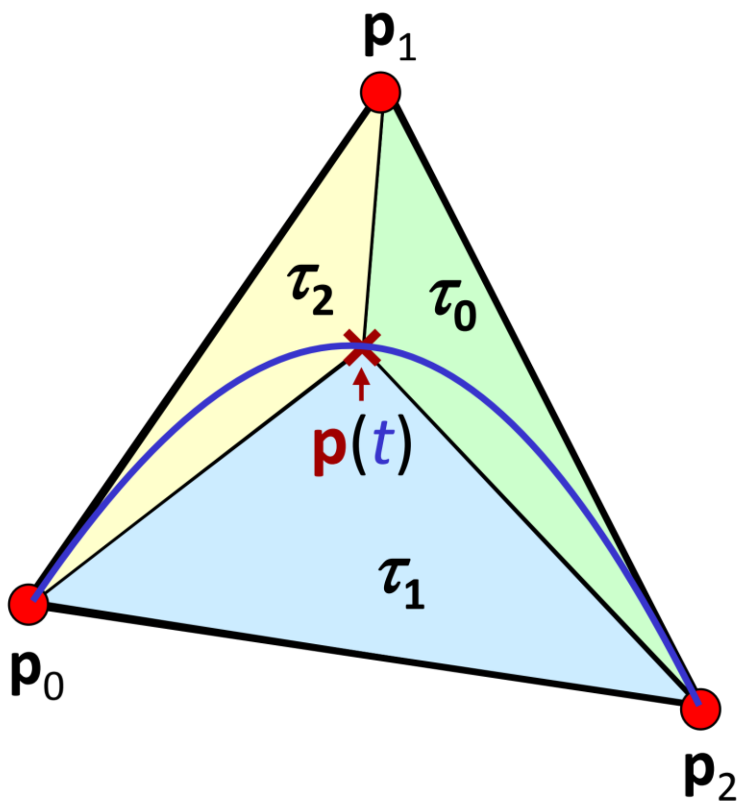
$$\tau_0 = \frac{\omega_0(1-t)^2}{\omega(t)} \Rightarrow 1-t = \sqrt{\frac{\tau_0(t)\omega(t)}{\omega_0}}$$

$$\tau_1 = \frac{2\omega_1 t(1-t)}{\omega(t)}$$

$$\tau_2 = \frac{\omega_2 t^2}{\omega(t)} \Rightarrow t = \sqrt{\frac{\tau_2(t)\omega(t)}{\omega_2}}$$

$$\tau_1 = \frac{2\omega_1 t(1-t)}{\omega(t)} = 2\,\frac{\omega_1}{\omega(t)}\sqrt{\frac{\tau_0(t)\omega(t)}{\omega_0}\,\frac{\tau_2(t)\omega(t)}{\omega_2}} = 2\omega_1\sqrt{\frac{\tau_0(t)\tau_2(t)}{\omega_0\omega_2}}$$

$$\Rightarrow \frac{\tau_1^2(t)}{\tau_2(t)\tau_0(t)} = \frac{4\omega_1^2}{\omega_0\omega_2}$$



Conversion to Implicit Form

More algebra ⋯

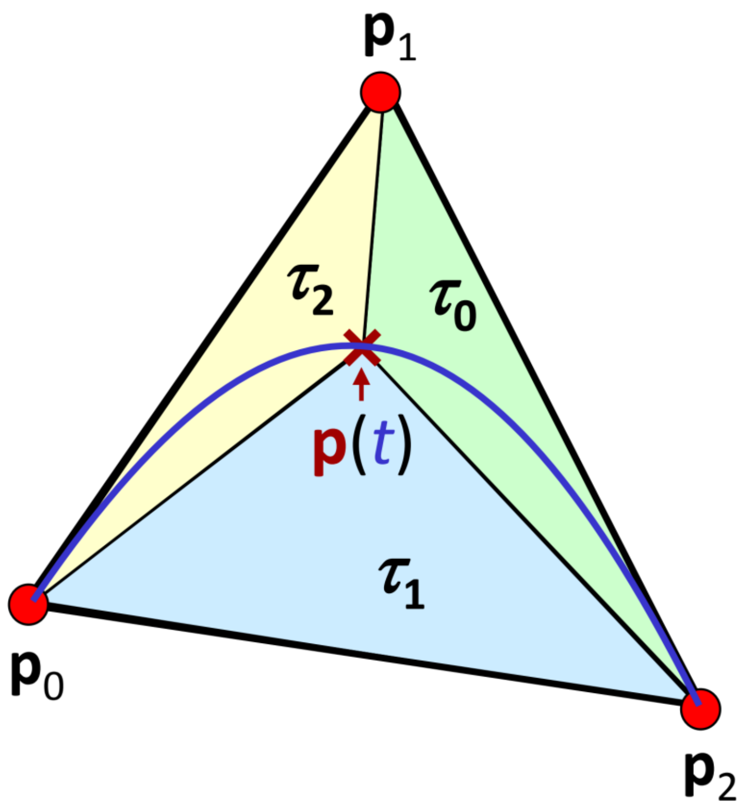
$$\frac{\tau_1^2(t)}{\tau_2(t)\tau_0(t)} = \frac{4\omega_1^2}{\omega_0\omega_2}$$

Using $\tau_2(t) = 1 - \tau_0(t) - \tau_1(t)$, we get

$$(\omega_0\omega_2)\tau_1^2(t) = 4\omega_1^2\tau_2(t)\tau_0(t) = 4\omega_1^2\big(1 - \tau_0(t) - \tau_1(t)\big)\tau_0(t)$$

$$= 4\omega_1^2\left(\tau_0(t) - \tau_0^2(t) - \tau_0(t)\tau_1(t)\right)$$

$$\Rightarrow (\omega_0\omega_2)\tau_1^2(t) + 4\omega_1^2\tau_1(t)\tau_0(t) + 4\omega_1^2\tau_0^2(t) - 4\omega_1^2\tau_0(t) = 0$$



Conversion to Implicit Form

More algebra ⋯

$$\frac{\tau_1^2(t)}{\tau_2(t)\tau_0(t)} = \frac{4\omega_1^2}{\omega_0\omega_2}$$

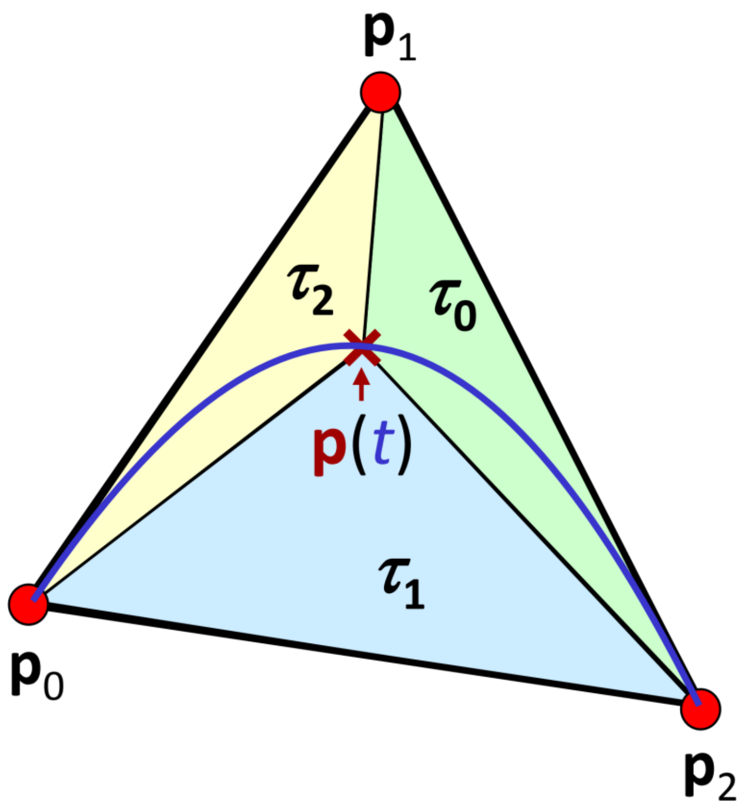
Using $\tau_2(t) = 1 - \tau_0(t) - \tau_1(t)$, we get

$$(\omega_0\omega_2)\tau_1^2(t) = 4\omega_1^2\tau_2(t)\tau_0(t) = 4\omega_1^2\big(1 - \tau_0(t) - \tau_1(t)\big)\tau_0(t)$$

$$= 4\omega_1^2\left(\tau_0(t) - \tau_0^2(t) - \tau_0(t)\tau_1(t)\right)$$

$$\Rightarrow (\omega_0\omega_2)\tau_1^2(t) + 4\omega_1^2\tau_1(t)\tau_0(t) + 4\omega_1^2\tau_0^2(t) - 4\omega_1^2\tau_0(t) = 0$$

$$ax^2 + bxy + cy^2 + 0x + ey + 0 = 0$$



Classification

Eigenvalue argument led to:

- Parabola requires $b^2 = 4ac$ in $ax^2 + bxy + cy^2 + dx + ey + f = 0$
- In our case:

$$(\omega_0\omega_2)\tau_1^2(t) + 4\omega_1^2\tau_1(t)\tau_0(t) + 4\omega_1^2\tau_0^2(t) - 4\omega_1^2\tau_0(t) = 0$$

  i.e.

  $4(\omega_0\omega_2)(4\omega_1^2) = (4\omega_1^2)^2$

  $\Leftrightarrow \omega_0\omega_2 = \omega_1^2$

- Standard form: $\omega_0 = \omega_2 = 1$

  $\Rightarrow \omega_1 = 1$

Classification

Similarly, it follows that

$\omega_1 < 1$ → Ellipse

$\omega_1 = 1$ → Parabola

$\omega_1 > 1$ → Hyperbola

Towards Dual Conic Sections

Rational quadratic curves – conic sections

- Consider a rational quadratic curve in normal form for $t \in [0,1]$:

$$\boldsymbol{x}(t) = \frac{(1-t)^2 \cdot \boldsymbol{b}_0 + 2 \cdot t \cdot (1-t) \cdot \omega \cdot \boldsymbol{b}_1 + t^2 \cdot \boldsymbol{b}_2}{(1-t)^2 + 2 \cdot t \cdot (1-t) \cdot \omega + t^2}$$

Dual Conic Sections

Rational quadratic curves – conic sections

- Dual conic section $t \in \mathbb{R} \setminus [0,1]$
- Choice of reparameterization

$$s(t) = \hat{t} = \frac{t}{2 \cdot t - 1} \Rightarrow (1 - \hat{t}) = \frac{t - 1}{2 \cdot t - 1}$$

$\hat{t}$ changes from 0 to $-\infty \Leftrightarrow t$ changes from 0 to $\frac{1}{2}$

$\hat{t}$ changes from ∞ to $1 \Leftrightarrow t$ changes from $\frac{1}{2}$ to 1

Dual Conic Sections

The following applies:

$$\boldsymbol{x}\big(s(t)\big) = \boldsymbol{x}(\hat{t})$$

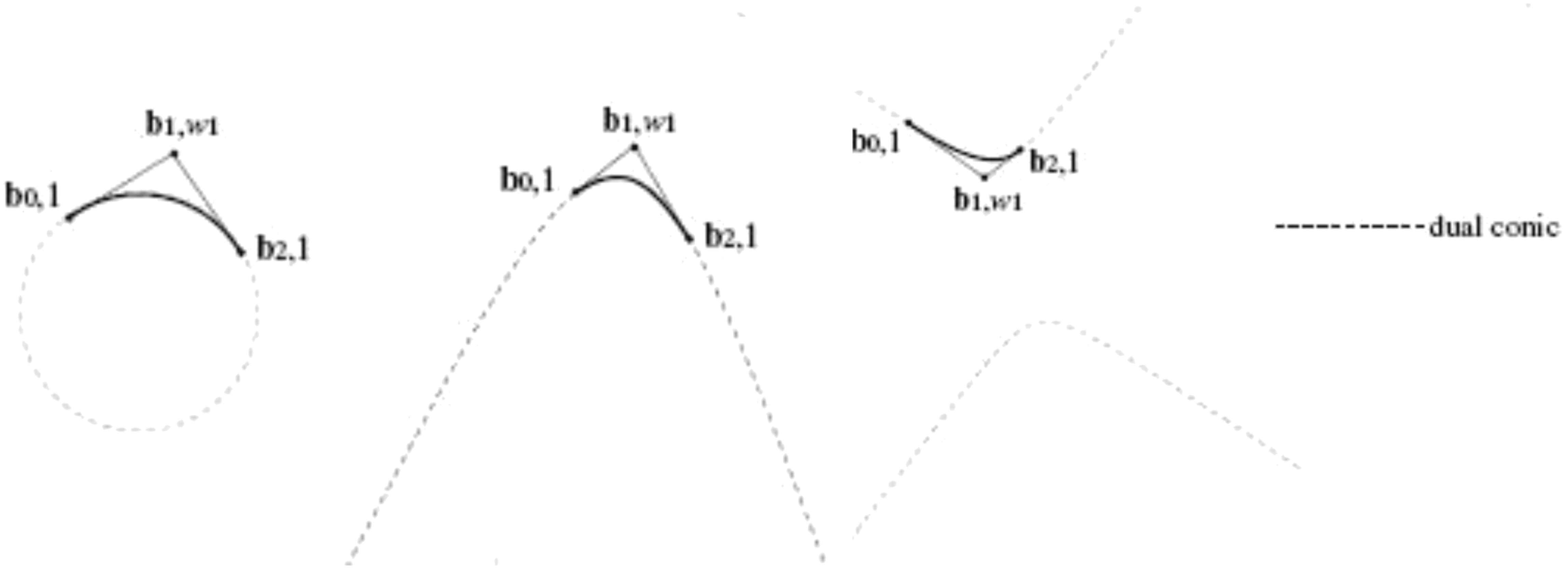
$$= \frac{(1-\hat{t})^2 \cdot b_0 + 2 \cdot \hat{t} \cdot (1-\hat{t}) \cdot \omega \cdot b_1 + \hat{t}^2 \cdot b_2}{(1-\hat{t})^2 + 2 \cdot \hat{t} \cdot (1-\hat{t}) \cdot \omega + \hat{t}^2}$$

$$= \frac{(1-t)^2 \cdot b_0 - 2 \cdot t \cdot (1-t) \cdot \omega \cdot b_1 + t^2 \cdot b_2}{(1-t)^2 - 2 \cdot t \cdot (1-t) \cdot \omega + t^2}$$

- →Dual conic section arises in Normal form by negation of ω

Dual Conic Sections

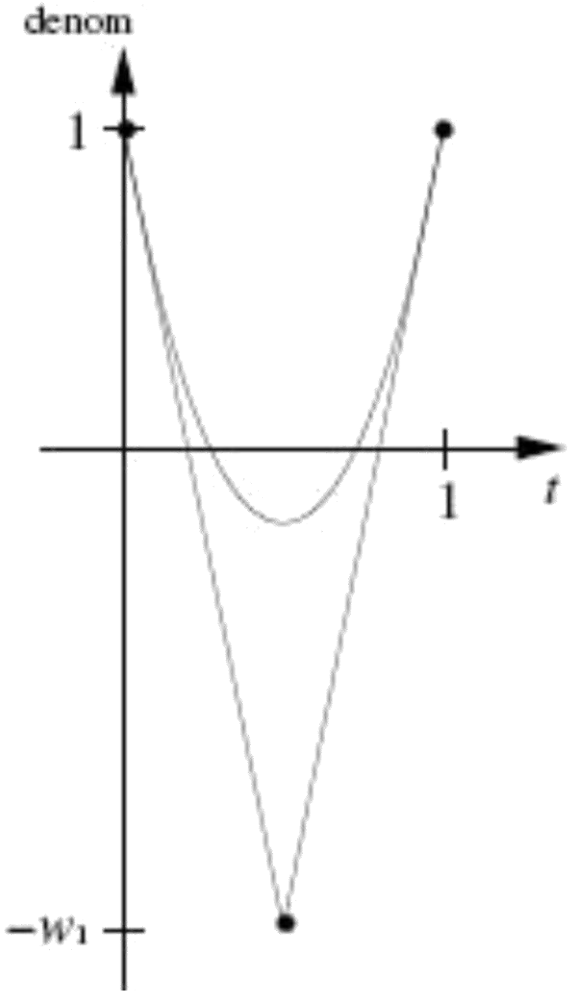
Examples:



Dual Conic Sections

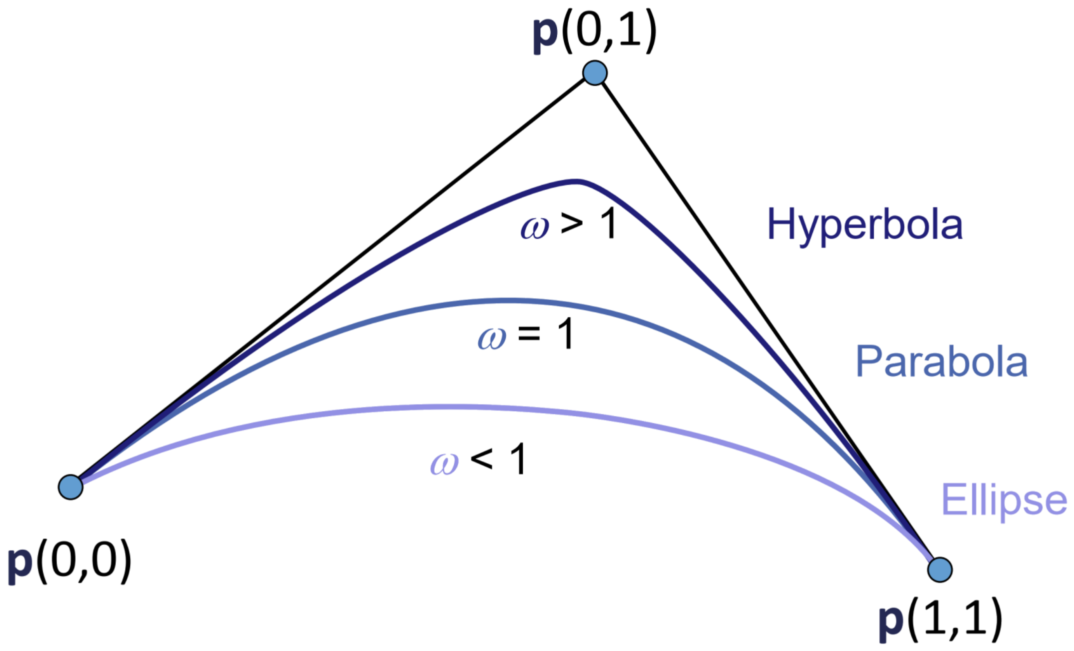
Classification of conic sections:

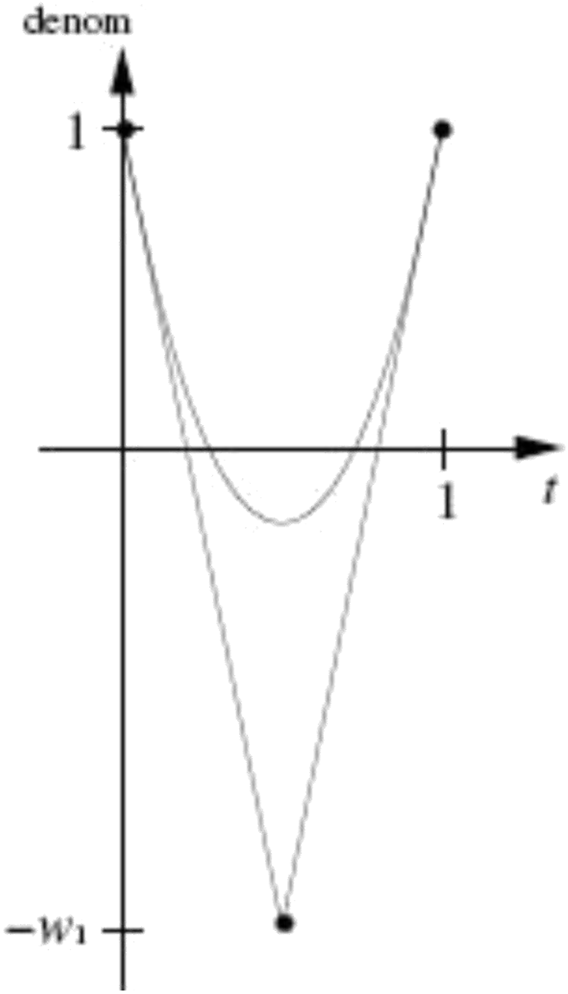
- By means of the dual conic section
- Consider singularities of the denominator function $(1-t)^2 - 2 \cdot t \cdot (1-t) \cdot \omega + t^2$ in $[0,1]$



Rational Bézier curves

- $\omega < 1 \rightarrow$ no singularities $\rightarrow$ ellipse
- $\omega = 1 \rightarrow$ one singularities $\rightarrow$ parabola
- $\omega > 1 \rightarrow$ two singularities $\rightarrow$ hyperbola





Circle in Bézier Form

- Quadratic rational polynomial:

$$f(t) = \frac{1}{1+t^2}\begin{pmatrix} 1-t^2 \\ 2t \end{pmatrix}, \qquad t = \tan\frac{\varphi}{2}, \qquad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

- Conversion to Bézier basis

$$B_0^{(2)} = (1-t)^2 = 1 - 2t + t^2 := \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$B_1^{(2)} = 2t(1-t) = 2t - 2t^2 := \begin{bmatrix} 0 & 2 & -2 \end{bmatrix}^{\mathrm{T}}$$

$$B_2^{(2)} = t^2 := \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

$$1 - t^2 := \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^{\mathrm{T}}$$

$$2t := \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$1 + t^2 := \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

Circle in Bézier Form

Conversion to Bézier basis: Method 1

$$B_0^{(2)} = (1-t)^2 = 1 - 2t + t^2 := [1 \quad -2 \quad 1]^T$$

$$B_1^{(2)} = 2t(1-t) = 2t - 2t^2 := [0 \quad 2 \quad -2]^T$$

$$B_2^{(2)} = t^2 := [0 \quad 0 \quad 1]^T$$

$$1 - t^2 := [1 \quad 0 \quad -1]^T$$

$$2t := [0 \quad 2 \quad 0]^T$$

$$1 + t^2 := [1 \quad 0 \quad 1]^T$$

Comparison yields:

$$1 - t^2 = B_0^{(2)} + B_1^{(2)}$$

$$2t = B_1^{(2)} + 2B_2^{(2)}$$

$$1 + t^2 = B_0^{(2)} + B_1^{(2)} + 2B_2^{(2)}$$

$$\boldsymbol{f}^{(hom)}(t) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} B_0^{(2)} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} B_1^{(2)} + \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} B_2^{(2)}$$

Circle in Bézier Form

Conversion to Bézier basis: Method 2

Use polar forms:

$$1 - t^2 \Rightarrow f_0 = 1 - t_1 t_2$$

$$2t \Rightarrow f_1 = t_1 + t_2$$

$$1 + t^2 \Rightarrow f_2 = 1 + t_1 t_2$$

And then evaluate at $(0,0)$, $(0,1)$, $(1,1)$

Circle in Bézier Form

- Result:

$$\boldsymbol{f}(t)=\frac{\binom{1}{0}B_0^{(2)}(t)+\binom{1}{1}B_1^{(2)}(t)+\binom{0}{2}B_2^{(2)}(t)}{B_0^{(2)}(t)+B_1^{(2)}(t)+2B_2^{(2)}(t)}$$

- Parameters:

$$t=\tan\frac{\varphi}{2}\Rightarrow\varphi=2\arctan t$$

$$t\in[0,1]\rightarrow\varphi\in\left[0,\frac{\pi}{2}\right]$$

Circle in Bézier Form

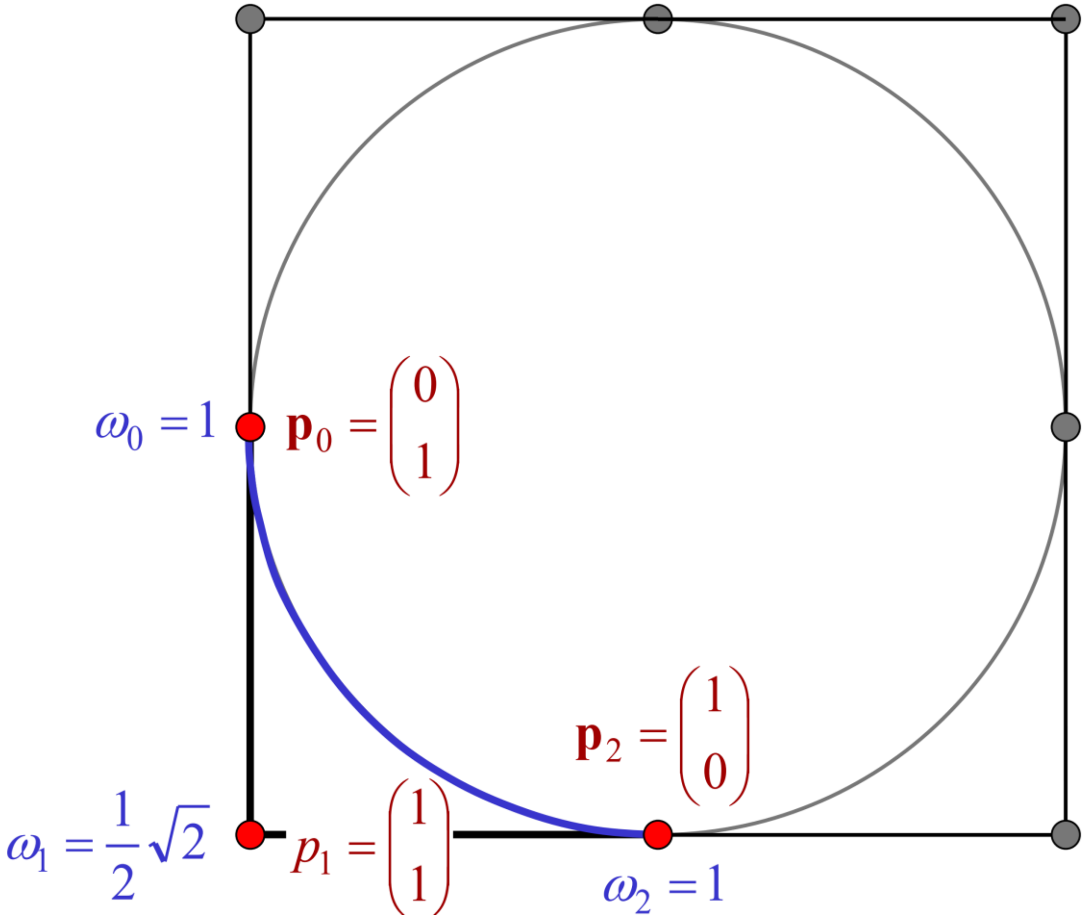
Standard Form:

$$\boldsymbol{f}(t) = \frac{B_0^{(2)}(\tilde{t})\boldsymbol{p}_0 + B_1^{(2)}(\tilde{t})\omega\boldsymbol{p}_1 + B_2^{(2)}(\tilde{t})\boldsymbol{p}_2}{B_0^{(2)}(\tilde{t}) + B_1^{(2)}(\tilde{t})\omega + B_2^{(2)}(\tilde{t})} \quad \text{with } \omega := \sqrt{\frac{1}{\omega_0\omega_2}}\,\omega_1$$

$$\boldsymbol{f}(t) = \frac{B_0^{(2)}\begin{pmatrix}1\\0\end{pmatrix} + \frac{\sqrt{2}}{2}B_1^{(2)}\begin{pmatrix}1\\1\end{pmatrix} + B_2^{(2)}\begin{pmatrix}0\\0\end{pmatrix}}{B_0^{(2)} + \frac{\sqrt{2}}{2}B_1^{(2)} + B_2^{(2)}}$$

Result: Circle in Bézier Form

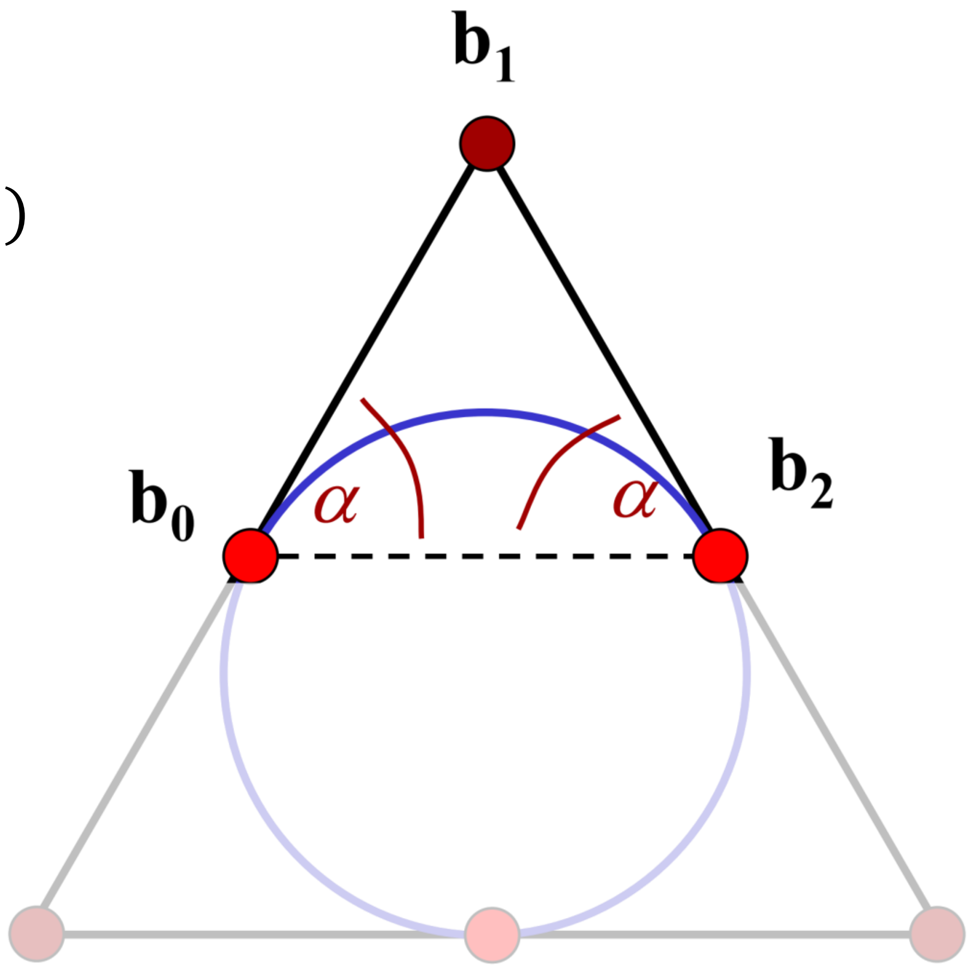
Final Result:



General Circle Segments

Circlar arcs:

- Let $\text{dist}(\boldsymbol{b}_0, \boldsymbol{b}_1) = \text{dist}(\boldsymbol{b}_1, \boldsymbol{b}_2)$
  and $\alpha = \text{angle}(\boldsymbol{b}_0, \boldsymbol{b}_2, \boldsymbol{b}_1) = \text{angle}(\boldsymbol{b}_2, \boldsymbol{b}_0, \boldsymbol{b}_1)$
- Then, $\boldsymbol{x}(t)$ is the circular arc for
$$\omega = \cos \alpha$$
- $\boldsymbol{x}(t)$ is not arc length parameterized!



Properties, Remarks

Continuity:

- The parameterization is only C^1, but G^∞
- No arc length parameterization possible
- *Even stronger:* No rational curve other than a straight line can have arc-length parameterization.

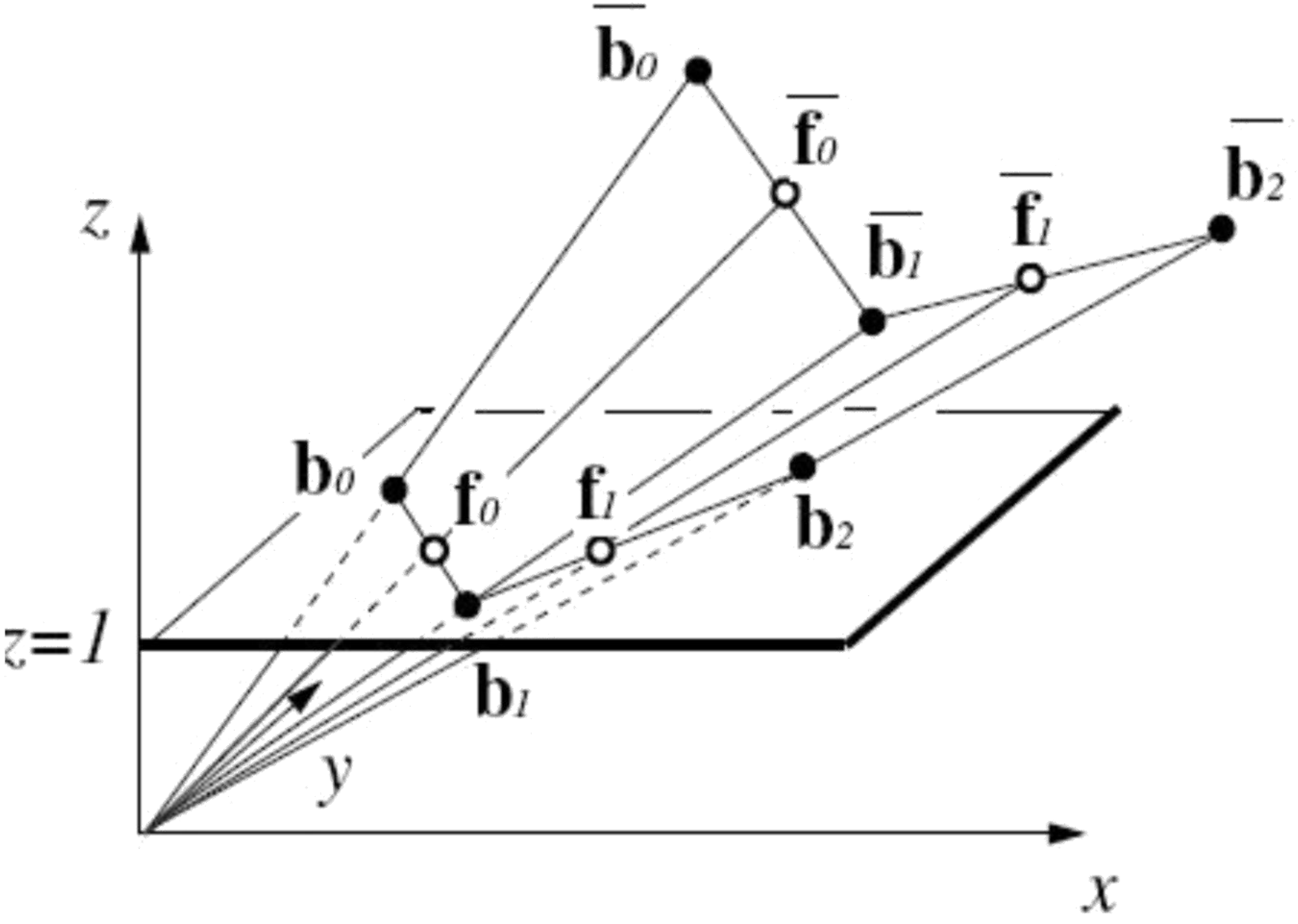
Circles in general degree Bézier splines:

- Simplest solution:
  - Form quadratic circle (segments)
  - Apply degree elevation to obtain the desired degree

Farin Points

$$\overline{f_i} = \frac{1}{2} \cdot \left(\overline{b_i} + \overline{b_{i+1}}\right)$$

$$f_i = \frac{\omega_i \cdot b_i + \omega_{i+1} \cdot b_{i+1}}{\omega_i + \omega_{i+1}}$$

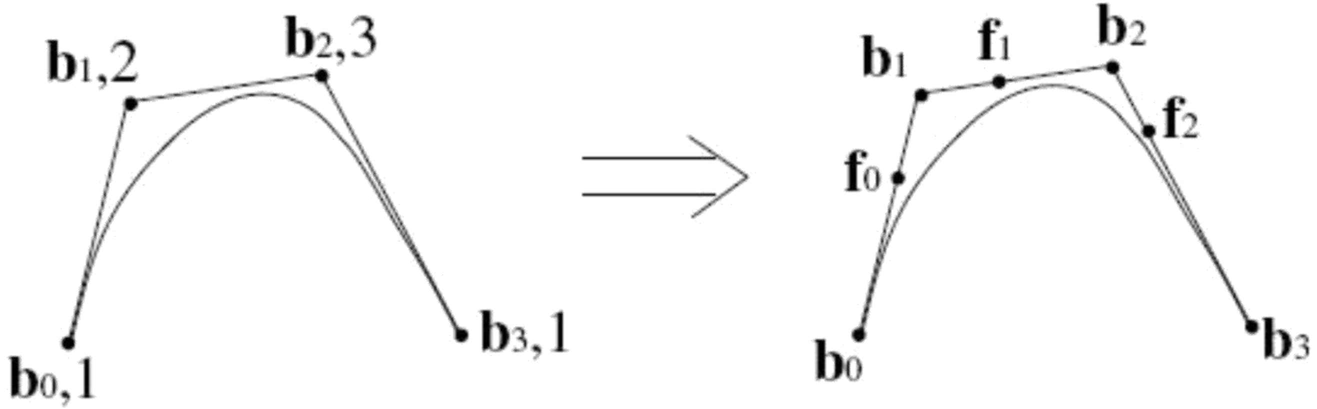


Farin Points

Not the weights themselves determine the curve shape, but the relation of the weights among each other!

The ratio $\frac{\omega_{i+1}}{\omega_i}$ is expressed by point f_i, at line segment $b_i \rightarrow b_{i+1}$ of the Bézier polygon. The following applies:

$$\frac{\omega_{i+1}}{\omega_i} = \frac{\|b_i - f_i\|}{\|b_{i+1} - f_i\|}$$



Farin Points

Alternative technique to specify weights:

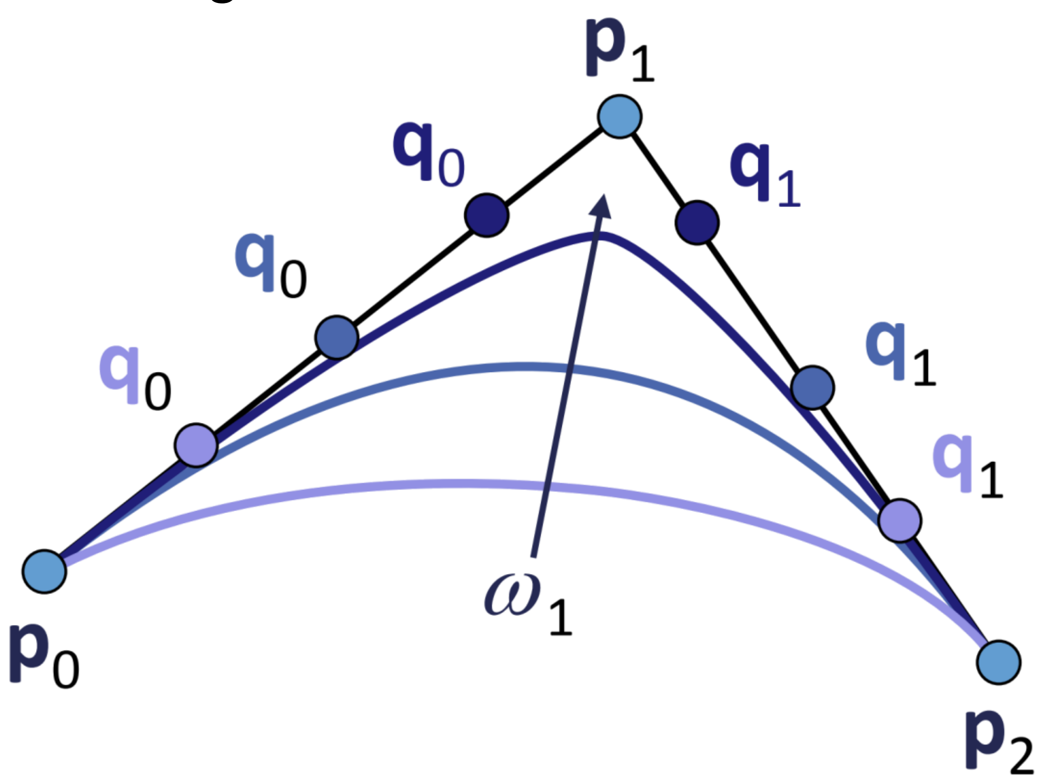
- Farin points or Weight points
- User interface: More intuitive in interactive design

Farin Points:

$$q_0 = \frac{\omega_0 p_0 + \omega_1 p_1}{\omega_0 + \omega_1}, q_1 = \frac{\omega_1 p_1 + \omega_2 p_2}{\omega_1 + \omega_2}$$

Standard Form

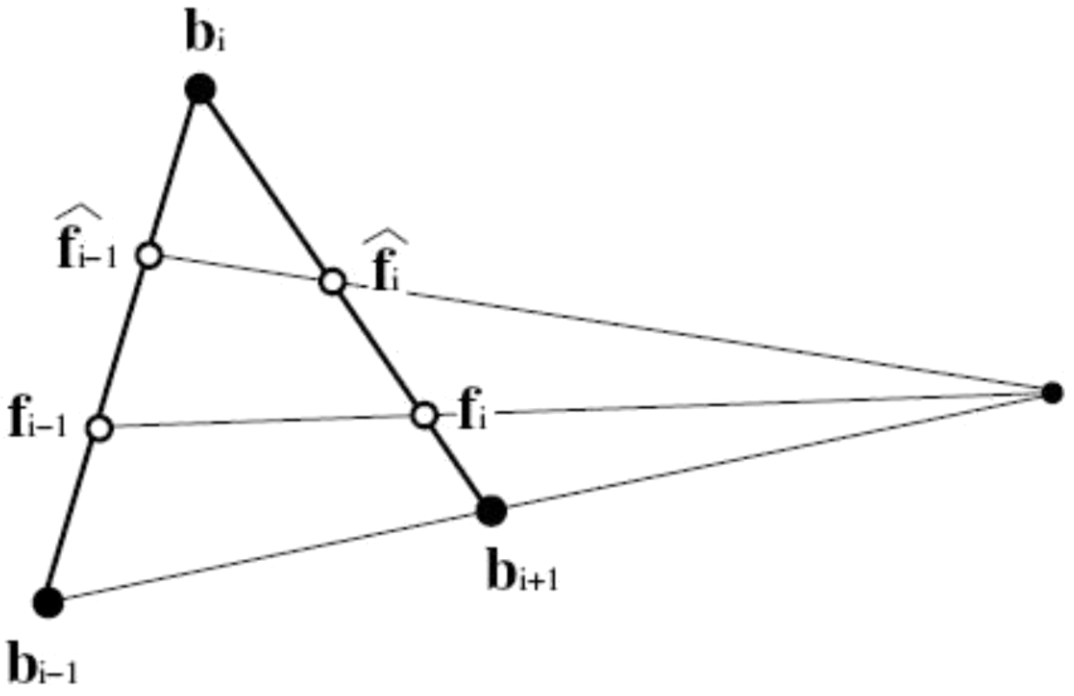
$$q_0 = \frac{p_0 + \omega_1 p_1}{1 + \omega_1}, q_1 = \frac{p_1 + \omega_1 p_2}{1 + \omega_1}$$



Farin Points

Farin Points and changing of weight:

- The change of the weight ω_i into $\widehat{\omega}_i$ under preservation of the other weights only changes the Farin points f_{i-1}, f_i to $\hat{f}_{i-1}$, $\widehat{f_i}$



Rational Curves: Rational Bézier Curves

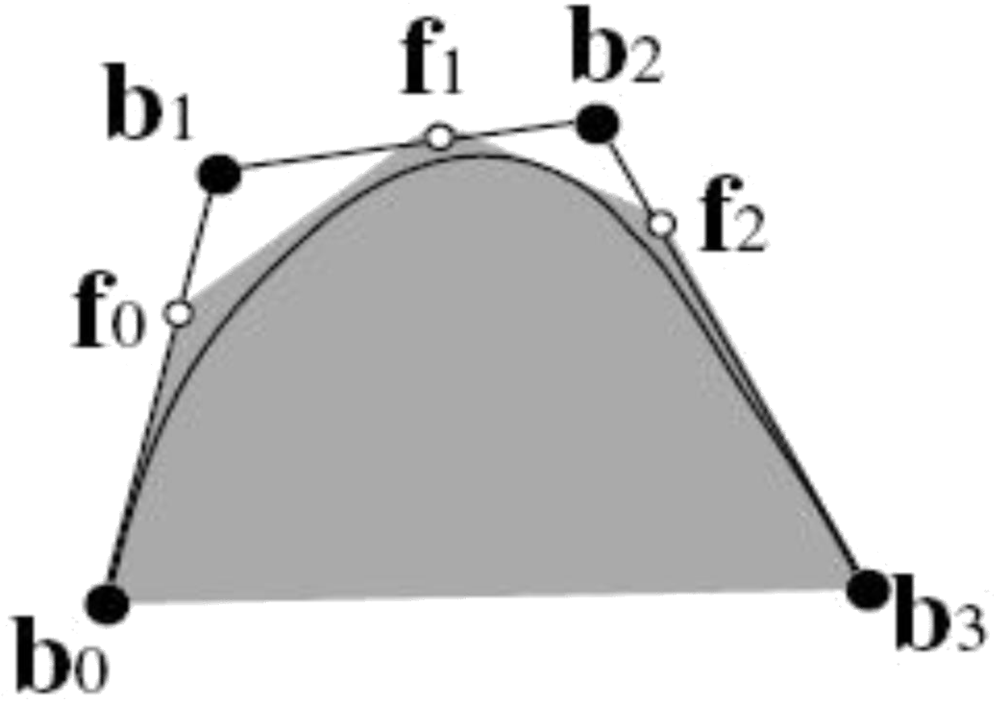
Properties of rational Bézier curves:

- (Let $\omega_i > 0$ for $i = 0, \ldots, n$)
- End point interpolation
- Tangent direction in the boundary points corresponds with the direction of the control polygon
- Variation diminishing property

Rational Curves: Rational Bézier Curves

Convex hull properties:

Tightened convex hull properties: the curve lies in the convex hull of $(b_0, f_0, \dots, f_{n-1}, b_n)$



Derivatives

Computing derivatives of rational Bézier curves:

- Straightforward: Apply quotient rule
- A simpler expression can be derived using an algebraic trick:

$$\boldsymbol{f}(t) = \frac{\sum_{i=0}^{n} B_i^{(d)}(t)\omega_i \boldsymbol{p}_i}{\sum_{i=0}^{n} B_i^{(d)}(t)\omega_{\boldsymbol{i}}} =: \frac{\boldsymbol{p}(t)}{\omega(t)}$$

$$\boldsymbol{f}(t) = \frac{\boldsymbol{p}(t)}{\omega(t)} \Rightarrow \boldsymbol{p}(t) = \boldsymbol{f}(t)\omega(t) \Rightarrow \boldsymbol{p}'(t) = \boldsymbol{f}'(t)\omega(t) + \boldsymbol{f}(t)\omega'(t)$$

$$\Rightarrow \boldsymbol{f}'(t)\omega(t) = \boldsymbol{p}'(t) - \boldsymbol{f}(t)\omega'(t) \Rightarrow \boldsymbol{f}'(t) = \frac{\boldsymbol{p}'(t) - \boldsymbol{f}(t)\omega'(t)}{\omega(t)}$$

Derivatives

At the end points:

$$\boldsymbol{f}'(t) = \frac{\boldsymbol{p}'(t) - \omega'(t)\boldsymbol{f}(t)}{\omega(t)}$$

$$\boldsymbol{f}'(0) = \frac{\boldsymbol{p}'(0) - \omega'(0)\boldsymbol{f}(0)}{\omega(0)}$$

$$= \frac{d(\omega_1\boldsymbol{p}_1 - \omega_0\boldsymbol{p}_0) - d(\omega_1 - \omega_0)\boldsymbol{p}_0}{\omega_0} = \frac{d}{\omega_0}(\omega_1\boldsymbol{p}_1 - \omega_0\boldsymbol{p}_0 - \omega_1\boldsymbol{p}_0 + \omega_0\boldsymbol{p}_0)$$

$$= d\,\frac{\omega_1}{\omega_0}(\boldsymbol{p}_1 - \boldsymbol{p}_0)$$

$$\boldsymbol{f}'(1) = d\,\frac{\omega_{d-1}}{\omega_d}(\boldsymbol{p}_d - \boldsymbol{p}_{d-1})$$

NURBS

Non-Uniform Rational B-Splines

NURBS

NURBS: Rational B-Splines

- Same idea:
  - Control points in homogenous coordinates
  - Evaluate curve in $(d + 1)$-dimensional space
    (same as before)
  - For display, divide by ω-component
    - (we can divide anytime)

NURBS

NURBS: Rational B-Splines

- Formally:($N_i^{(d)}$:B-spline basis function i of degree d)

$$\boldsymbol{f}(t) = \frac{\sum_{i=1}^{n} N_i^{(d)}(t)\omega_i \boldsymbol{p}_i}{\sum_{i=1}^{n} N_i^{(d)}(t)\omega_{\boldsymbol{i}}}$$

- Knot sequences etc. all remain the same
- de Boor algorithm – similar to rational de Casteljau alg.
  - option 1. – apply separately to numerator, denominator
  - option 2. – normalize weights in each intermediate result

    - the second option is numerically more stable