# 计算机辅助几何设计
# 2023秋学期

## Spline Surfaces

Tensor Product Surfaces · Total Degree Surfaces

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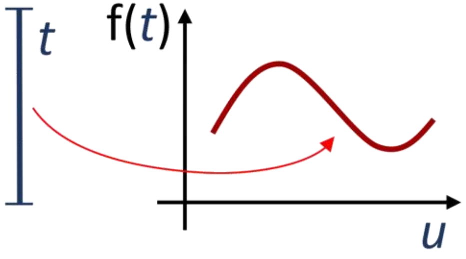
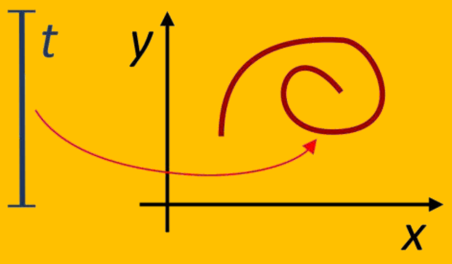
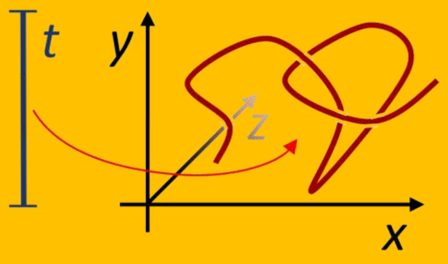
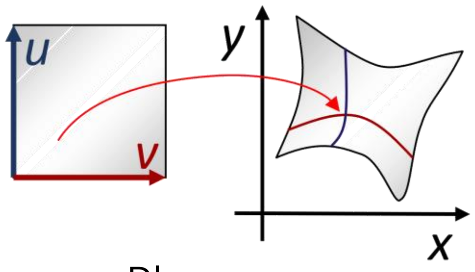
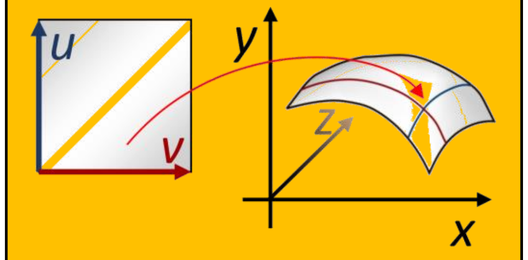
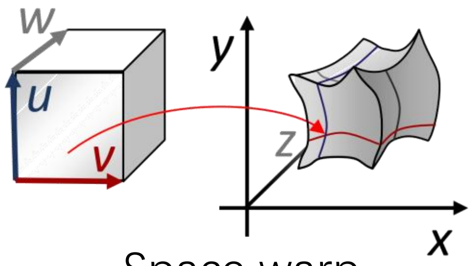
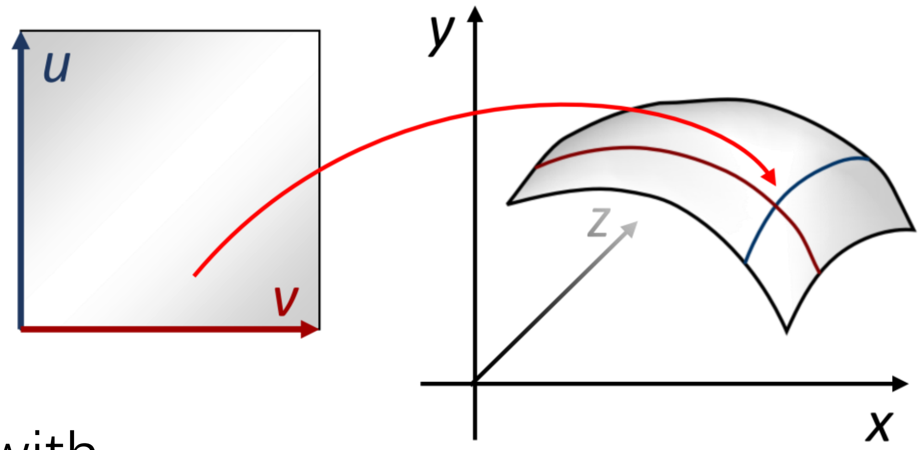
| | Output: 1D | Output: 2D | Output: 3D |
|---|---|---|---|
| Input: 1D |  Function graph |  Plane curve |  Space curve |
| Input: 2D | |  Plane warp |  Surface |
| Input: 3D | | |  Space warp |

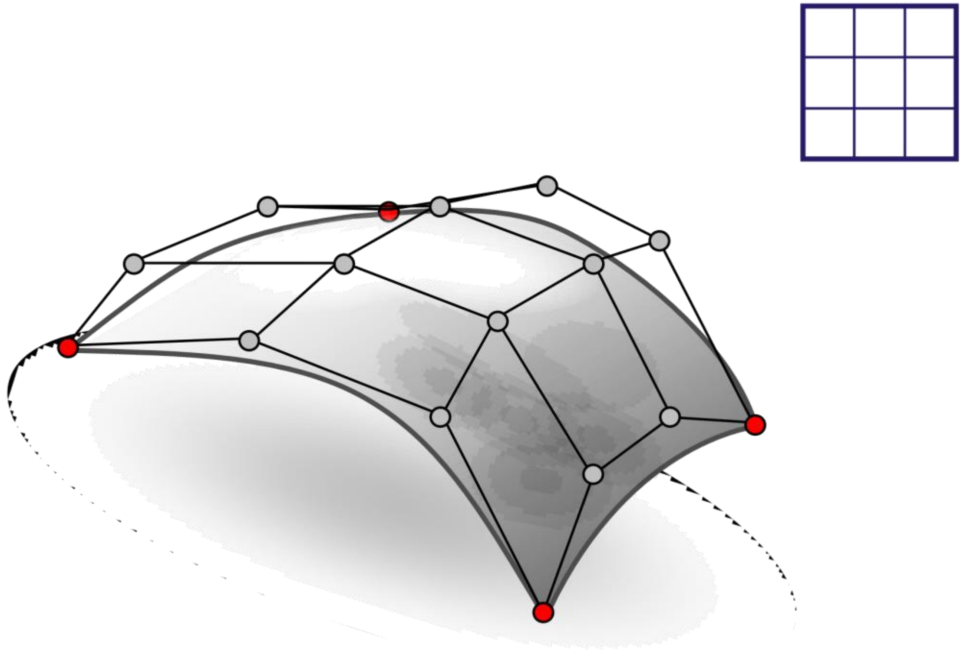
# Spline Surfaces

**Parametric spline surfaces:**

- Two parameter coordinates $(u, v)$
- Piecewise bivariate polynomials
  (rational surfaces → homogeneous coords)
- Assemble multiple pieces to form a surface with continuity
- Each piece is called *spline patch*

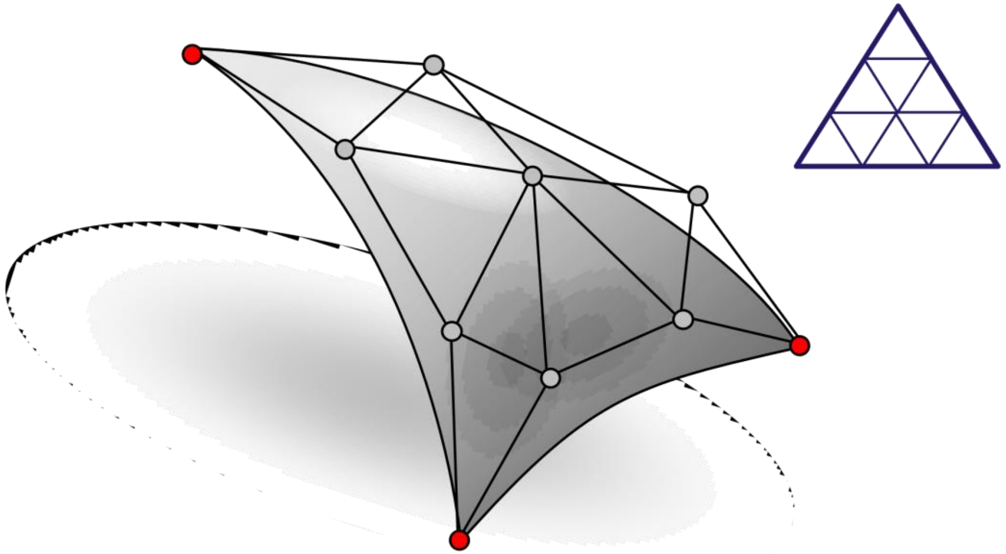


# Spline Surfaces



## Two different approaches

- Tensor product surfaces
  - Simple construction
  - Everything carries over from curve case
  - Quad patches
  - Degree anisotropy
- Total degree surfaces
  - Not as straightforward
    (blossoming will help)
  - Isotropic degree
  - Triangle patches
  - “natural” generalization of curves



# Tensor Product Surfaces

# Tensor Product Surfaces

## Simple Idea

- Given a basis for a one dimensional function space on the interval $t \in [t_0, t_1] \to \mathbb{R}^d$:
$$\boldsymbol{B}^{(curv)} \coloneqq \{b_1(t), \dots, b_n(t)\}$$
- Build a new basis with two parameters by taking all possible products:
$$\boldsymbol{B}^{(surf)} \coloneqq \{b_1(u)b_1(v), b_1(u)b_2(v), \dots, b_n(u)b_n(v)\}$$

# Tensor Product Surfaces

## Tensor product basis

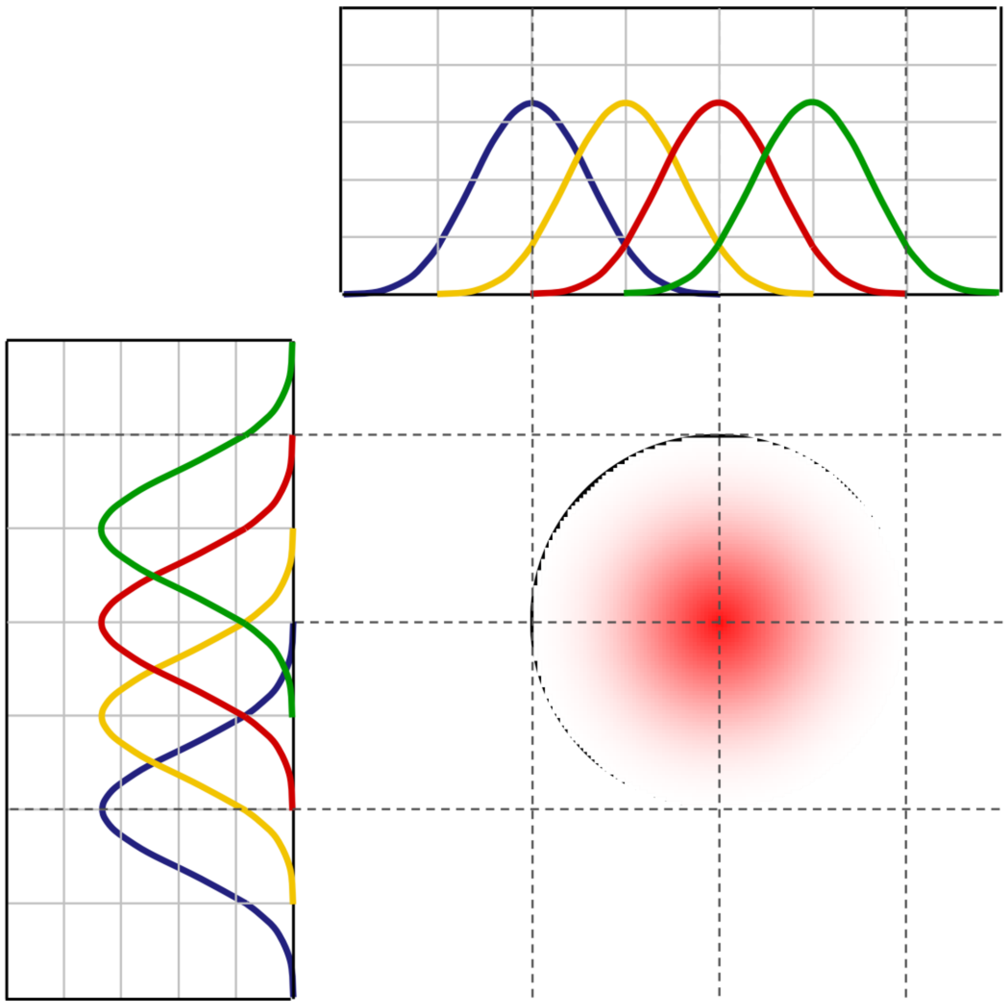
| | $b_1(u)$ | $b_2(u)$ | $b_3(u)$ | $b_4(u)$ |
|---|---|---|---|---|
| $b_1(v)$ | $b_1(v)b_1(u)$ | $b_1(v)b_2(u)$ | $b_1(v)b_3(u)$ | $b_1(v)b_4(u)$ |
| $b_2(v)$ | $b_2(v)b_1(u)$ | $b_2(v)b_2(u)$ | $b_2(v)b_3(u)$ | $b_2(v)b_4(u)$ |
| $b_3(v)$ | $b_3(v)b_1(u)$ | $b_3(v)b_2(u)$ | $b_3(v)b_3(u)$ | $b_3(v)b_4(u)$ |
| $b_4(v)$ | $b_4(v)b_1(u)$ | $b_4(v)b_2(u)$ | $b_4(v)b_3(u)$ | $b_4(v)b_4(u)$ |

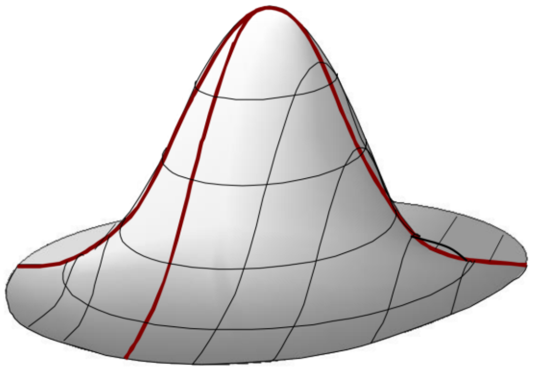
# Monomial Example

**Tensor product basis of cubic monomials**

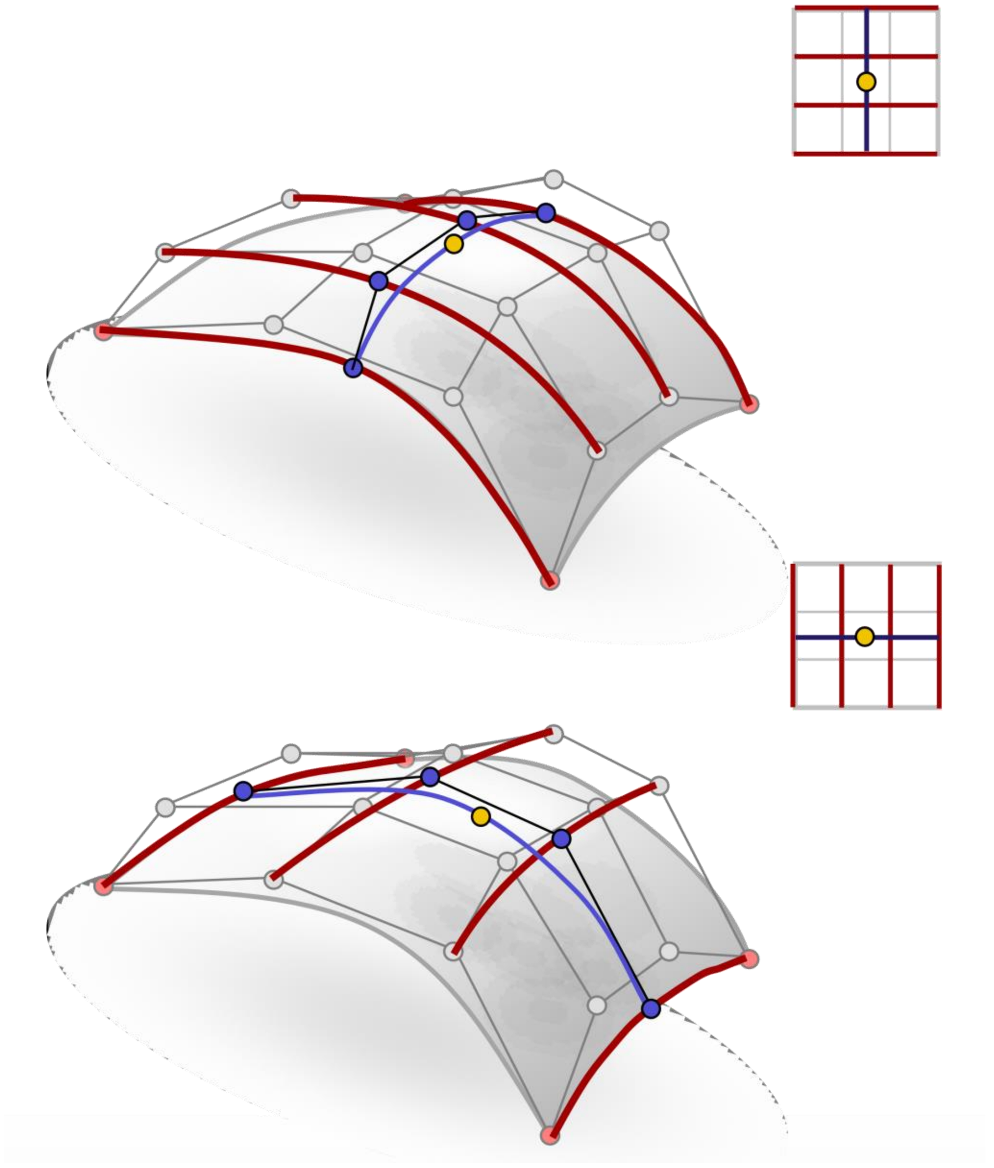
| | $1$ | $u$ | $u^2$ | $u^3$ |
|---|---|---|---|---|
| $1$ | $1$ | $u$ | $u^2$ | $u^3$ |
| $v$ | $v$ | $vu$ | $vu^2$ | $vu^3$ |
| $v^2$ | $v^2$ | $v^2u$ | $v^2u^2$ | $v^2u^3$ |
| $v^3$ | $v^3$ | $v^3u$ | $v^3u^2$ | $v^3u^3$ |

# Example





# Tensor Product Surfaces



## Tensor Product Surfaces

$$\boldsymbol{f}(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i(u) b_j(v) \boldsymbol{p}_{i,j}$$

$$= \sum_{i=1}^{n} b_i(u) \sum_{j=1}^{n} b_j(v) \boldsymbol{p}_{i,j}$$

$$= \sum_{j=1}^{n} b_j(v) \sum_{i=1}^{n} b_i(u) \boldsymbol{p}_{i,j}$$

- “Curves of Curves”
- Order does not matter

# Properties

**Properties of tensor product surfaces:**

- Linear invariance: Obvious
- Affine invariance?
  - Needs partition of unity property
  - Assume basis $B^{(curv)} := \{b_1(t), \dots, b_n(t)\}$ forms a partition of unity, i.e.: $\sum_{i=1}^{n} b_i(v) = 1$
  - Then we get:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} b_i(u) b_j(v) = \sum_{i=1}^{n} b_i(u) \sum_{j=1}^{n} b_j(v) = \sum_{j=1}^{n} b_j(v) \cdot 1 = 1$$

- Affine invariance for tensor product surfaces is induced by the corresponding property of the employed curve basis

# Properties

## Properties of tensor product surfaces:

- Convex Hull?
  - Assume basis $\boldsymbol{B}^{(curv)} := \{b_1(t), \ldots, b_n(t)\}$ forms a partition of unity and it is nonnegative ($\geq 0$) on $t \in [t_0, t_1]$
  - Obviously, we then have:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \underbrace{b_i(u)}_{\geq 0} \underbrace{b_j(v)}_{\geq 0} \geq 0$$

  - So we have the convex hull property on $[t_0, t_1]^2$
- The convex hull property for tensor product surface is induced by the property of the employed curve basis

# Partial Derivatives

**Computing partial derivatives:**

- First derivatives:

$$\frac{\partial}{\partial u}\sum_{i=1}^{n}\sum_{j=1}^{n} b_i(u)b_j(v)\boldsymbol{p}_{i,j} = \sum_{j=1}^{n} b_j(v)\sum_{i=1}^{n}\left(\frac{d}{du}b_i\right)(u)\boldsymbol{p}_{i,j}$$

$$\frac{\partial}{\partial v}\sum_{i=1}^{n}\sum_{j=1}^{n} b_i(u)b_j(v)\boldsymbol{p}_{i,j} = \sum_{i=1}^{n} b_i(u)\sum_{j=1}^{n}\left(\frac{d}{dv}b_j\right)(v)\boldsymbol{p}_{i,j}$$

- Just spline-curve combinations of curve derivatives

# Partial Derivatives

**Computing partial derivatives:**

- Second derivatives:

$$\frac{\partial}{\partial u^2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_i(u)b_j(v)\boldsymbol{p}_{i,j} = \sum_{j=1}^{n} b_j(v)\sum_{i=1}^{n}\left(\frac{d^2}{du^2}b_i\right)(u)\boldsymbol{p}_{i,j}$$

$$\frac{\partial^2}{\partial u \partial v}\sum_{i=1}^{n}\sum_{j=1}^{n} b_i(u)b_j(v)\boldsymbol{p}_{i,j} = \frac{\partial}{\partial v}\sum_{j=1}^{n} b_j(v)\sum_{i=1}^{n}\left(\frac{d}{du}b_i\right)(u)\boldsymbol{p}_{i,j}$$

$$= \sum_{j=1}^{n}\left(\frac{d}{dv}b_j\right)(v)\sum_{i=1}^{n}\left(\frac{d}{du}b_i\right)(u)\boldsymbol{p}_{i,j}$$

# Partial Derivatives

**Computing partial derivatives:**

- General derivatives:

$$\frac{\partial^{r+s}}{\partial u^r \partial v^s} \sum_{i=1}^{n} \sum_{j=1}^{n} b_i(u) b_j(v) \boldsymbol{p}_{i,j} = \sum_{j=1}^{n} \left( \frac{d^s}{dv^s} b_i \right)(v) \sum_{i=1}^{n} \left( \frac{d^r}{du^r} b_i \right)(u) \boldsymbol{p}_{i,j}$$

$$= \sum_{i=1}^{n} \left( \frac{d^r}{du^r} b_i \right)(u) \sum_{j=1}^{n} \left( \frac{d^s}{dv^s} b_j \right)(v) \boldsymbol{p}_{i,j}$$

# Normal Vectors

**We can compute normal vectors from partial derivatives:**

$$\boldsymbol{n}(u,v) = \frac{\left(\sum_{j=1}^{n} b_j(v) \sum_{i=1}^{n} \frac{d}{du} b_i(u) \boldsymbol{p}_{i,j}\right) \times \left(\sum_{j=1}^{n} \frac{d}{dv} b_j(v) \sum_{i=1}^{n} b_i(u) \boldsymbol{p}_{i,j}\right)}{\left\|\left(\sum_{j=1}^{n} b_j(v) \sum_{i=1}^{n} \frac{d}{du} b_i(u) \boldsymbol{p}_{i,j}\right) \times \left(\sum_{j=1}^{n} \frac{d}{dv} b_j(v) \sum_{i=1}^{n} b_i(u) \boldsymbol{p}_{i,j}\right)\right\|}$$

- Problem: degenerate cases
  - Collinear tangents
  - Irregular parametrization
- Need extra code to handle special cases

# Tensor Product Surfaces

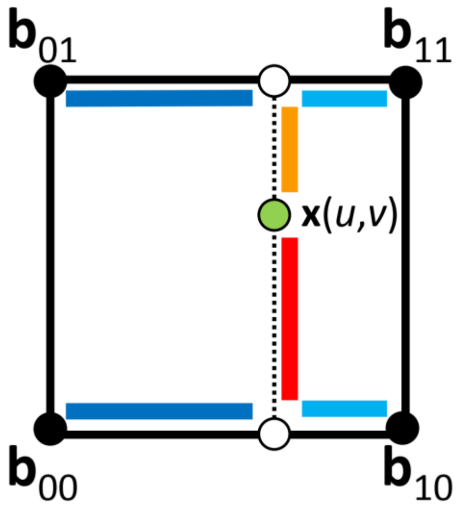
## Tensor Product Bézier Surfaces

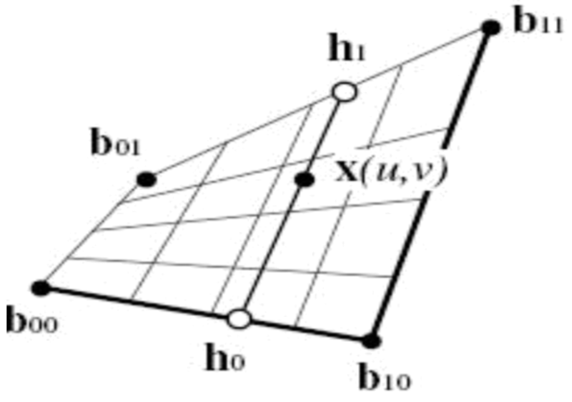
## Tensor Product Bézier Spline Surfaces

# Tensor Product Bézier Surfaces

Bézier curves:
repeated linear interpolation

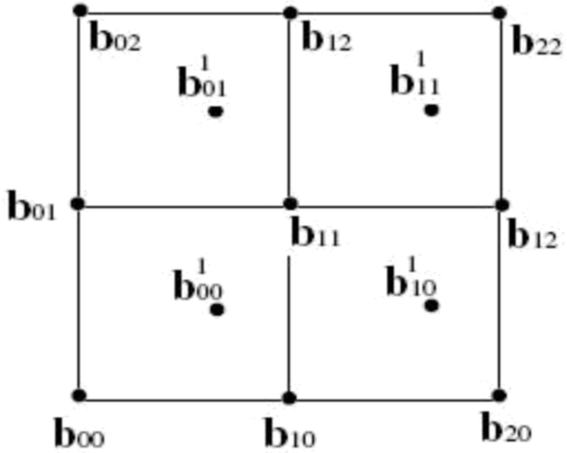
> *now a different setup:*
> 4 points $\boldsymbol{b}_{00}, \boldsymbol{b}_{10}, \boldsymbol{b}_{11}, \boldsymbol{b}_{01}$
> Parameter area $[0,1] \times [0,1]$



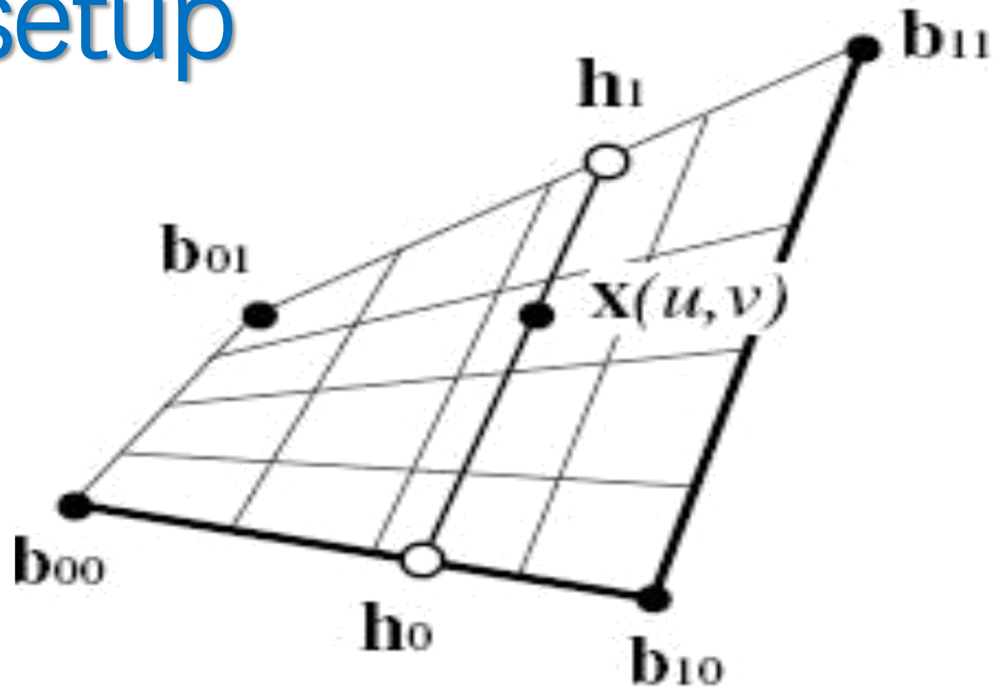




bilinear interpolation:
repeated linear interpolation

repeated bilinear interpolation:
Gives us tensor product Bézier surfaces
(example) shows quadratic Bézier Surfaces)



# Some formulas for this setup




$$\boldsymbol{h}_0 = (1-u)\boldsymbol{b}_{00} + u\boldsymbol{b}_{10}$$

$$\boldsymbol{h}_1 = (1-u)\boldsymbol{b}_{01} + u\boldsymbol{b}_{11}$$

$$\begin{aligned}\boldsymbol{x}(u,v) &= (1-v)\boldsymbol{h}_0 + v\boldsymbol{h}_1 \\ &= (1-v)[(1-u)\boldsymbol{b}_{00} + u\boldsymbol{b}_{10}] + v[(1-u)\boldsymbol{b}_{01} + u\boldsymbol{b}_{11}]\end{aligned}$$

$$\boldsymbol{x}(u,v) = (1-u)(1-v)\boldsymbol{b}_{00} + u(1-v)\boldsymbol{b}_{10} + (1-u)v\boldsymbol{b}_{01} + uv\boldsymbol{b}_{11}$$

# Some formulas for this setup

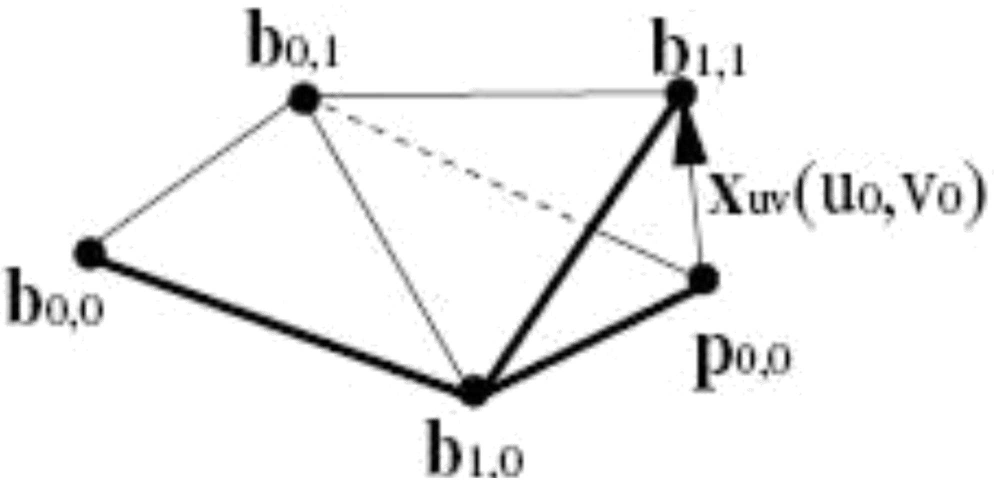
Derivatives of bilinear surfaces

$$\boldsymbol{x}_u(u,v) = (1-v)(\boldsymbol{b}_{10} - \boldsymbol{b}_{00}) + v(\boldsymbol{b}_{11} - \boldsymbol{b}_{01})$$

$$\boldsymbol{x}_v(u,v) = (1-u)(\boldsymbol{b}_{01} - \boldsymbol{b}_{00}) + u(\boldsymbol{b}_{11} - \boldsymbol{b}_{10})$$

$$\boldsymbol{x}_{uu}(u,v) = \boldsymbol{x}_{vv}(u,v) = 0$$

$$\boldsymbol{x}_{uv}(u,v) = \boldsymbol{b}_{00} - \boldsymbol{b}_{10} - \boldsymbol{b}_{01} + \boldsymbol{b}_{11}$$



# Some formulas for this setup

## Biquadratic surfaces

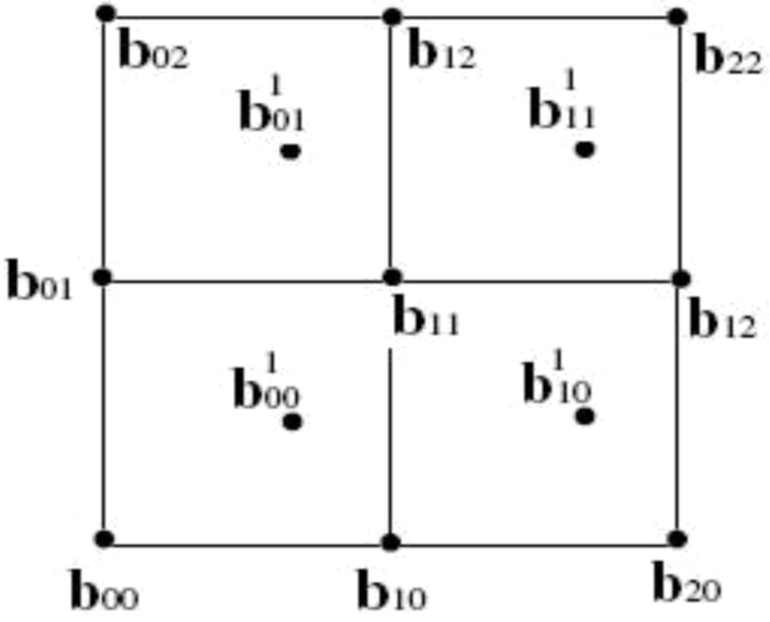
$$\boldsymbol{b}_{00}^{1} = (1-u)(1-v)\boldsymbol{b}_{00} + u(1-v)\boldsymbol{b}_{10} + (1-u)v\boldsymbol{b}_{01} + uv\boldsymbol{b}_{11}$$

$$\boldsymbol{b}_{10}^{1} = (1-u)(1-v)\boldsymbol{b}_{10} + u(1-v)\boldsymbol{b}_{20} + (1-u)v\boldsymbol{b}_{11} + uv\boldsymbol{b}_{21}$$

$$\boldsymbol{b}_{01}^{1} = (1-u)(1-v)\boldsymbol{b}_{01} + u(1-v)\boldsymbol{b}_{11} + (1-u)v\boldsymbol{b}_{02} + uv\boldsymbol{b}_{12}$$

$$\boldsymbol{b}_{11}^{1} = (1-u)(1-v)\boldsymbol{b}_{11} + u(1-v)\boldsymbol{b}_{21} + (1-u)v\boldsymbol{b}_{12} + uv\boldsymbol{b}_{22}$$

$$\begin{aligned}\boldsymbol{x}(u,v) \\ &= (1-u)(1-v)\boldsymbol{b}_{00}^{1} + u(1-v)\boldsymbol{b}_{10}^{1} + (1-u)v\boldsymbol{b}_{01}^{1} + uv\boldsymbol{b}_{11}^{1} \\ &= \sum_{i=0}^{2}\sum_{j=0}^{2} B_i^2(u)B_j^2(v)\boldsymbol{b}_{i,j}\end{aligned}$$



# Bézier Patches

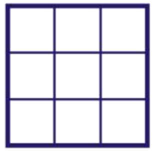
**Bézier Patches:**

- Use tensor product Bernstein basis

$$\boldsymbol{f}(u,v) = \sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u)B_j^{(d)}(v)\boldsymbol{p}_{i,j}$$

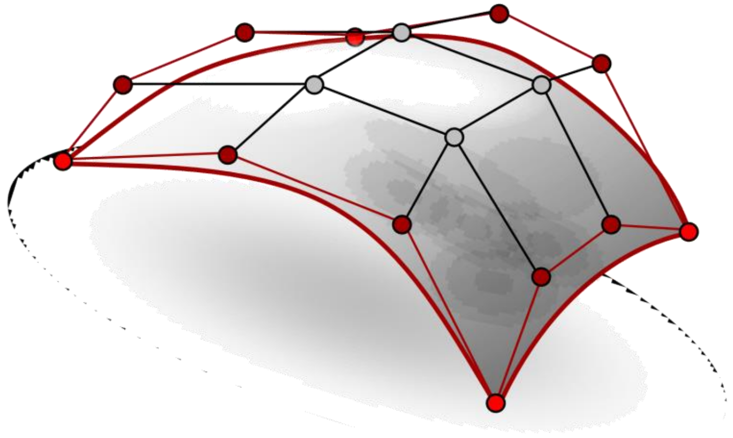
- We get automatically:
  - Affine invariance
  - Convex hull property

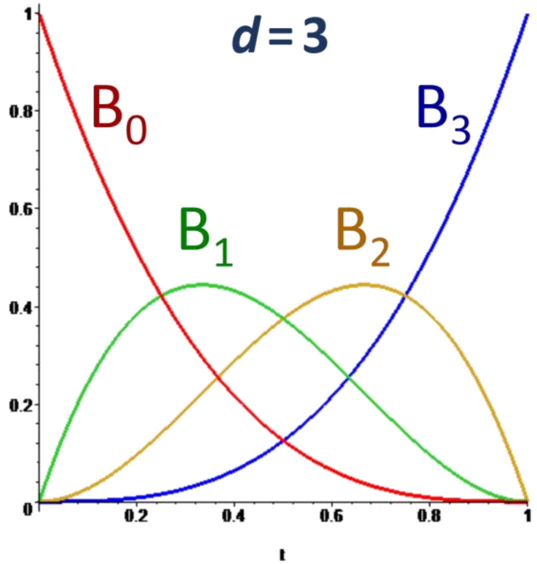
# Bézier Patches

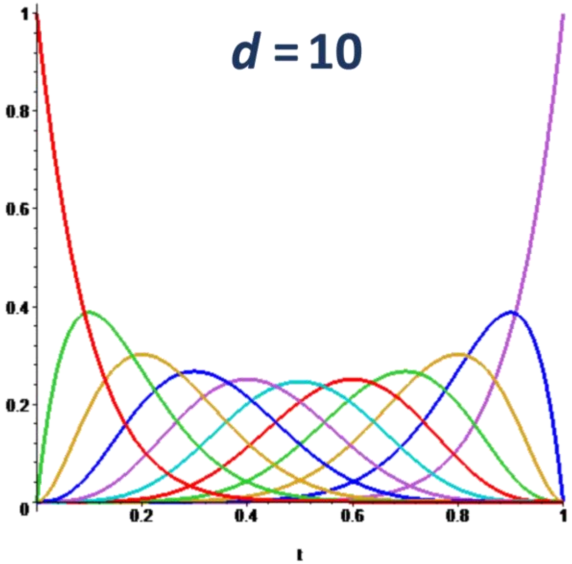


## Bézier Patches:

- Remember endpoint interpolation:
  - Boundary curves are Bézier curves of the boundary control points





# Bézier Patches

**Bézier Patches:**

- Tangent vectors:
  - First derivatives at boundary points are proportional to differences of control points:

$$\frac{\partial}{\partial u}\boldsymbol{f}(u,v)\Big|_{u=0} = \sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(v)B'^{(d)}_j(0)\boldsymbol{p}_{i,j}$$

$$= d\sum_{j=0}^{d} B_j^{(d)}(v)\left(\boldsymbol{p}_{1,j} - \boldsymbol{p}_{0,j}\right)$$

$$\frac{\partial}{\partial u}\boldsymbol{f}(u,v)\Big|_{u=1} = d\sum_{j=0}^{d} B_j^{(d)}(v)\left(\boldsymbol{p}_{d,j} - \boldsymbol{p}_{d-1,j}\right)$$

# Continuity Conditions

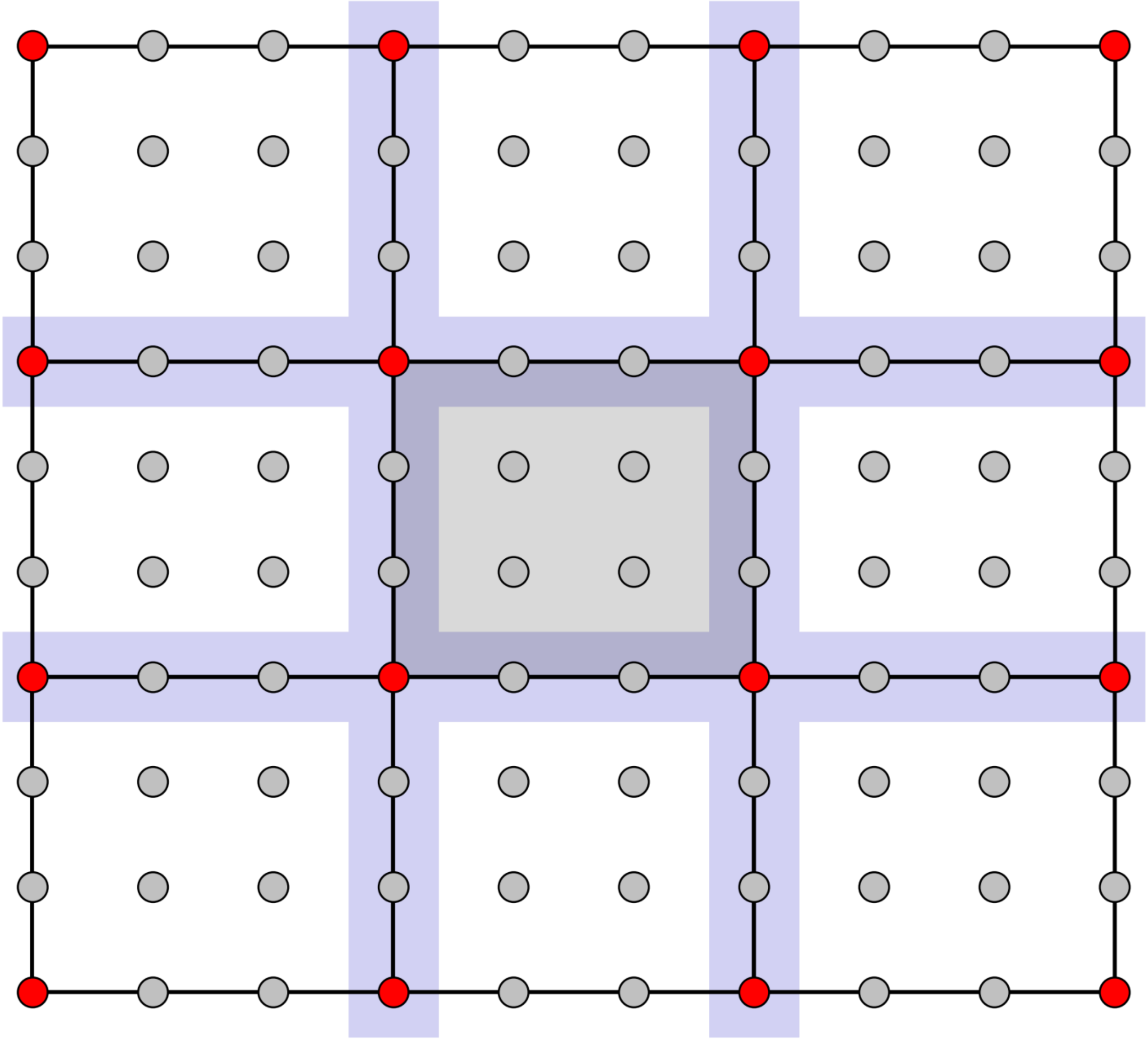
**For $C^0$ continuity:**

- Boundary control points must match

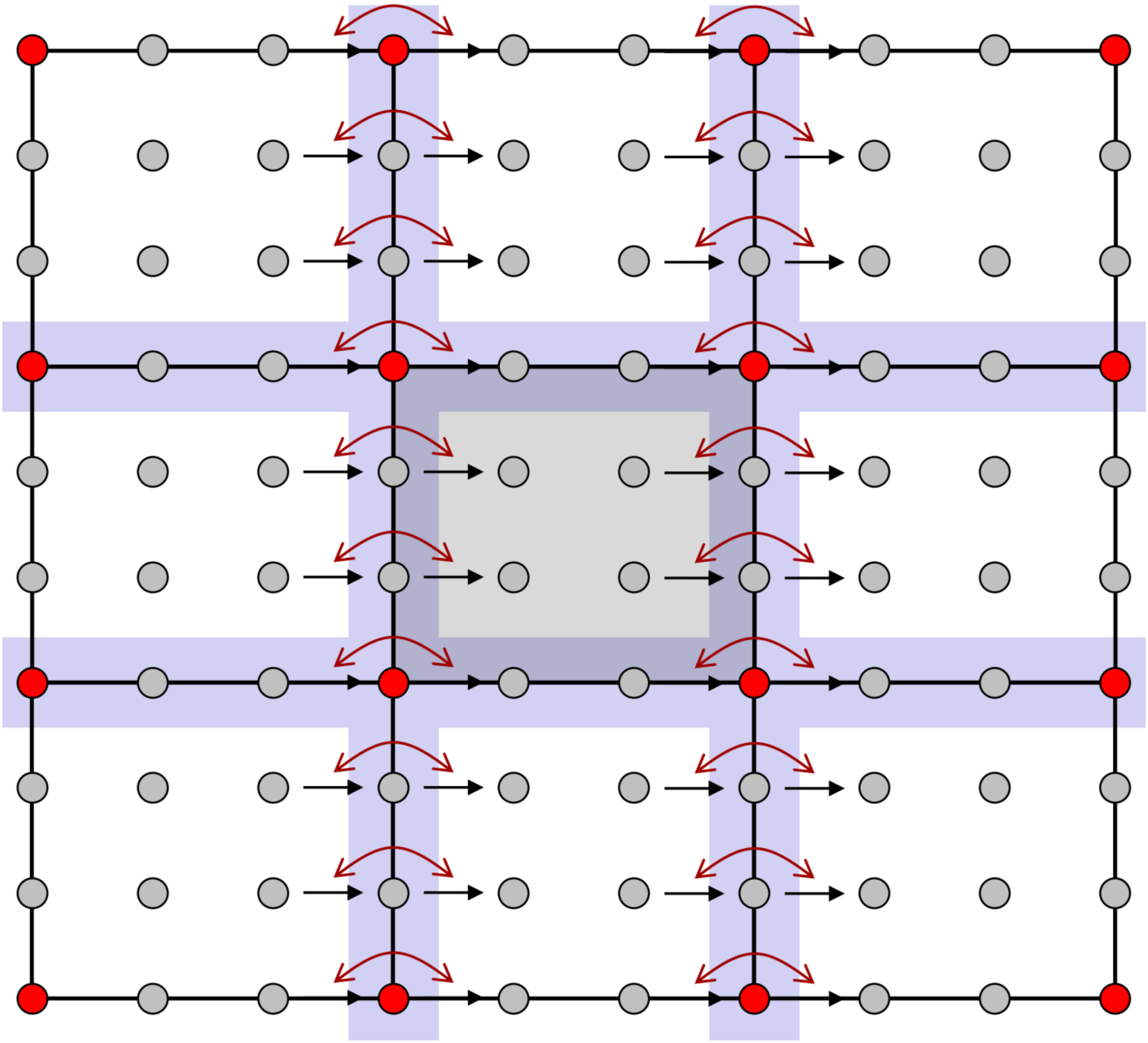
**For $C^1$ continuity:**

- Difference vectors must match at the boundary

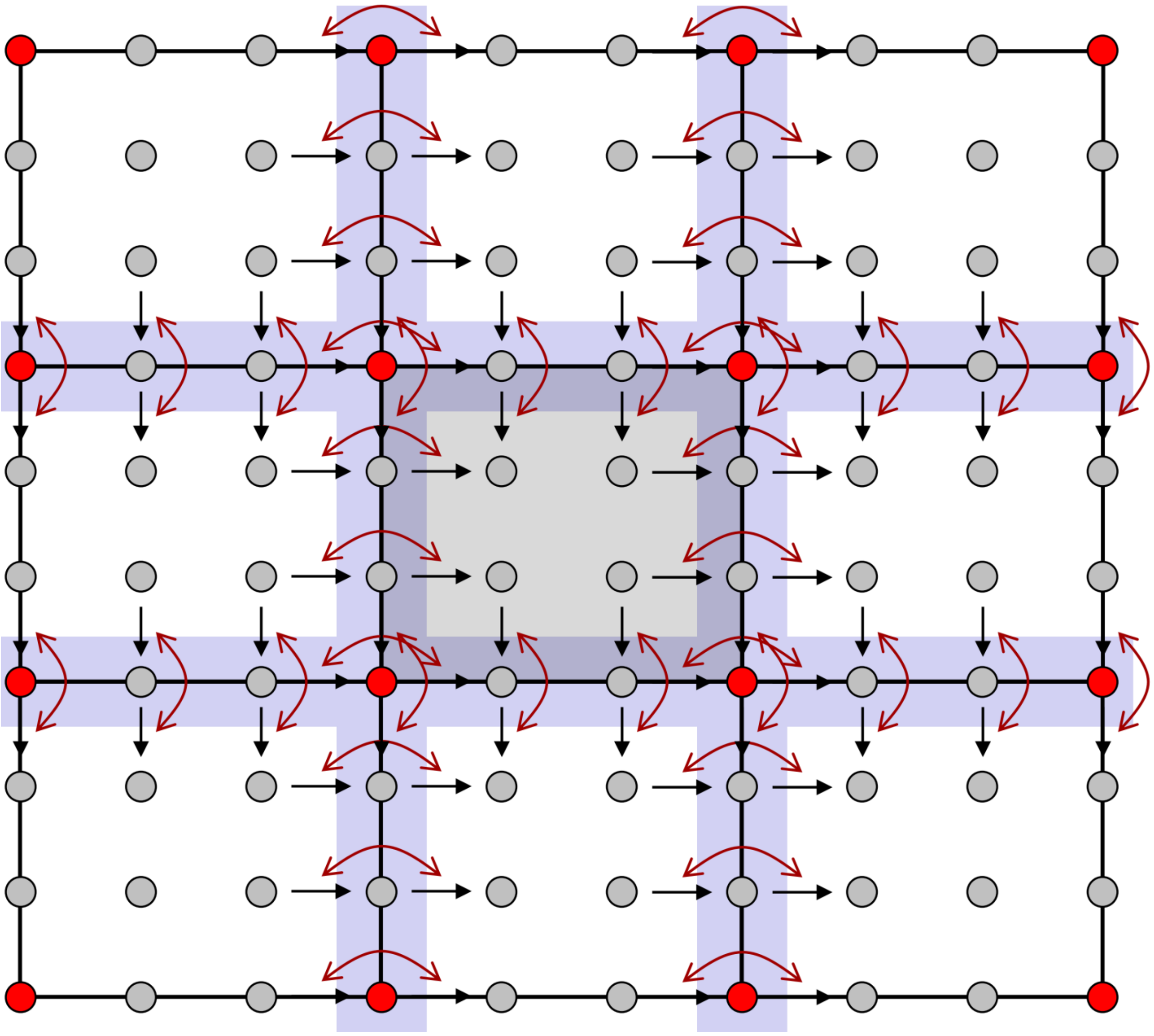
# $C^0$ Continuity



# $C^1$ Continuity



# $C^1$ Continuity



# Polars & Blossoms

## Blossoms for tensor product surfaces:

- Polar form of a polynomial tensor product surfaces of degree $d$:

$$F\colon\ \mathbb{R}\times\mathbb{R}\to\mathbb{R}^n \qquad F(u,v)$$

$$\boldsymbol{f}\colon\ \mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^n \qquad \boldsymbol{f}(u_1,\dots,u_d;v_1,\dots,v_d)$$

- Required properties:
  - **Diagonality**: $\boldsymbol{f}(u,\dots,u;v,\dots,v) = F(u,v)$
  - **Symmetry**: $\boldsymbol{f}(u_1,\dots,u_d;v_1,\dots,v_d) = f\left(u_{\pi(1)},\dots,u_{\pi(d)};v_{\mu(1)},\dots,v_{\mu(d)}\right)$
    for all permutations of indices $\pi$, $\mu$
  - **Multi-affine:** $\sum\alpha_k = 1$

$$\Rightarrow \boldsymbol{f}\left(u_1,\dots,\sum\alpha_k u_i^{(k)},\dots,u_d;v_1,\dots,v_d\right)$$

$$= \alpha_1\boldsymbol{f}\left(u_1,\dots,u_i^{(1)},\dots,u_d;v_1,\dots,v_d\right) + \dots + \alpha_n\boldsymbol{f}\left(u_1,\dots,u_i^{(n)},\dots,u_d;v_1,\dots,v_d\right)$$

and $\boldsymbol{f}\left(u_1,\dots,u_d;v_1,\dots,\sum\alpha_k v_i^{(k)},\dots,,v_d\right)$

$$= \alpha_1\boldsymbol{f}\left(u_1,\dots,u_d;v_1,\dots,v_i^{(1)},\dots,v_d\right) + \dots + \alpha_n\boldsymbol{f}\left(u_1,\dots,u_d;v_1,\dots,v_i^{(n)},\dots,v_d\right)$$

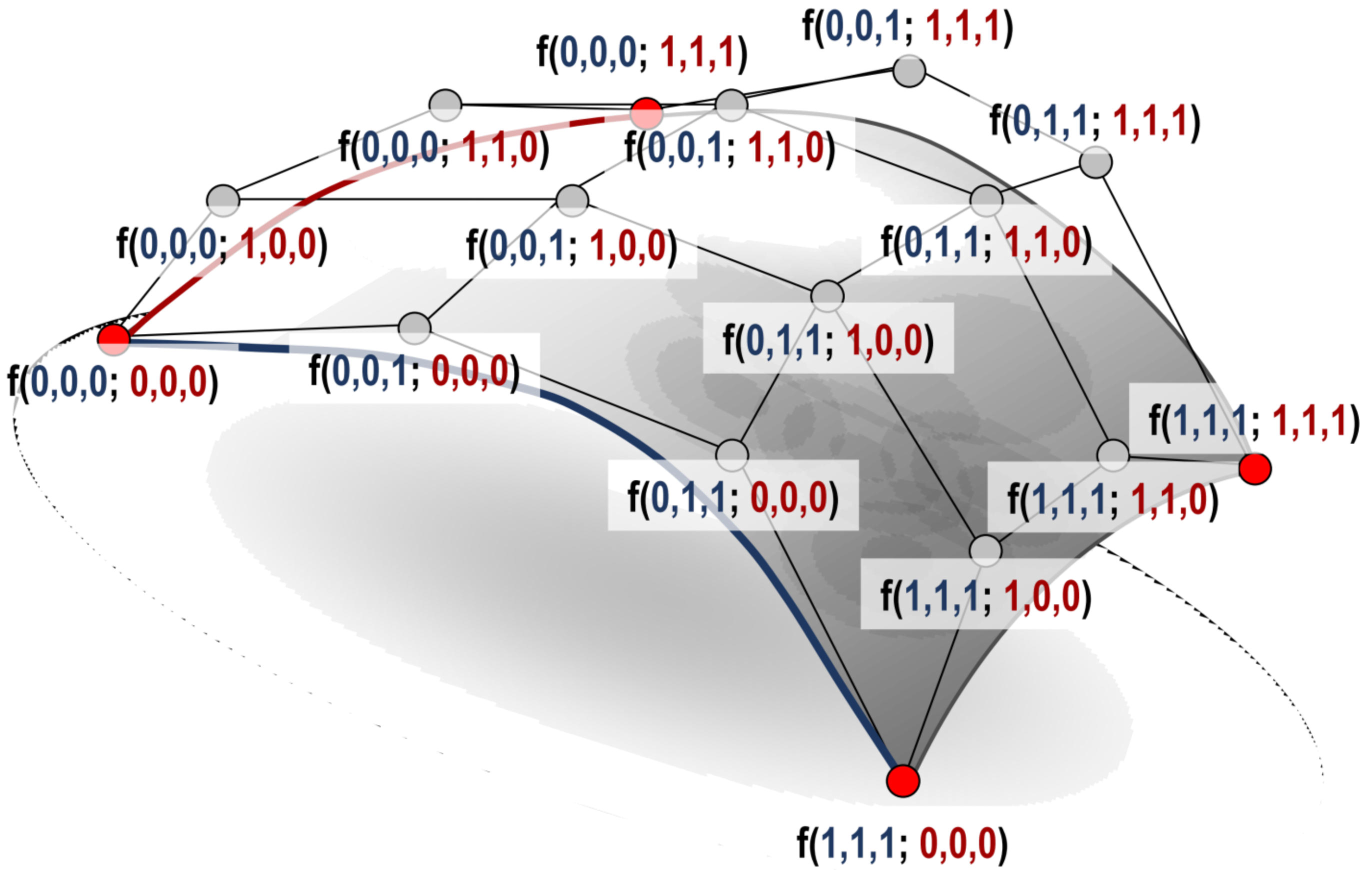
# Short Summary

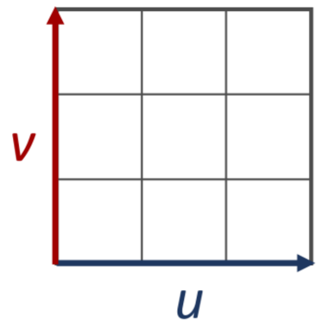
**Polar forms for tensor product surfaces:**

- Polar separately in $u$ and $v$
- Notation: $\boldsymbol{f}(\underbrace{u_1, \dots, u_d}_{u\text{-parameters}}; \underbrace{v_1, \dots, v_d}_{v\text{-parameters}})$
- Can be used to derive properties/algorithms similar to the curve case

# Bézier Control Points

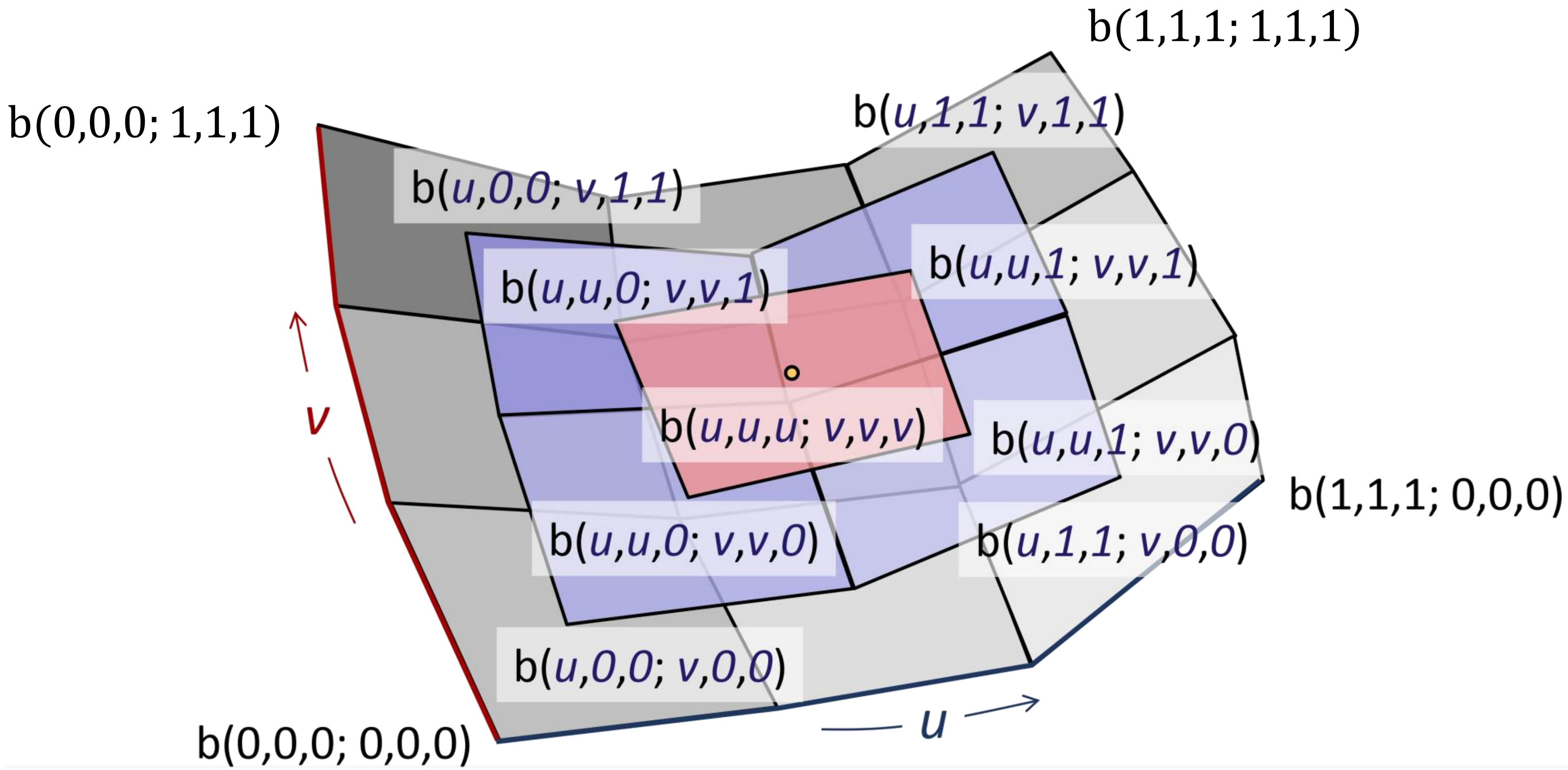
## Bézier control points in blossom notation:



# de Casteljau Algorithm

## de Casteljau algorithm for tensor product surfaces



# Tensor Product Surfaces

## Tensor Product B-Spline Surfaces

# B-Spline Patches

## B-Spline Patches

- More general than Bézier patches
  (we get Bézier patches as a special case)
- First, we fix a degree $d$
- Then, we need knot sequences in $u$ and $v$ direction:
  $(u_1, \ldots, u_n), (v_1, \ldots, v_m)$
- And a corresponding array of control points

$$\begin{matrix} d_{0,0} & \cdots & d_{n-d+1,0} \\ \cdots & & \cdots \\ d_{0,m-d+1} & \cdots & d_{n-d+1,m-d+1} \end{matrix}$$

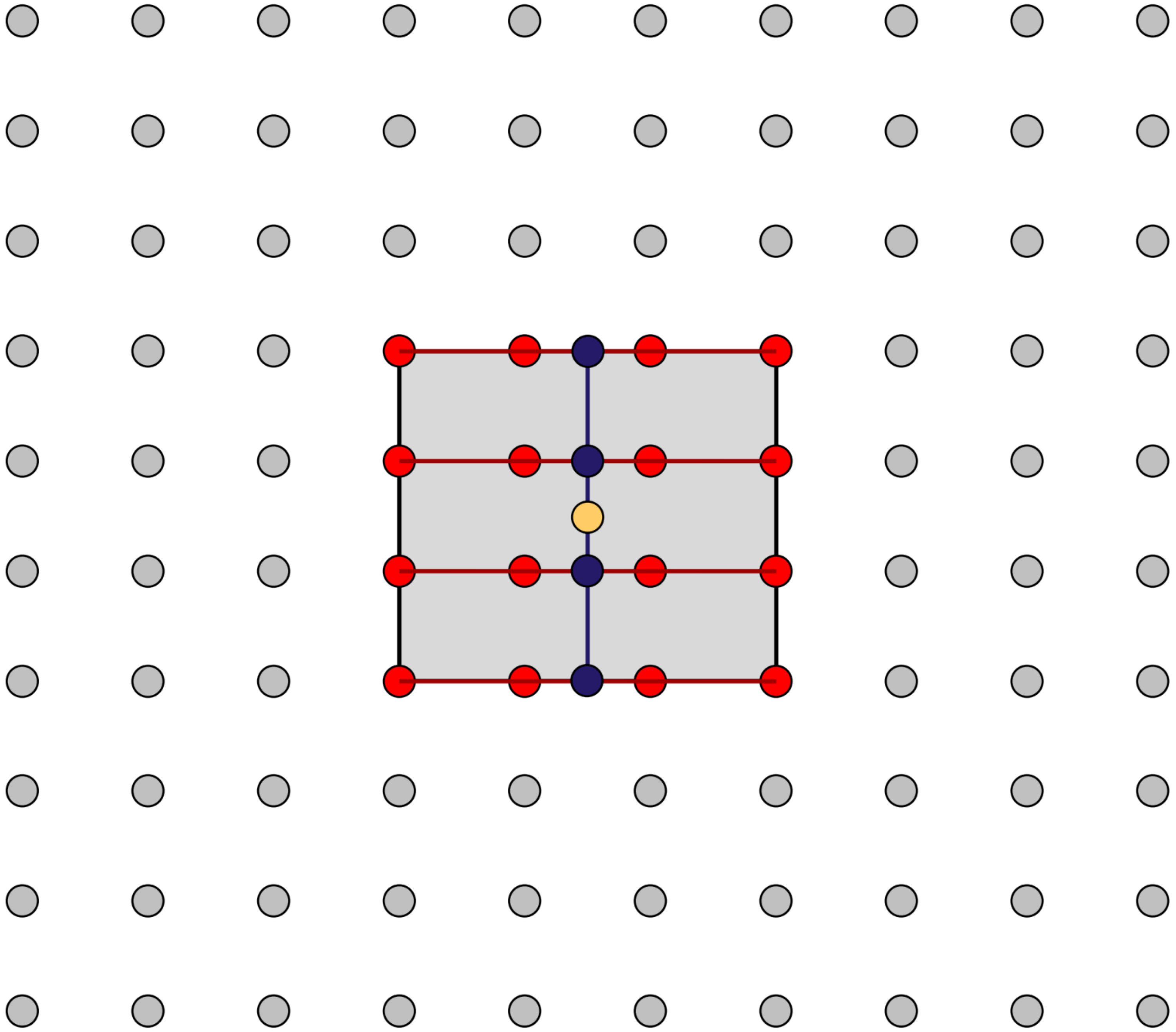
# B-Spline Patches

**Then, obtain a parametric B-spline patch as:**

$$\boldsymbol{f}(u,v) = \sum_{i=0}^{d} \sum_{j=0}^{d} N_i^{(d)}(u) N_j^{(d)}(v) \boldsymbol{p}_{i,j}$$

- We can evaluate the patches using the de Boor Algorithm:
  - "Curves of curves" idea
  - Determine the knots/control points influencing $(u, v)$, These will be no more than $(d+1) \times (d+1)$ points
  - Compute $(d+1)$ $v$-direction control points along $u$-direction, Performing $(d+1)$ curve evaluations
  - Then evaluate the curve in $v$-direction
  - (or the other way around, interchanging $u, v$-directions)

# Illustration



# B-Spline Patches

**Alternative:**

- 2D de Boor algorithm
- Works similar to the 2D de Casteljau algorithm but with different weights (we can use tensor–product blossoming to derive the weights)

# Tensor Product Surfaces

## Rational Patches

# Rational Patches

## Rational Patches

- We can use rational Bézier/B-splines to create the patches ("rational Bézier patches" / "NURBS-patches")
- Idea:
  - Form a parametric surface in 4D, homogenous space
  - Then project to $\omega = 1$ to obtain the surface in Euclidian 3D space
- In short: Just use homogeneous coordinates everywhere

# Rational Patch

**Rational Bézier Patch:**

$$\boldsymbol{f}^{(hom)}(u,v) = \sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u)B_j^{(d)}(v)\begin{pmatrix}\omega_{i,j}\boldsymbol{p}_{i,j}\\ \omega_{i,j}\end{pmatrix}$$

$$\boldsymbol{f}^{(Eucl)}(u,v) = \frac{\sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u)B_j^{(d)}(v)\omega_{i,j}\boldsymbol{p}_{i,j}}{\sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u)B_j^{(d)}(v)\omega_{i,j}}$$

# Rational Patch

**Rational B-Spline Patch:**

$$\boldsymbol{f}^{(hom)}(u,v) = \sum_{i=0}^{d}\sum_{j=0}^{d} N_i^{(d)}(u) N_j^{(d)}(v) \begin{pmatrix} \omega_{i,j}\boldsymbol{p}_{i,j} \\ \omega_{i,j} \end{pmatrix}$$

$$\boldsymbol{f}^{(Eucl)}(u,v) = \frac{\sum_{i=0}^{d}\sum_{j=0}^{d} N_i^{(d)}(u) N_j^{(d)}(v) \omega_{i,j}\boldsymbol{p}_{i,j}}{\sum_{i=0}^{d}\sum_{j=0}^{d} N_i^{(d)}(u) N_j^{(d)}(v) \omega_{i,j}}$$

# Remark: Rational Patches

**Observation:**

- Euclidian surface is not a tensor product surface
  - Denominator depends on both $u$ and $v$
- Homogeneous space: 4D surface is a tensor product surface.

$$\boldsymbol{f}^{(Eucl)}(u,v) = \frac{\sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u) B_j^{(d)}(v)\omega_{i,j}\boldsymbol{p}_{i,j}}{\sum_{i=0}^{d}\sum_{j=0}^{d} B_i^{(d)}(u) B_j^{(d)}(v)\omega_{i,j}}$$

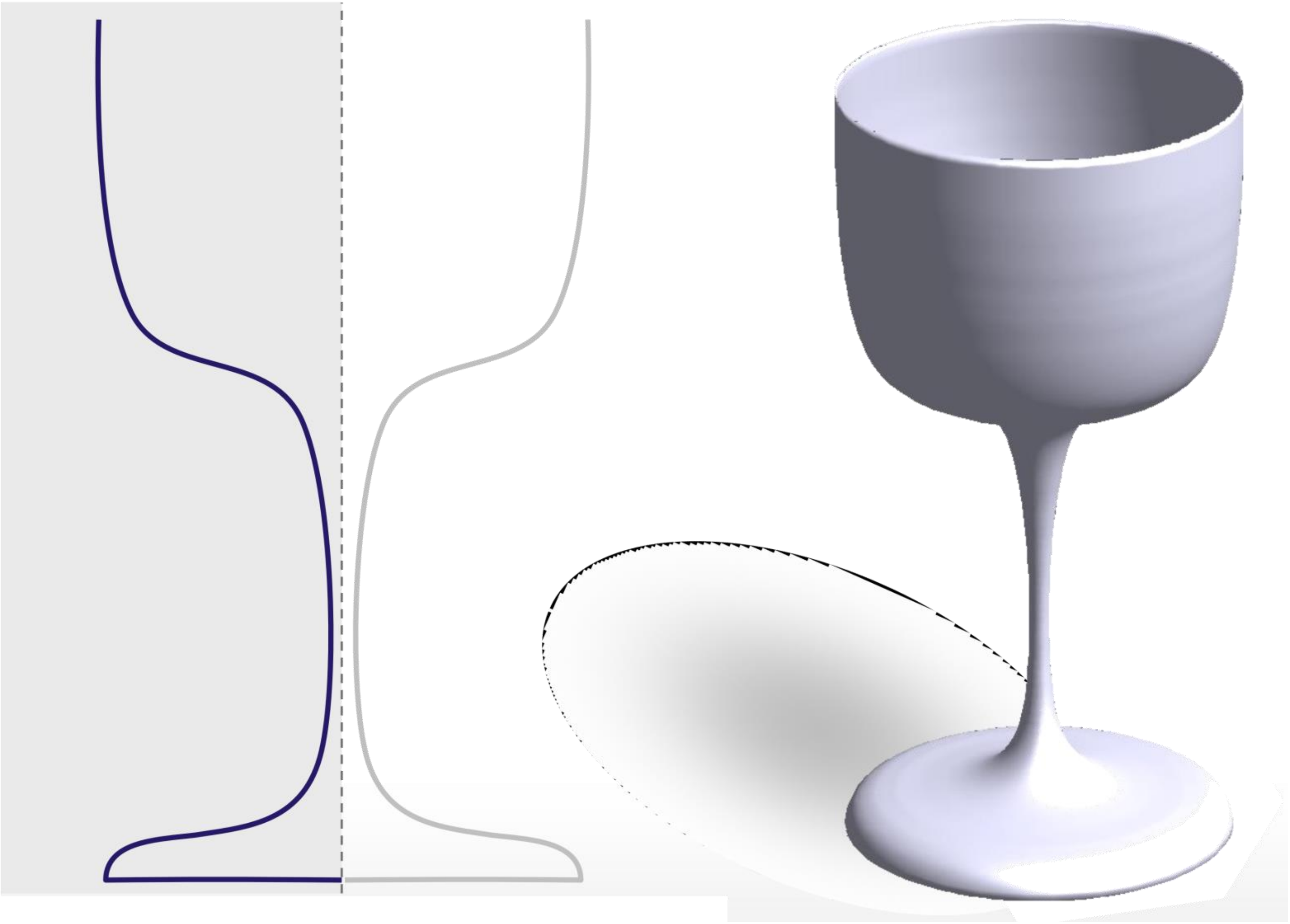
$$\boldsymbol{f}^{(Eucl)}(u,v) = \frac{\sum_{i=0}^{d}\sum_{j=0}^{d} N_i^{(d)}(u) N_j^{(d)}(v)\omega_{i,j}\boldsymbol{p}_{i,j}}{\sum_{i=0}^{d}\sum_{j=0}^{d} N_i^{(d)}(u) N_j^{(d)}(v)\omega_{i,j}}$$

# Surfaces of Revolution

**Advantages of rational patches:**

- Rational patches can represent surfaces of revolution exactly.
- Examples:
  - Cylinders
  - Cones
  - Spheres
  - Ellipsoids
  - Tori
- Question: given a cross section curve, how do we get the control points for the 3D surface?

# Surfaces of Revolution



# Surfaces of Revolution

**Given:**

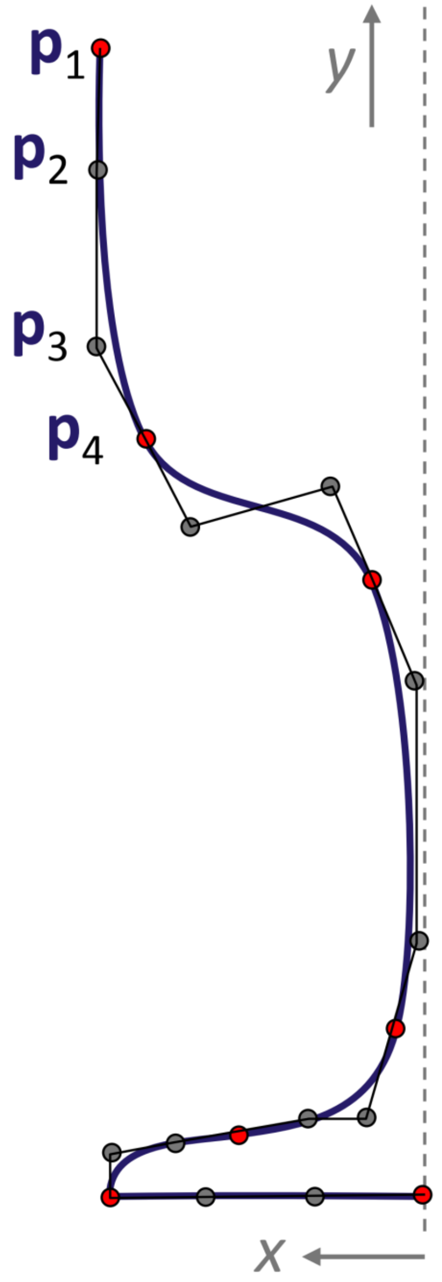
- Control points $p_1, \ldots, p_n$ of curve ("generatrix")

**We want to compute:**

- Control points $p_{i,j}$ of a rational surface

**Such that:**

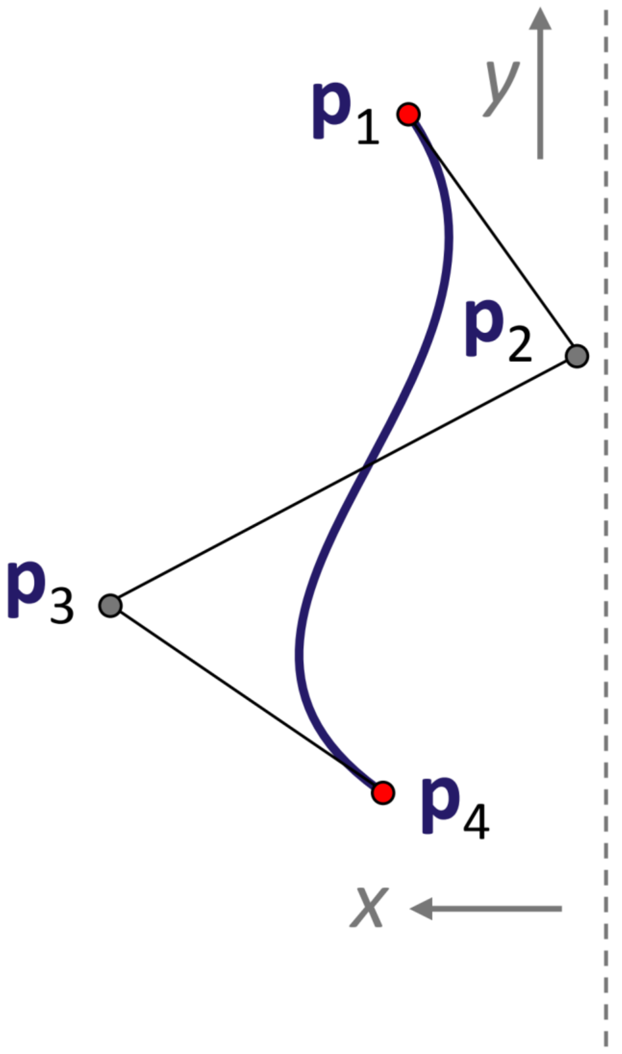
- The surface describes the surface of revolution that we obtain by rotating the curve around the $y$ axis (w.l.o.g.)



# Surfaces of Revolution

**Simplification:**

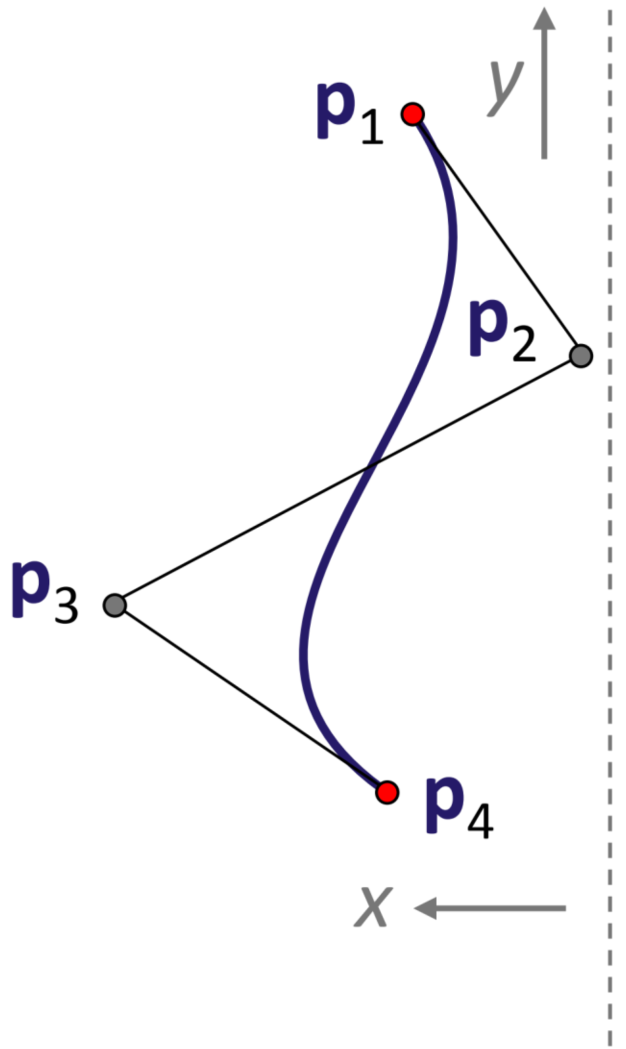
- We look only at a single rational Bézier segment
- Applying the scheme to multiple segments together is straightforward
- The same idea also works for B-splines



# Surfaces of Revolution

**Construction:**

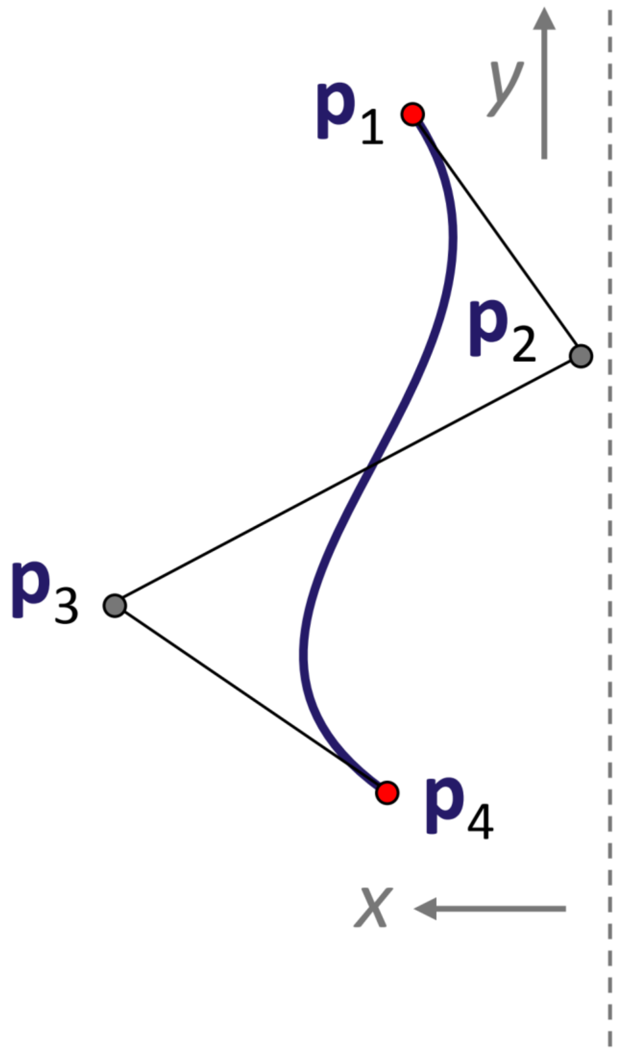
- We are given control points
  $\boldsymbol{p}_1, \dots, \boldsymbol{p}_{d+1}$
  ($d$ is the degree in $u$ direction)
- We introduce a new parameter $v$
- In $v$ direction, we use quadratic Bézier curves (2nd degree basis in $v$-direction)



# Surfaces of Revolution

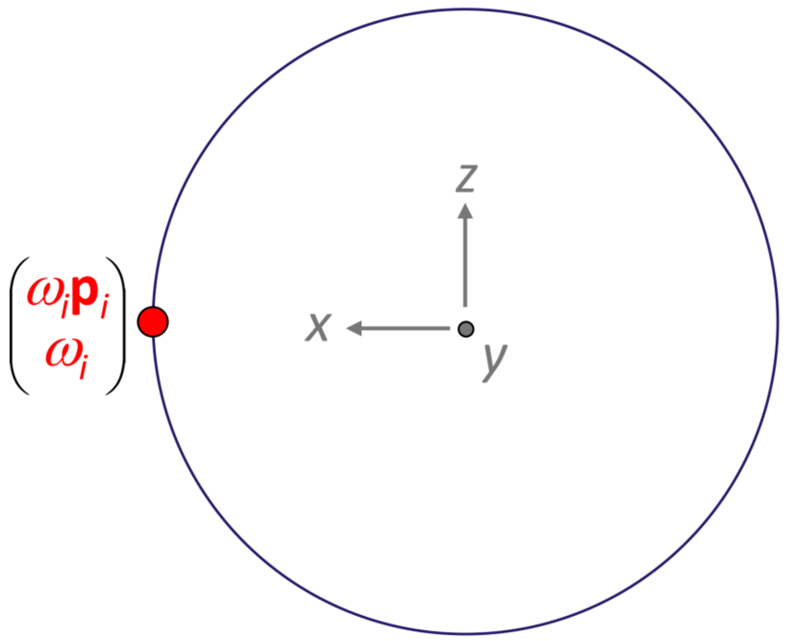
**Key Idea:**

- For $u$-direction curves: control points (and thus the curves) must move on circles around the $y$-axis
- Circles must have the same parametrization (this is easy)
- This means, the control points rotate around the y-axis
- Affine invariance will make the whole curve rotate, we get the desired surface of revolution



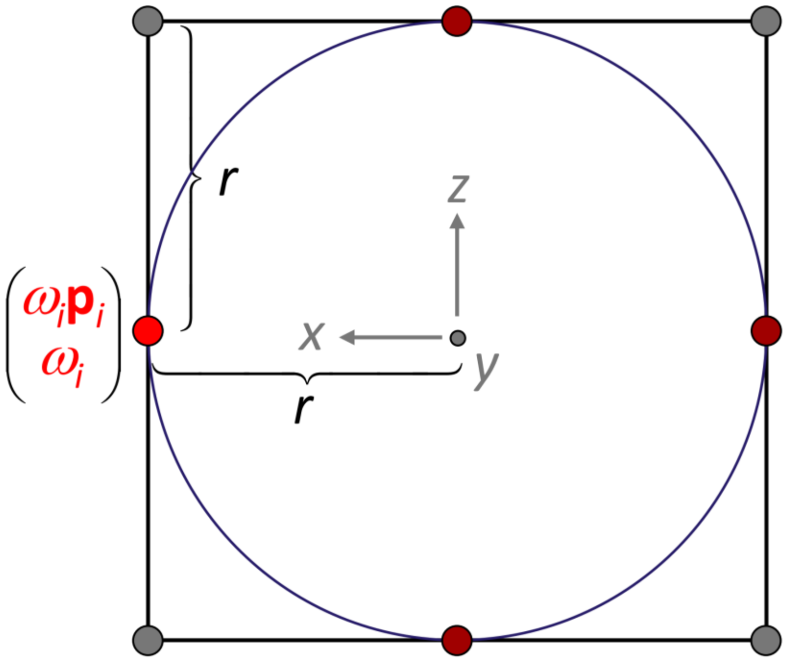
# Surface of Revolution

**Making one point rotate around the y-axis:**



# Surface of Revolution

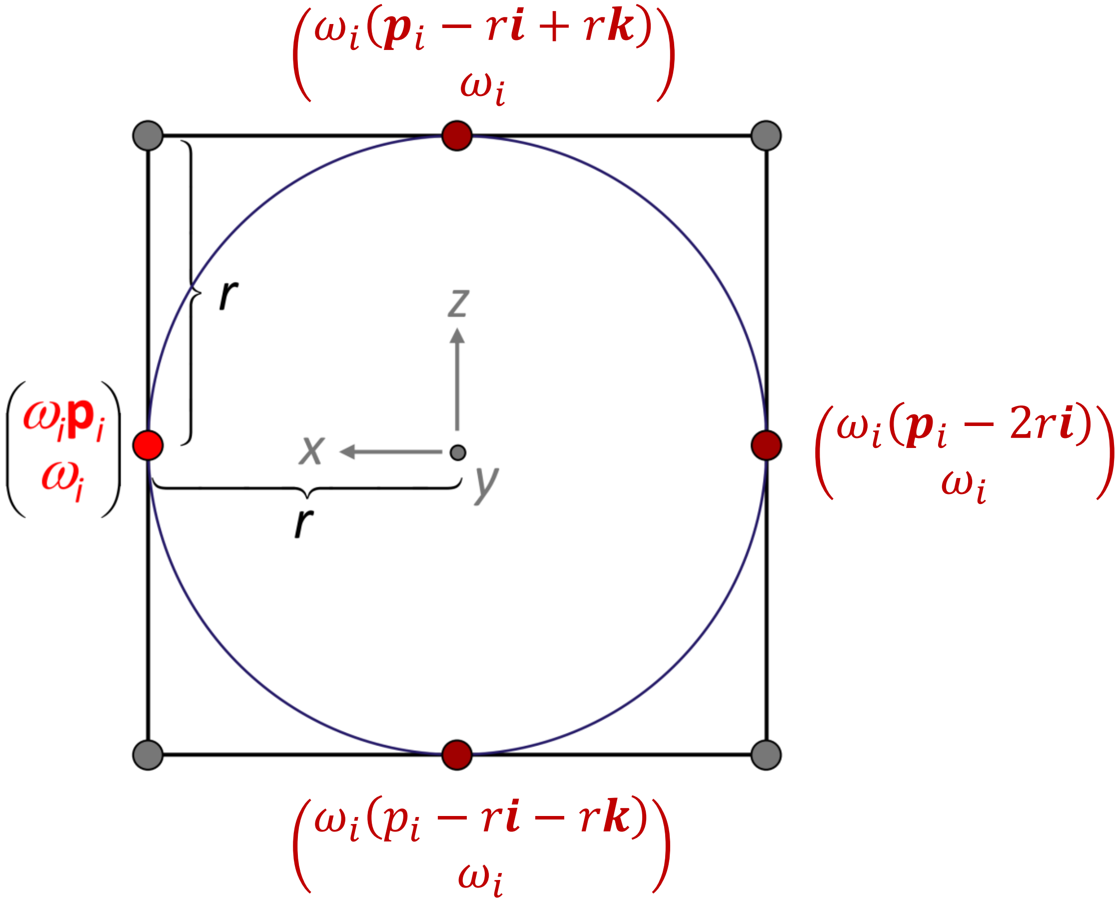
**Making one point rotate around the y-axis:**



# Surface of Revolution

**Making one point rotate around the y-axis:**

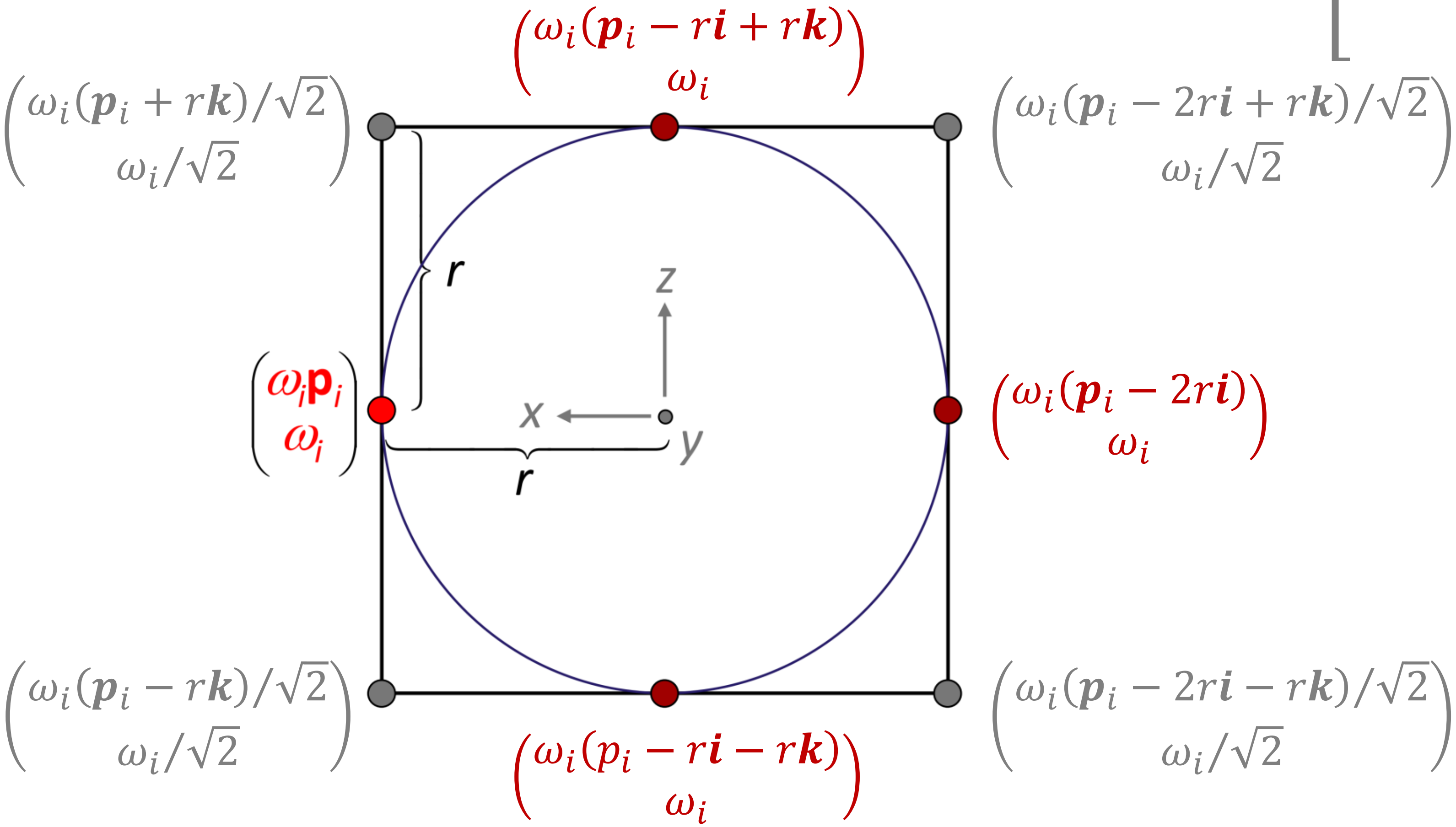
$$\left[\boldsymbol{i} := \begin{pmatrix}1\\0\\0\end{pmatrix}, \boldsymbol{k} := \begin{pmatrix}0\\0\\1\end{pmatrix}\right]$$



# Surface of Revolution

**Making one point rotate around the y-axis:**

$$\left[ \boldsymbol{i} := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{k} := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$$



# Remark

**What we get:**

- We obtain 4 segments, i.e. 4 patches for each Bézier segment
- A similar construction with 3 segments exists as well

**Does the scheme yield a circle for weights $\neq 1$ in the generatrix curve?**

- Common factors in weights cancel out
- Therefore, we still obtain a circle at these points
- Parametrization does not change either

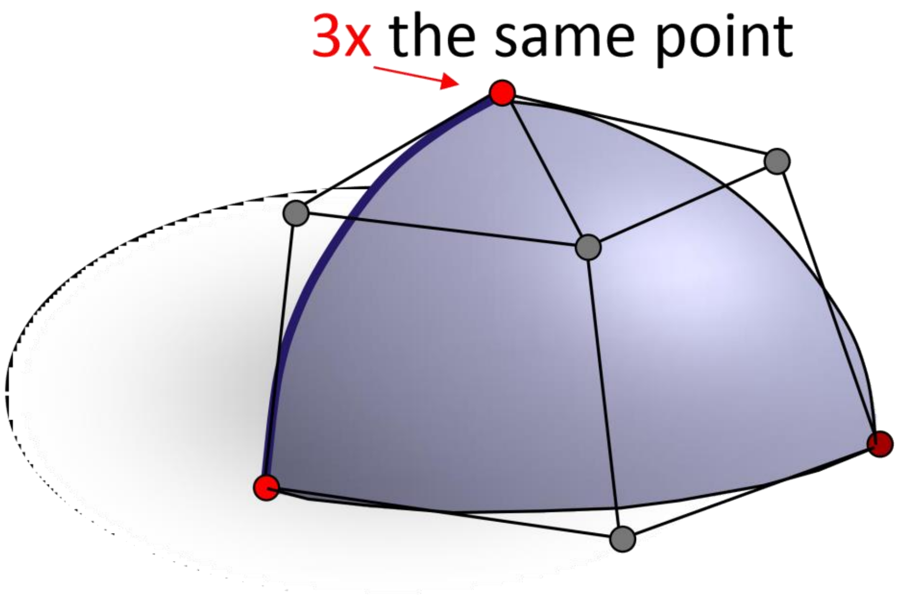
# Benefit

**With this construction, it is straightforward to create:**

- Spheres
- Tori
- Cylinders
- Cones

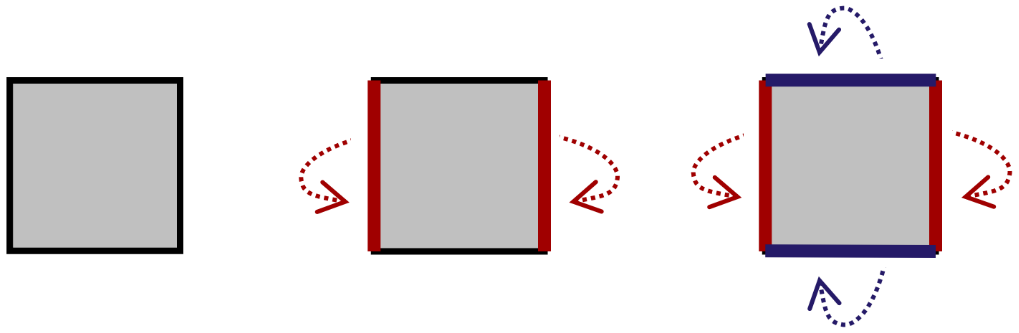
**And affine transformations of these (e.g. ellipsoids)**

# Parametrization Restrictions




**Remaining problem:**

- The sphere and the cone are not regularly parametrized (double control points)
- Might cause trouble (normal computation, tessellation)
- In general: no sphere, or $n$-tori ($n > 1$) can be parametrized without degeneracies
- What works: open surfaces, cylinders, tori



# Curves on Surfaces, trimmed NURBS

**Quad patch problem:**

- All of our shapes are parameterized over rectangular regions
- General boundary curves are hard to create
- Topology fixed to a disc (or cylinder, torus)
- No holes in  the middle
- Assembling complicated shapes is painful
  - Lots of pieces
  - Continuity conditions for assembling pieces become complicated
  - Cannot use $C^2$ B-splines continuity along boundaries when using multiple pieces

# Curves on Surfaces, trimmed NURBS

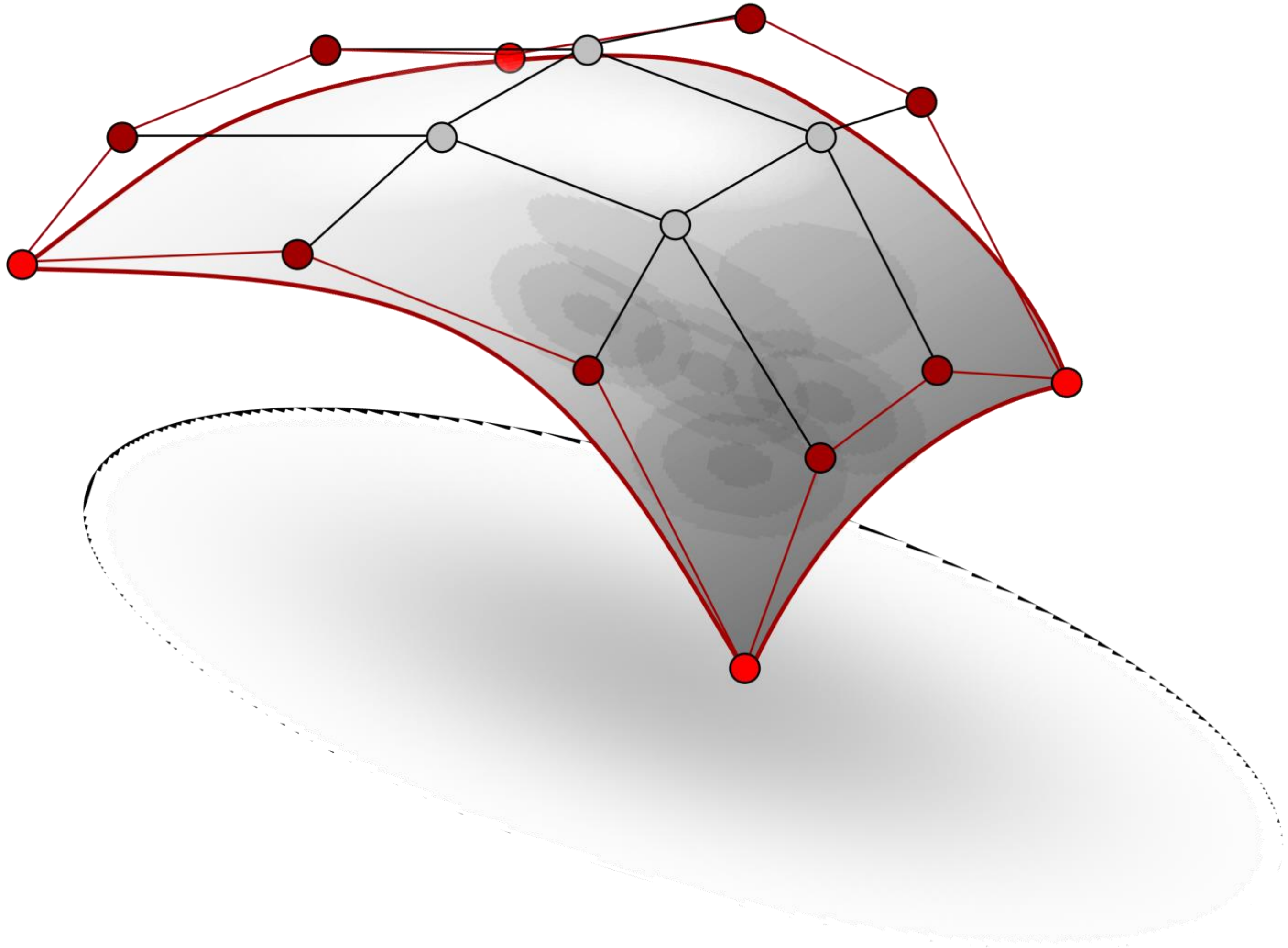
**Consequence:**

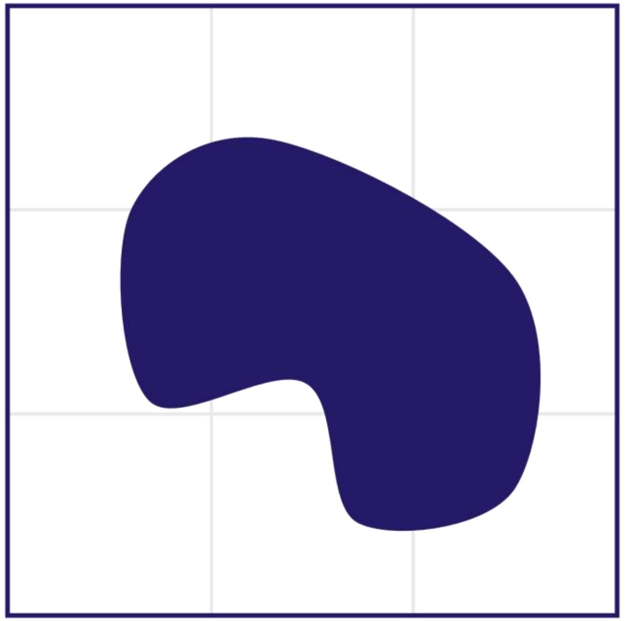
- We need more control over the parameter domain
- One solution is *trimming* using *curves on surfaces (CONS)*
- Standard tool in CAD: *trimmed NURBS*

**Basic idea:**

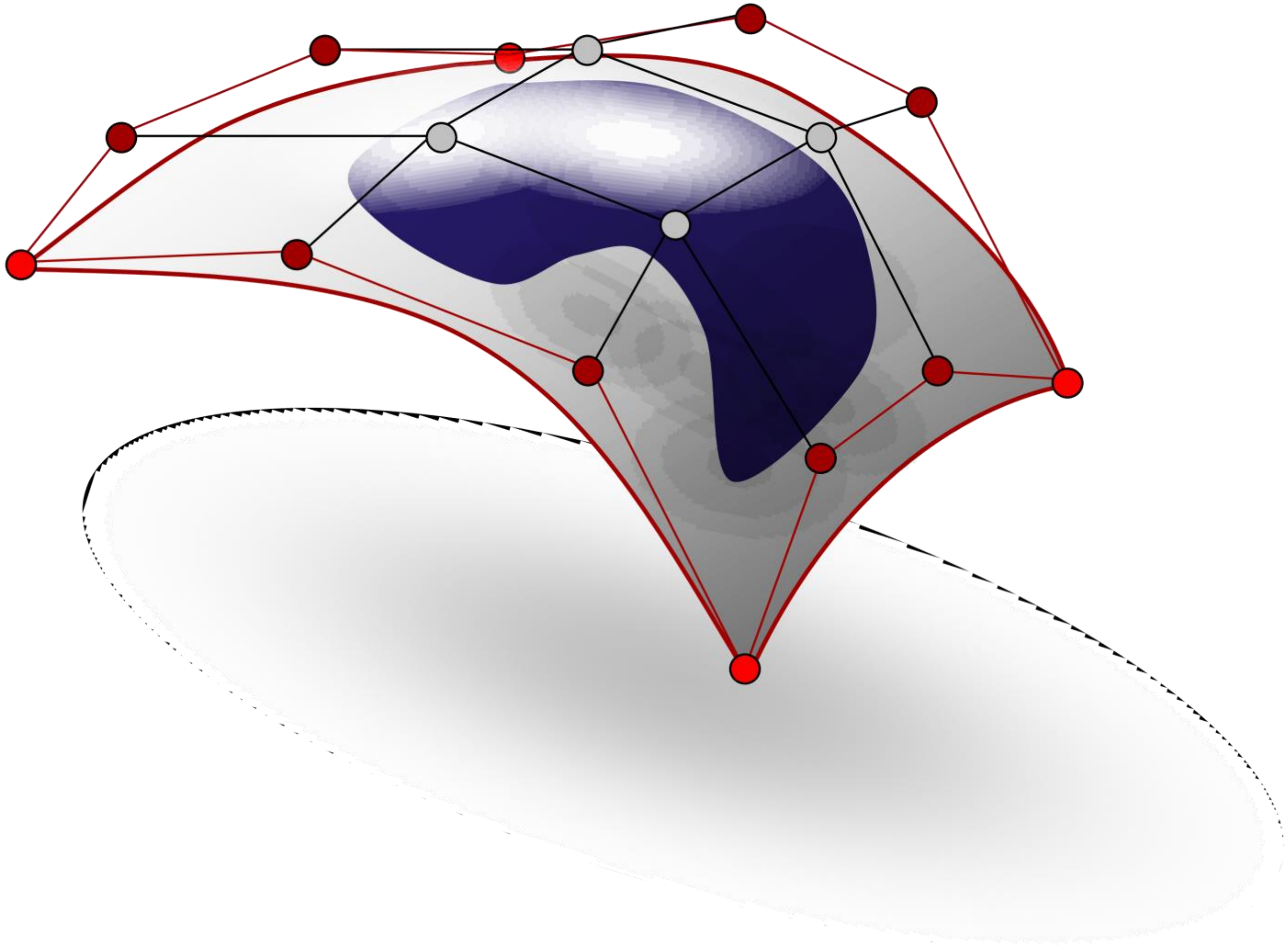
- Specify a curve in the parameter domain that encapsulates one (or more) pieces of area
- Tessellate the parameter domain accordingly to cut out the trimmed piece (rendering)

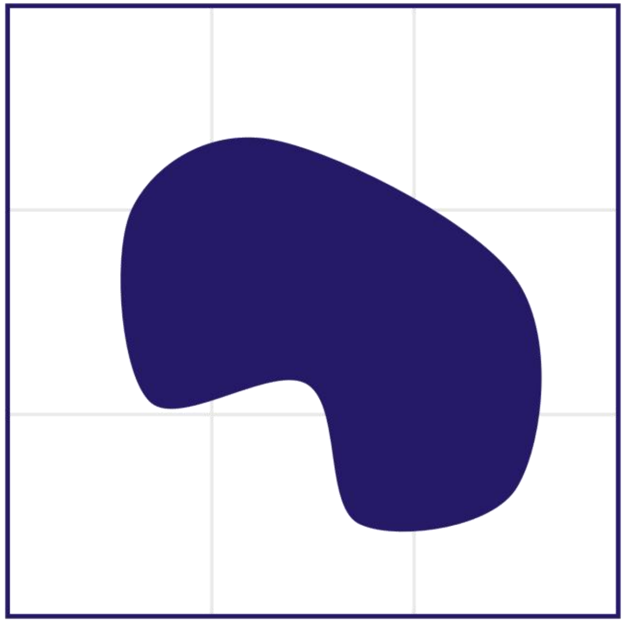
# Curves-on-Surfaces (CONS)



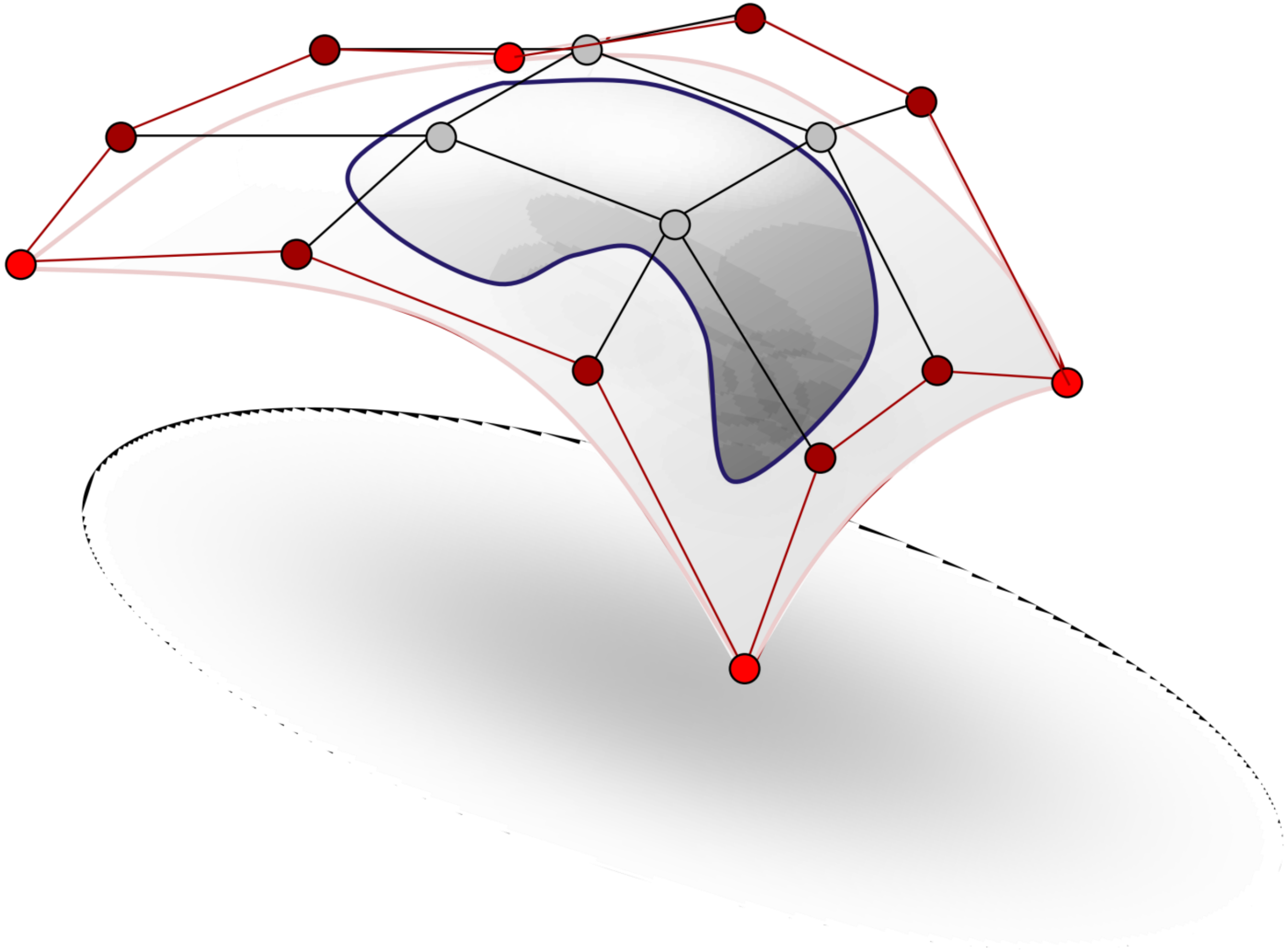


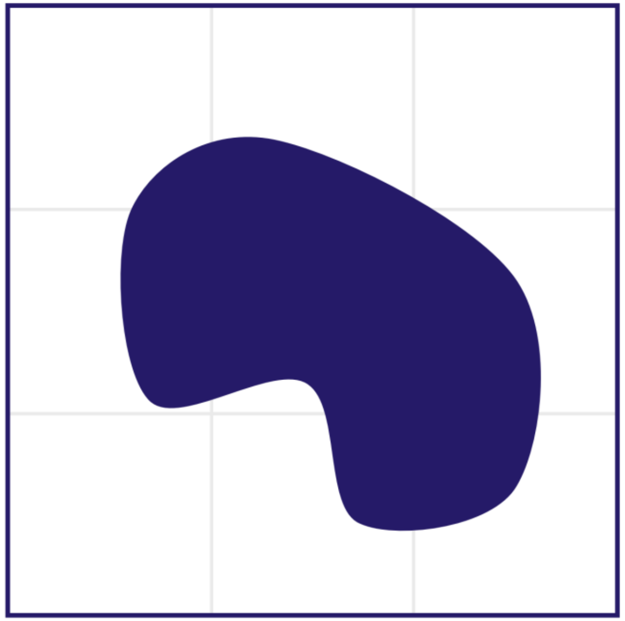
# Curves-on-Surfaces (CONS)





# Curves-on-Surfaces (CONS)

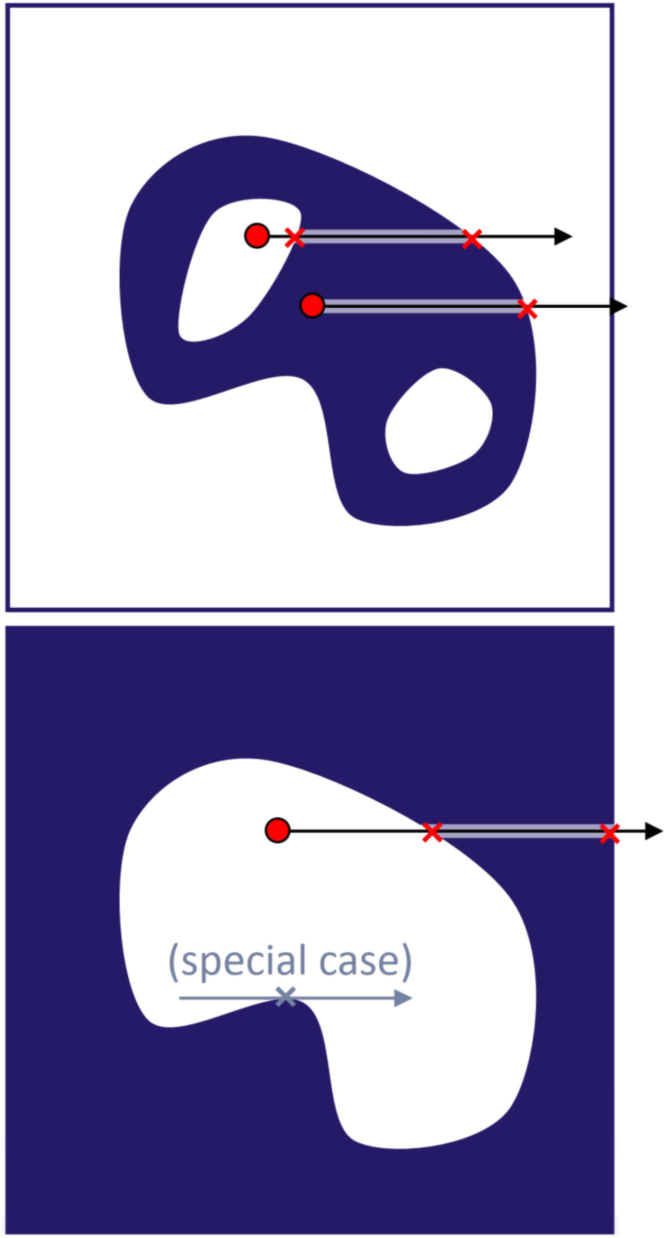




# General Shapes

**General shapes with holes:**

- Draw multiple curves
- Inside / outside test:
  - If any ray in the parameter domain intersects the boundary curves an odd number of times, the point is inside
  - Outside otherwise
  - Implementation needs to take care of special cases (critical points with respect to normal of the ray)
  - Nasty, but doable

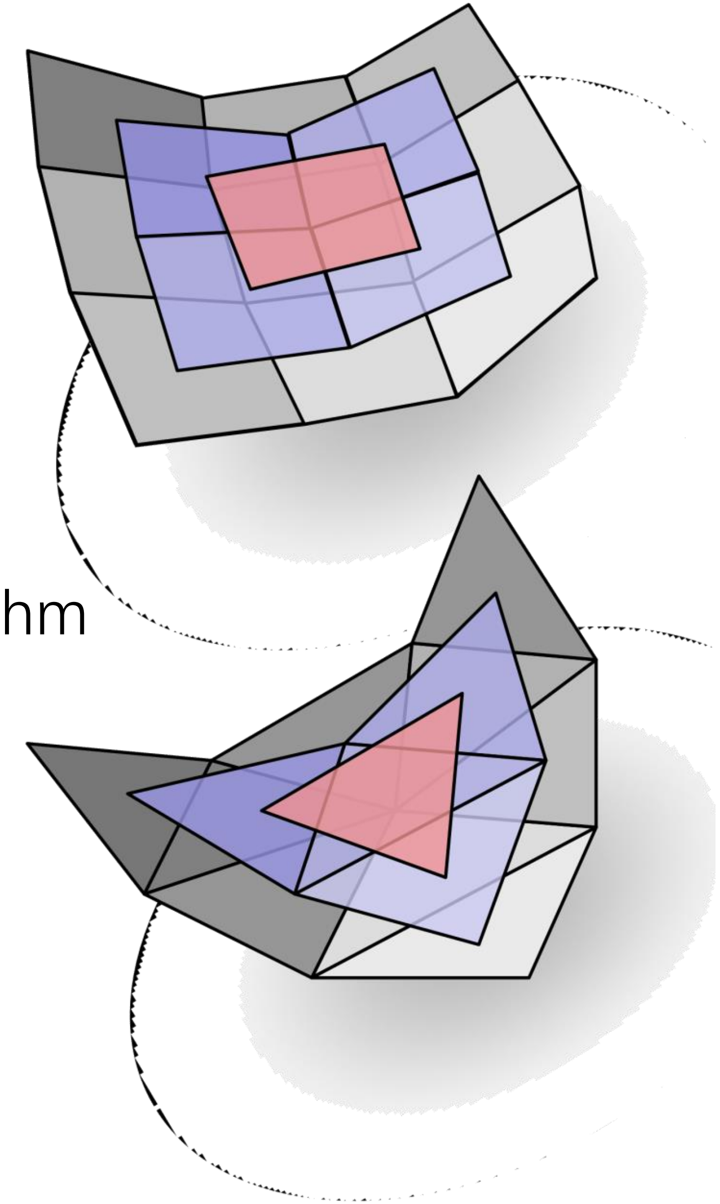


# Total Degree Surfaces

# Bézier Triangles

**Alternative surface definition:** Bézier triangles

- Constructed according to given total degree
  - Completely symmetric: degree anisotropy
- Can be derived using a triangular de Casteljau algorithm
  - Blossoming formalism is very helpful for defining Bézier Triangles
  - Barycentric interpolation of blossom values



# Blossoms for Total Degree Surfaces

**Blossom with points as arguments:**

- Polar form degree $d$ with points as input and output:

  $\boldsymbol{F}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$

  $\boldsymbol{f}: \mathbb{R}^{d \times n} \rightarrow \mathbb{R}^{m}$

  points as arguments

- Required Properties:
  - Diagonality: $\boldsymbol{f}(\boldsymbol{t}, \boldsymbol{t}, \ldots, \boldsymbol{t}) = \boldsymbol{F}(\boldsymbol{t})$
  - Symmetry: $\boldsymbol{f}(\boldsymbol{t}_1, \boldsymbol{t}_2, \ldots, \boldsymbol{t}_d) = \boldsymbol{f}\left(\boldsymbol{t}_{\pi(1)}, \boldsymbol{t}_{\pi(2)}, \ldots, \boldsymbol{t}_{\pi(d)}\right)$

    for all permutations of indices $\pi$

- Multi-affine: $\sum \alpha_k = 1$

  $$\Rightarrow \boldsymbol{f}\left(\boldsymbol{t}_1, \ldots, \sum \alpha_k \boldsymbol{t}_i^{(k)}, \ldots, \boldsymbol{t}_d\right)$$

  $$= \alpha_1 \boldsymbol{f}\left(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_i^{(1)}, \ldots, \boldsymbol{t}_d\right) + \cdots + \alpha_n \boldsymbol{f}\left(\boldsymbol{t}_1, \ldots, \boldsymbol{t}_i^{(n)}, \ldots, \boldsymbol{t}_d\right)$$

# Example

**Example:** bivariate monomial basis

- In powers of $(u, v)$:

  $B = \{1, u, v, u^2, uv, v^2\}$

- Blossom form: multilinear in $(u_1, u_2, v_1, v_2)$

  $$B = \{1, \frac{1}{2}(u_1 + u_2), \frac{1}{2}(v_1 + v_2), u_1 u_2, \frac{1}{4}(u_1 v_1 + u_1 v_2 + u_2 v_1 + u_2 v_2), v_1 v_2\}$$

# Barycentric Coordinates

**Barycentric Coordinates:**

- Planar case:

  Barycentric combinations of 3 points

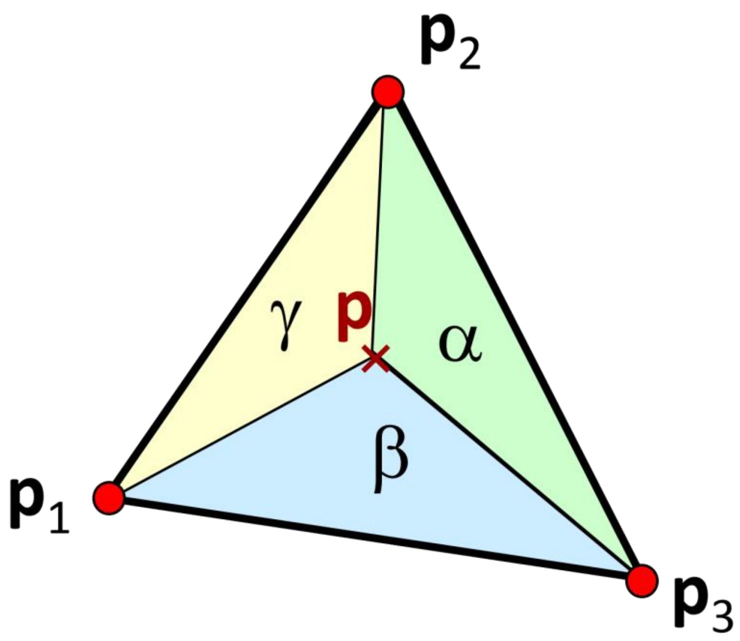
  $\boldsymbol{p} = \alpha \boldsymbol{p}_1 + \beta \boldsymbol{p}_2 + \gamma \boldsymbol{p}_3$, with $\alpha + \beta + \gamma = 1$

  $\gamma = 1 - \alpha - \beta$

- Area formulation

  $\gamma = 1 - \alpha - \beta$

  $$\alpha = \frac{area(\Delta(\boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}))}{area(\Delta(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3))}, \beta = \frac{area(\Delta(\boldsymbol{p}_1, \boldsymbol{p}_3, \boldsymbol{p}))}{area(\Delta(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3))}, \gamma = \frac{area(\Delta(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}))}{area(\Delta(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3))}$$

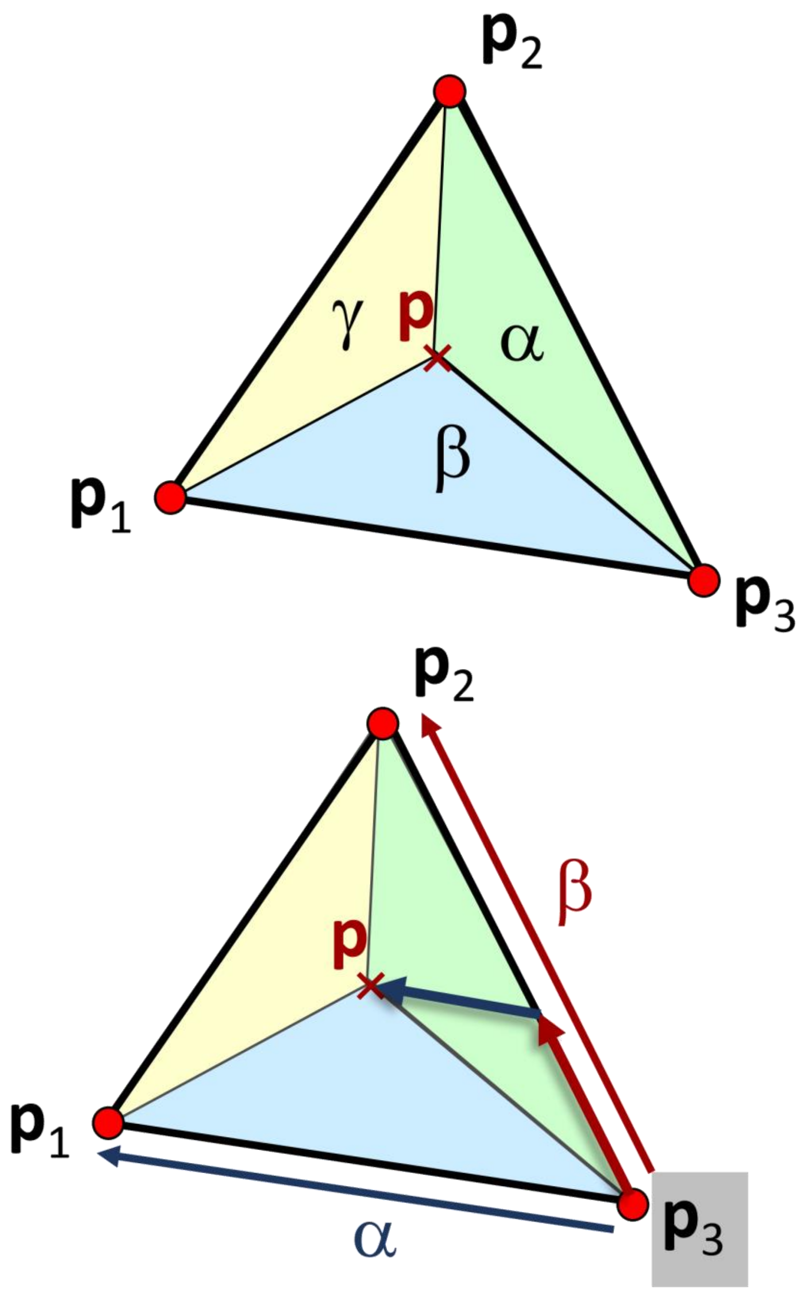


# Barycentric Coordinates

**Barycentric Coordinates:**

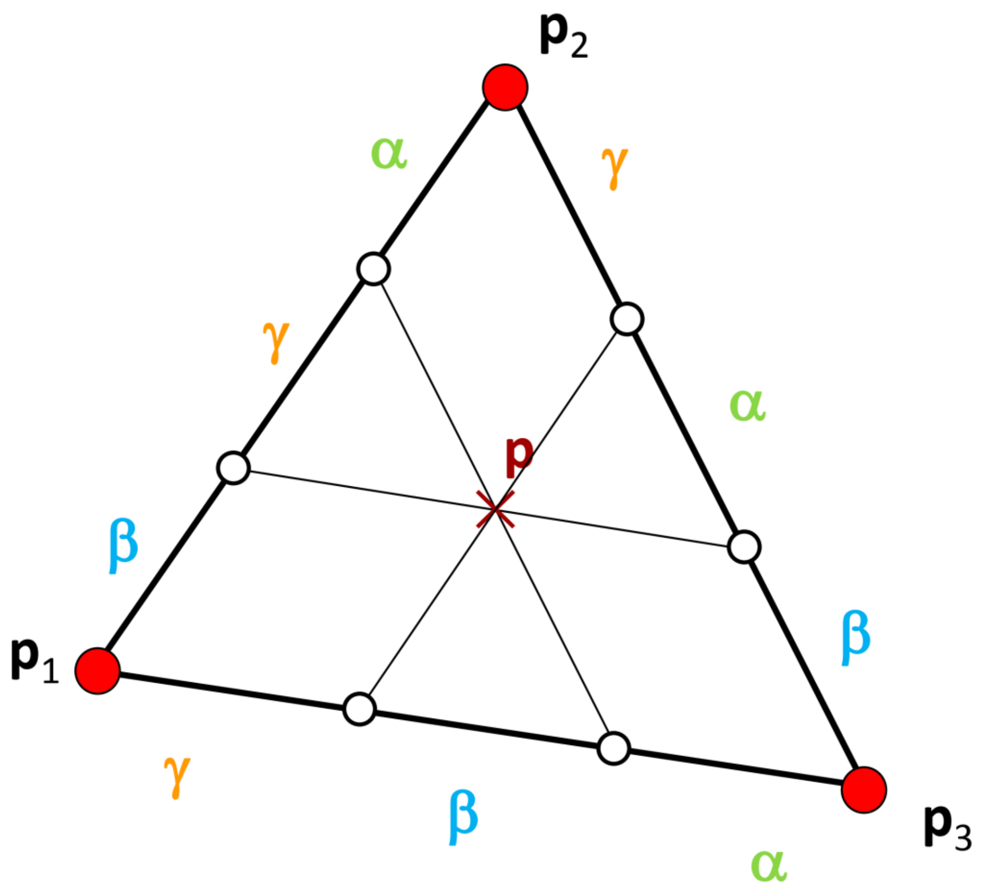
- Linear formulation:

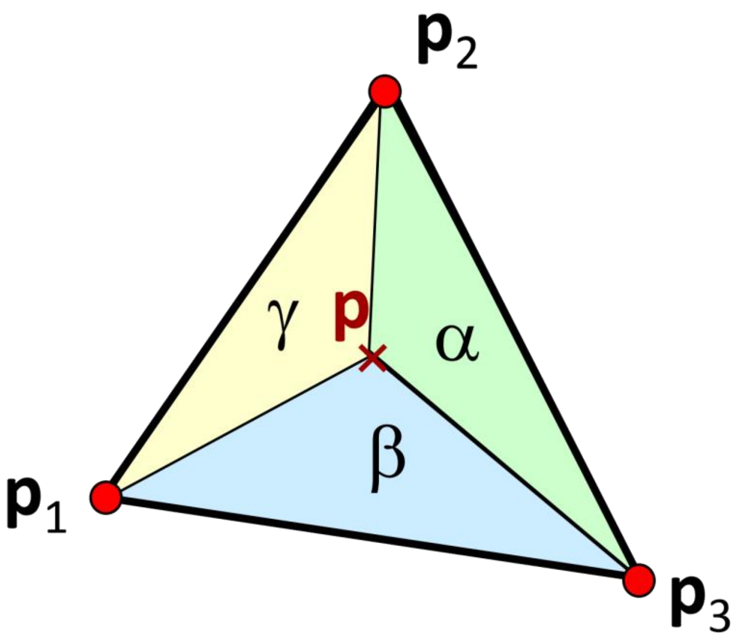
$$\boldsymbol{p} = \alpha \boldsymbol{p}_1 + \beta \boldsymbol{p}_2 + \gamma \boldsymbol{p}_3$$
$$= \alpha \boldsymbol{p}_1 + \beta \boldsymbol{p}_2 + (1 - \alpha - \beta) \boldsymbol{p}_3$$
$$= \alpha \boldsymbol{p}_1 + \beta \boldsymbol{p}_2 + \boldsymbol{p}_3 - \alpha \boldsymbol{p}_3 - \beta \boldsymbol{p}_3$$
$$= \boldsymbol{p}_3 + \underline{\alpha(\boldsymbol{p}_1 - \boldsymbol{p}_3)} + \underline{\beta(\boldsymbol{p}_2 - \boldsymbol{p}_3)}$$



# Barycentric Coordinates

$\boldsymbol{p} = \alpha\boldsymbol{p}_1 + \beta\boldsymbol{p}_2 + \gamma\boldsymbol{p}_3$, with $\alpha + \beta + \gamma = 1$



# Bézier Triangles: Overview

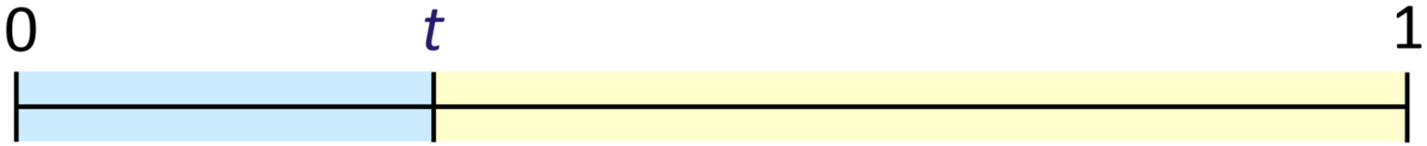
**Bézier Triangles:** Main Ideas

- Use 3D points as inputs to the blossoms
- These are Barycentric coordinates of a parameter triangle $\{a, b, c\}$
- Use 3D points as outputs
- Form control points by multiplying parameter points, just as in the curve case: $\boldsymbol{p}(\underbrace{a, \dots, a}_{i}, \underbrace{b, \dots, b}_{j}, \underbrace{c, \dots, c}_{k})$
- De Casteljau Algorithm: compute polynomial values $p(x, \dots, x)$ by barycentric interpolation

# Plugging in the Barycentric Coord's

**Analog:** 2D curves in barycentric coordinates

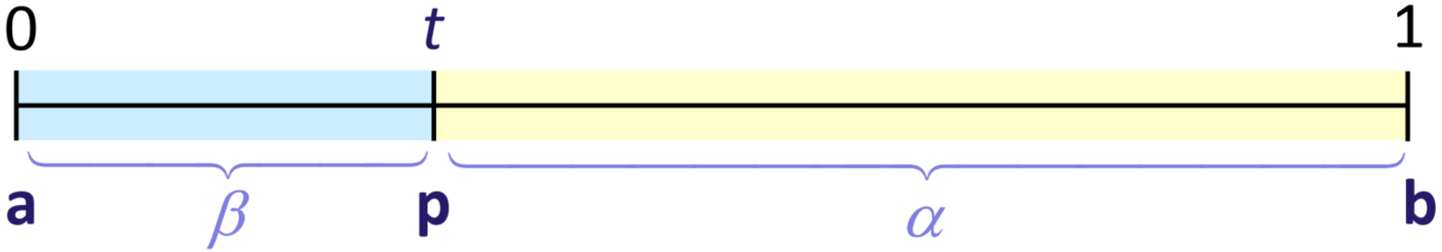
- Barycentric coordinates for 2D curves:



# Plugging in the Barycentric Coord's

**Analog:** 2D curves in barycentric coordinates

- Barycentric coordinates for 2D curves:



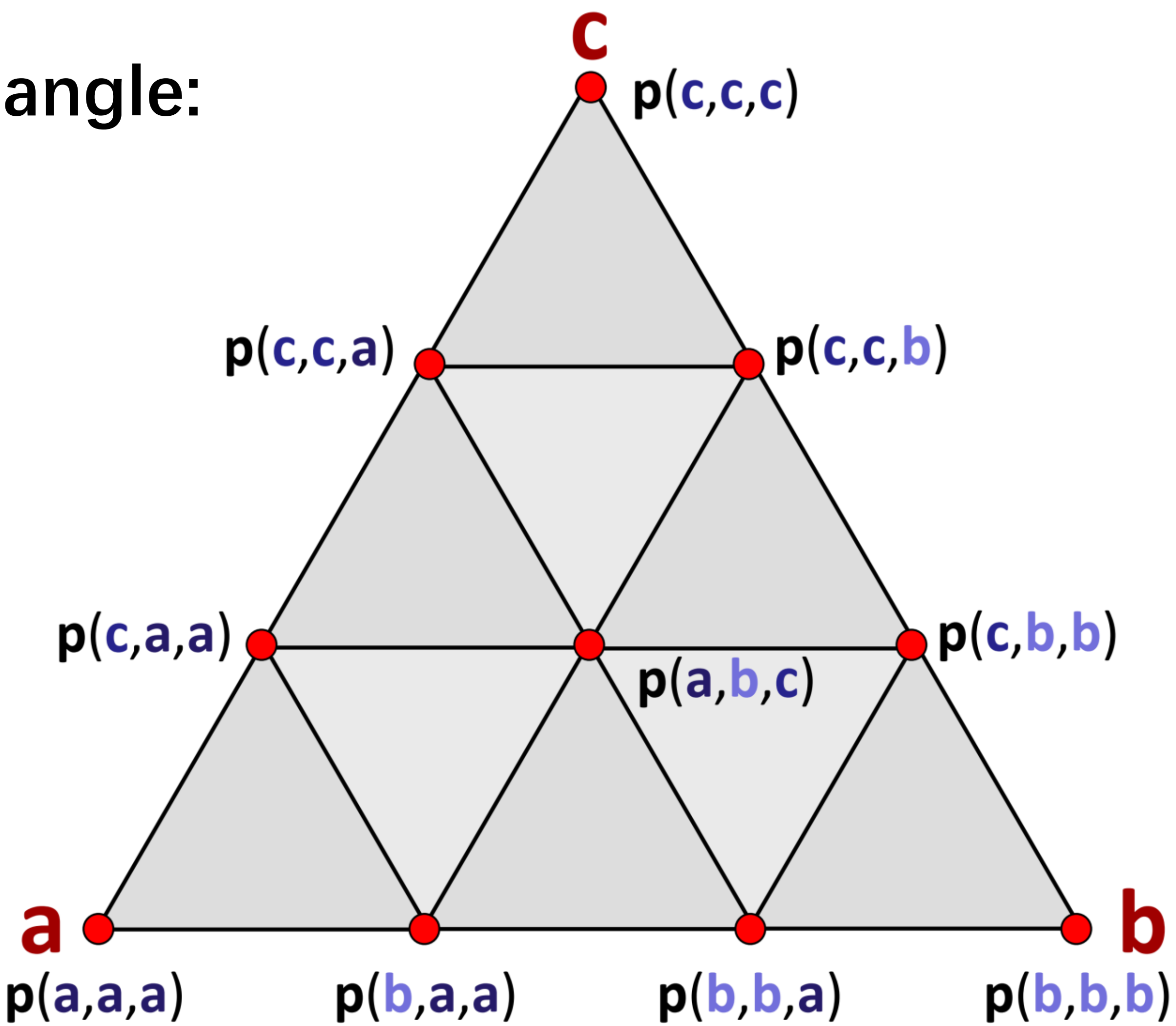

- $p = \alpha a + \beta b, \quad \alpha + \beta = 1$
- Bézier splines:

$$\boldsymbol{F}(t) = \sum_{i=0}^{d} \binom{d}{i} (1-t)^i t^{d-i} \boldsymbol{f}(\underbrace{\boldsymbol{a}, \dots, \boldsymbol{a}}_{i}, \underbrace{\boldsymbol{b}, \dots, \boldsymbol{b}}_{d-i}) \quad \text{(standard form)}$$

$$\boldsymbol{F}(\boldsymbol{p}) = \sum_{\substack{i+j=d \\ i \geq 0, j \geq 0}} \frac{d!}{i!j!} \alpha^i \beta^j \boldsymbol{f}(\underbrace{\boldsymbol{a}, \dots, \boldsymbol{a}}_{i}, \underbrace{\boldsymbol{b}, \dots, \boldsymbol{b}}_{j}) \quad \text{(barycentric form)}$$

# Example

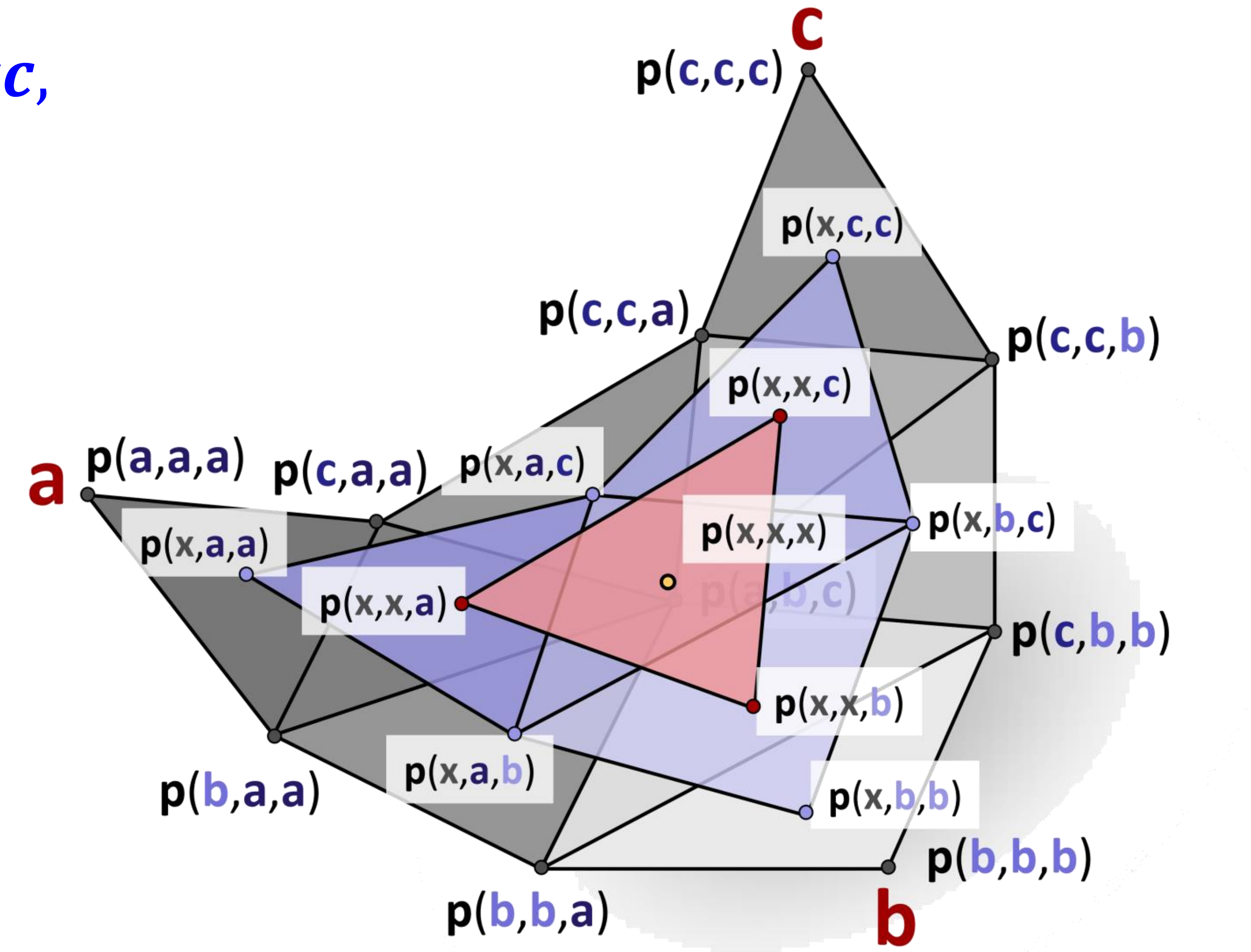
Cubic Bézier Triangle:



# De Casteljau Algorithm

$$\boldsymbol{x} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c},$$

$$\alpha + \beta + \gamma = 1$$



# Bernstein Form

**Writing this recursion out, we obtain:**

$$F(\boldsymbol{x}) = \sum_{\substack{i+j+k=d \\ i,j,k\geq 0}} \frac{d!}{i!\,j!\,k!} \alpha^i \beta^j \gamma^k \boldsymbol{f}(\underbrace{a,\dots,a}_{i},\underbrace{b,\dots,b}_{j},\underbrace{c,\dots,c}_{k})$$

$$\boldsymbol{x} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c},$$

$$\alpha + \beta + \gamma = 1$$

- This is the *Bernstein form* of a Bézier triangle surface
- (Proof by induction)

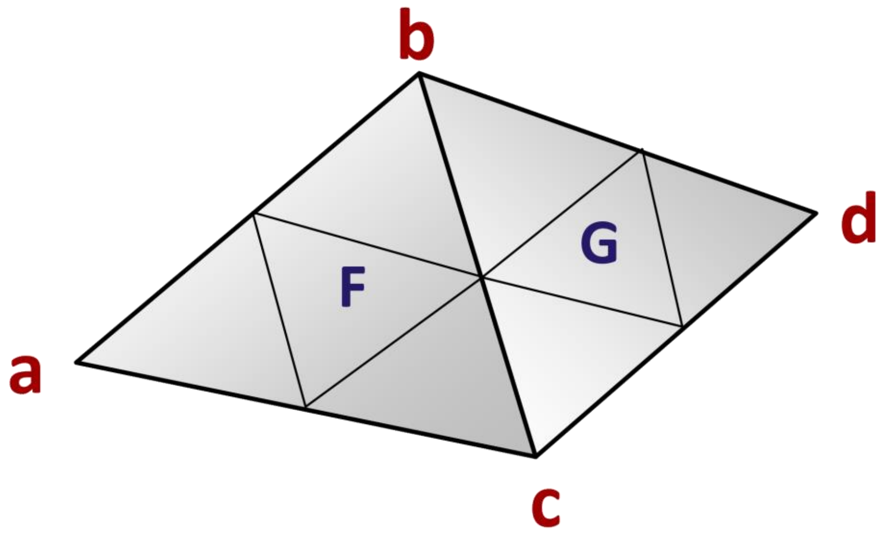
# Continuity

**We need to assemble Bézier triangles continuously:**

- What are the conditions for $C^0$, $C^1$ continuity?
- As an example, we will look at the quadratic case⋯
- (Try the cubic case as an exercise)

# Continuity

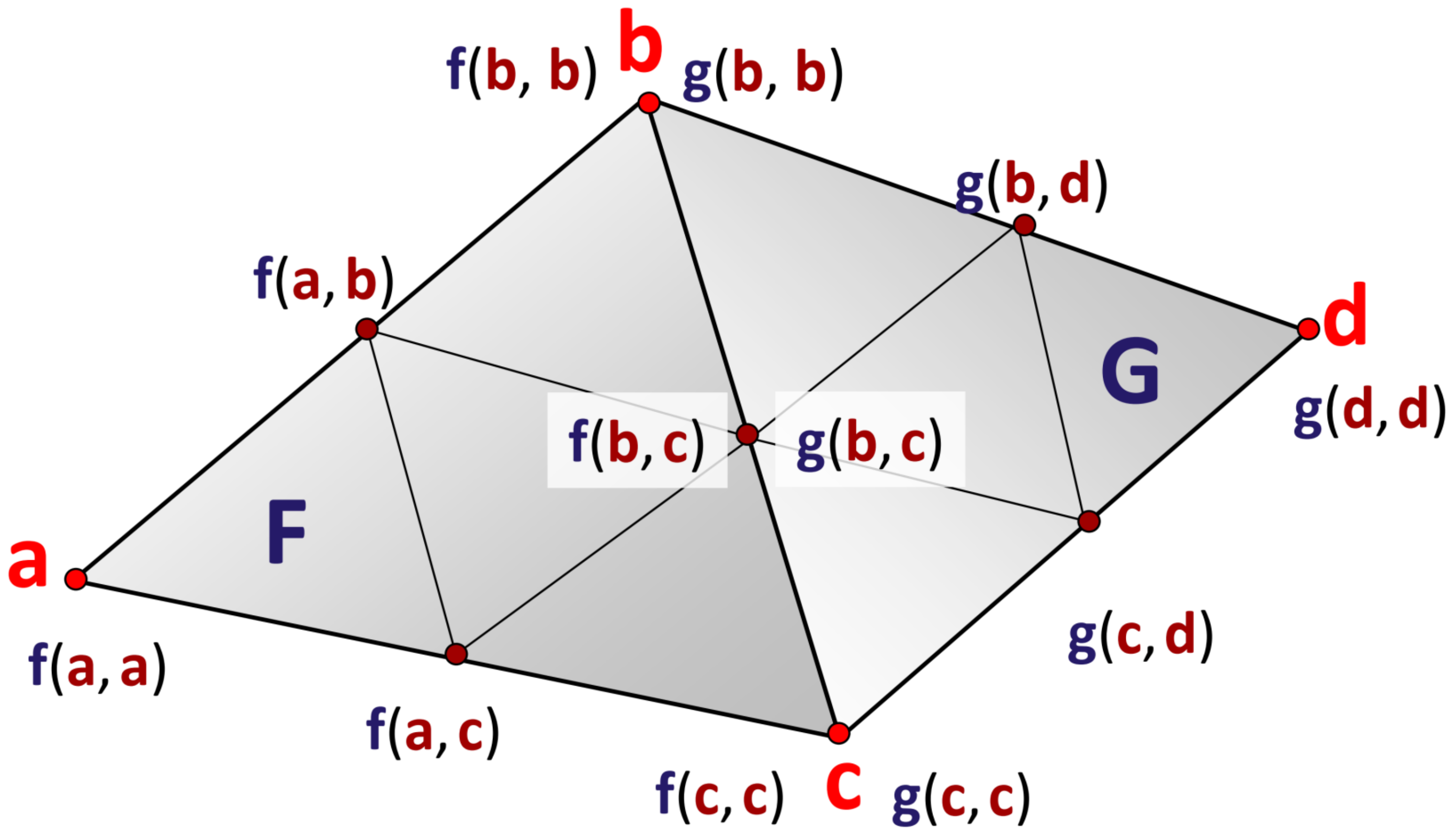
**Situation:**




- Two Bézier triangles meet along a common edge.
  - Parametrization: $T_1 = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}, T_2 = \{\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{d}\}$
  - Polynomial surfaces $\boldsymbol{F}(T_1)$, $\boldsymbol{G}(T_2)$
  - Control points:
    - $\boldsymbol{F}(T_1)$: $\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{a}), \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}), \boldsymbol{f}(\boldsymbol{b}, \boldsymbol{b}), \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{c}), \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{c}), \boldsymbol{f}(\boldsymbol{b}, \boldsymbol{c})$
    - $\boldsymbol{G}(T_2)$: $\boldsymbol{g}(\boldsymbol{d}, \boldsymbol{d}), \boldsymbol{g}(\boldsymbol{d}, \boldsymbol{b}), \boldsymbol{g}(\boldsymbol{b}, \boldsymbol{b}), \boldsymbol{g}(\boldsymbol{d}, \boldsymbol{c}), \boldsymbol{g}(\boldsymbol{c}, \boldsymbol{c}), \boldsymbol{g}(\boldsymbol{b}, \boldsymbol{c})$

# Continuity

Situation:



# Continuity

## $C^0$ continuity:

- The points on the boundary have to agree:

$$f(b,b) = g(b,b)$$
$$f(b,c) = g(b,c)$$
$$f(c,c) = g(c,c)$$

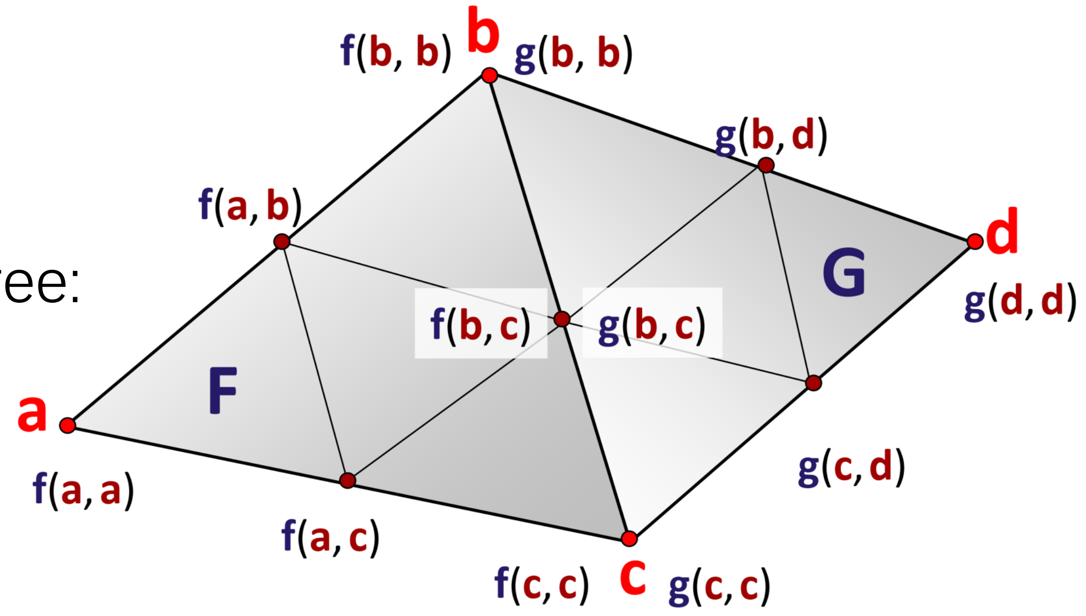
- Proof: Let $x := \beta b + \gamma c,\ \beta + \gamma = 1$

$$
\begin{aligned}
f(x,x) &= \beta f(b,x) + \gamma f(c,x) \\
&= \beta^2 f(b,b) + 2\beta\gamma f(b,c) + \gamma^2 f(c,c)
\end{aligned}
$$

$$\quad\ \ \| \qquad\qquad\quad \| \qquad\qquad\quad \|$$

$$g(b,b) \qquad g(b,c) \qquad g(c,c)$$

$$
\begin{aligned}
&= \beta^2 g(b,b) + 2\beta\gamma g(b,c) + \gamma^2 g(c,c) \\
&= \beta g(b,x) + \gamma g(c,x) = g(x,x)
\end{aligned}
$$



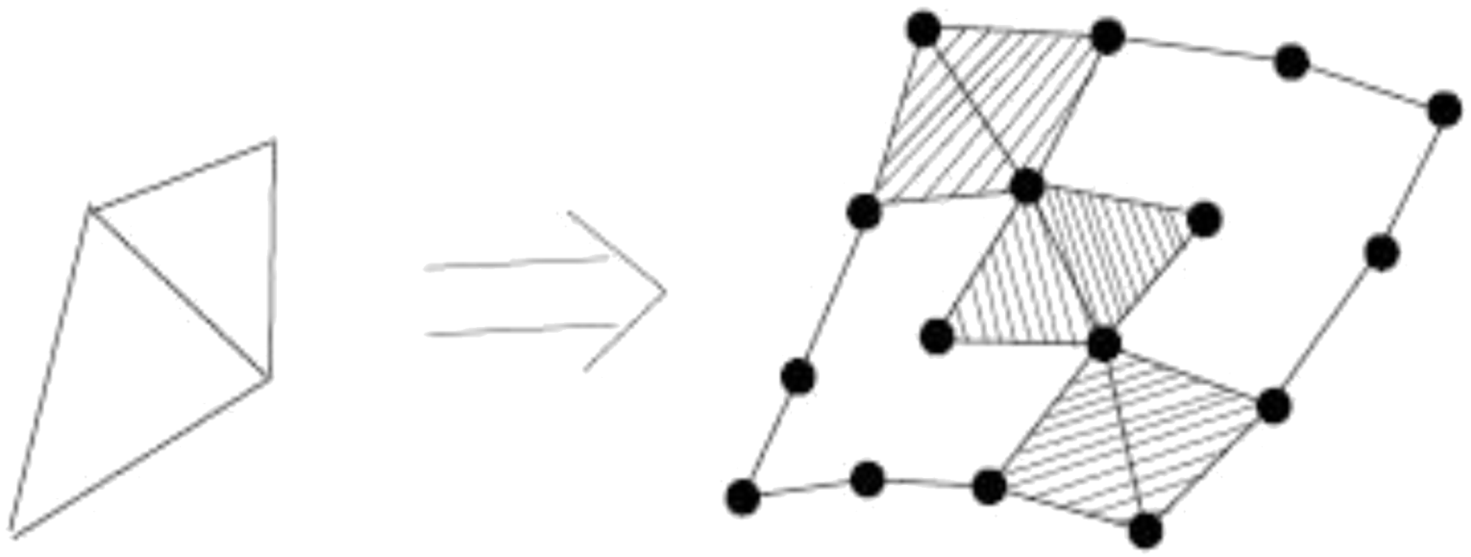
# Continuity

**$C^1$ continuity:**

- We need $C^0$ continuity.

*In addition:*

- Points at hatched quadrilaterals are coplanar
- Hatched quadrilaterals are an affine image of the same parameter quadrilateral



# Continuity

**$C^1$ continuity:**

- We need $C^0$ continuity.

*In addition:*

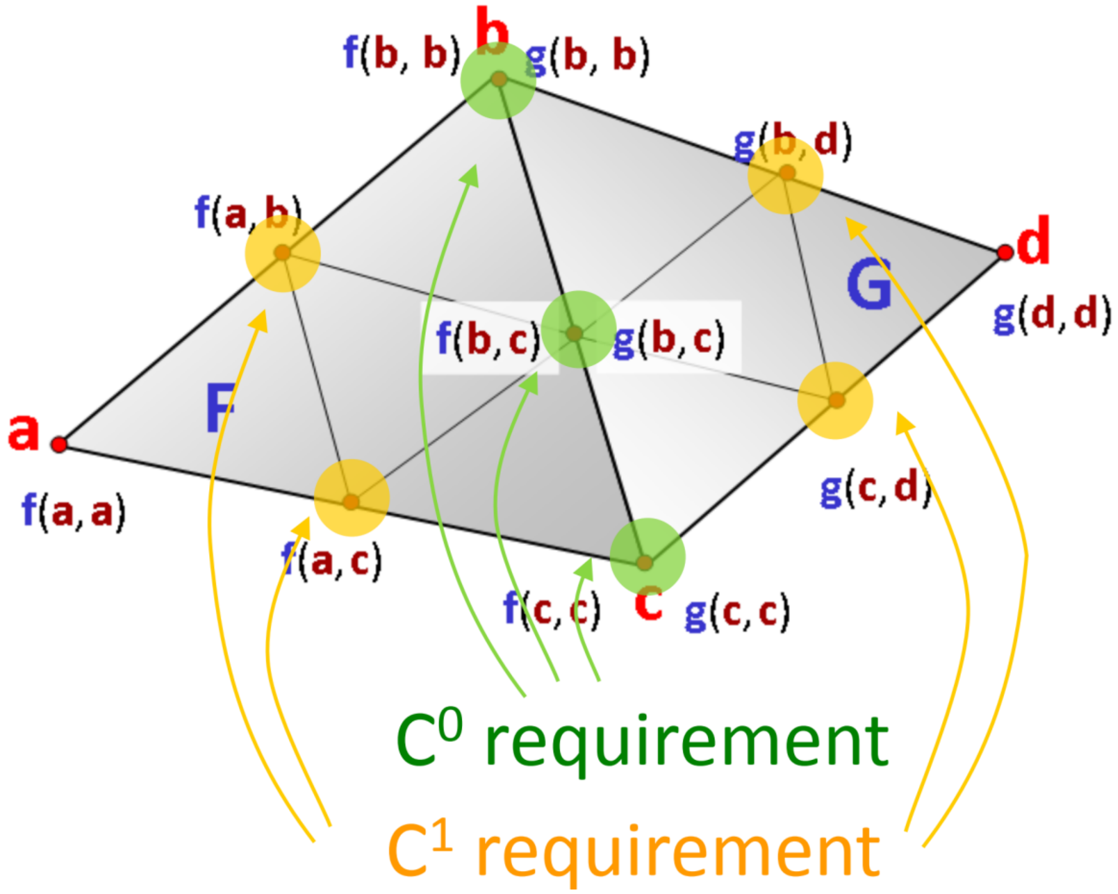
The blossoms have to agree partially:

$$f(a,b) = g(a,b)$$

$$f(b,d) = g(b,d)$$

$$f(a,c) = g(a,c)$$

$$f(c,d) = g(c,d)$$



# Continuity

## $C^1$ continuity: Proof

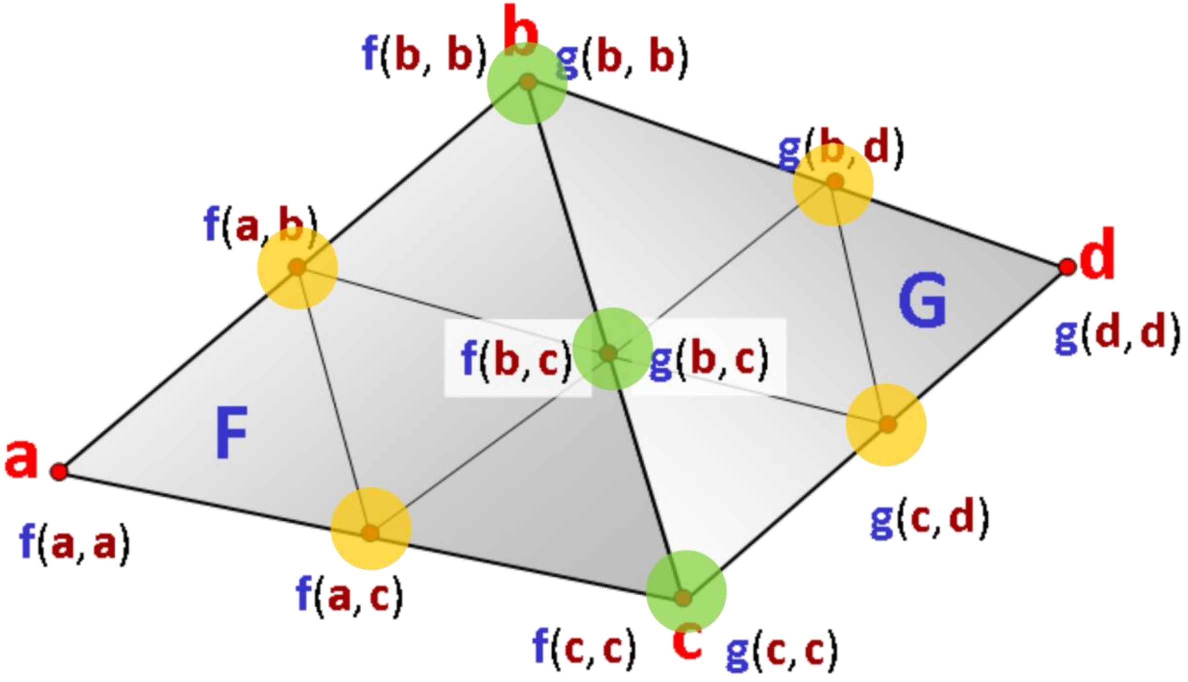
- Derivatives:

  $$\frac{\partial}{\partial \hat{\boldsymbol{d}}} \boldsymbol{F}(\boldsymbol{x})|_{\boldsymbol{x}=\boldsymbol{p}} = \boldsymbol{f}(\boldsymbol{p}, \hat{\boldsymbol{d}})$$

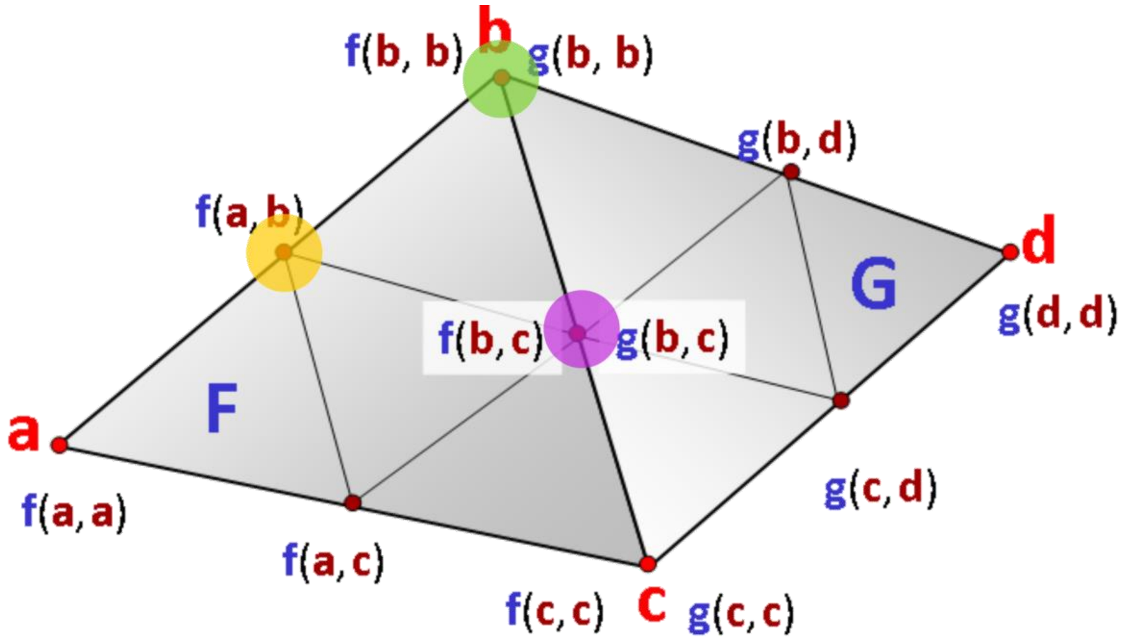
  (similar to the curve case)
- $C^1$-Continuity:

  $$\forall \boldsymbol{x} \in \mathbb{R}^3: \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{p}, \boldsymbol{x})$$
- We have to show

  $$\forall \boldsymbol{x} \in \mathbb{R}^3: \begin{cases} \boldsymbol{f}(\boldsymbol{b}, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{b}, \boldsymbol{x}) \\ \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{c}, \boldsymbol{x}) \end{cases}$$
- $\Rightarrow C^1$ continuity follows for all boundary points (by interp.)



# Continuity




## $C^1$ continuity: Proof

- So we have to show

$$\forall \boldsymbol{x} \in \mathbb{R}^3 : \begin{cases} \boldsymbol{f}(\boldsymbol{b}, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{b}, \boldsymbol{x}) \\ \boldsymbol{f}(\boldsymbol{c}, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{c}, \boldsymbol{x}) \end{cases}$$

- Proof:

Write $\boldsymbol{x} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c}$ (coordinate system)

$$\boldsymbol{f}(\boldsymbol{b}, \boldsymbol{x}) = \alpha\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) + \beta\boldsymbol{f}(\boldsymbol{b}, \boldsymbol{b}) + \gamma\boldsymbol{f}(\boldsymbol{b}, \boldsymbol{c})$$

$$\boldsymbol{g}(\boldsymbol{b}, \boldsymbol{x}) = \alpha\boldsymbol{g}(\boldsymbol{a}, \boldsymbol{b}) + \beta\boldsymbol{g}(\boldsymbol{b}, \boldsymbol{b}) + \gamma\boldsymbol{g}(\boldsymbol{b}, \boldsymbol{c})$$

$$\boldsymbol{f}(\boldsymbol{b}, \boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{b}, \boldsymbol{x}) \Leftrightarrow \alpha\boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) + \beta\boldsymbol{f}(\boldsymbol{b}, \boldsymbol{b}) + \gamma\boldsymbol{f}(\boldsymbol{b}, \boldsymbol{c})$$

$$= \alpha\boldsymbol{g}(\boldsymbol{a}, \boldsymbol{b}) + \beta\boldsymbol{g}(\boldsymbol{b}, \boldsymbol{b}) + \gamma\boldsymbol{g}(\boldsymbol{b}, \boldsymbol{c})$$

$$\Leftrightarrow \boldsymbol{f}(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{g}(\boldsymbol{a}, \boldsymbol{b})$$

$C^0$ $C^0$

(same for the other conditions)

# Continuity

**So what does this mean?**

- The blossoms have to agree partially:

$$f(a,b) = g(a,b)$$
$$f(b,d) = g(b,d)$$
$$f(a,c) = g(a,c)$$
$$f(c,d) = g(c,d)$$

- The points must be coplanar (with edge points):
  $f(a,b), g(b,d), g(b,b), g(b,c)$
- The points in $F$ must be affine images of the points in $G$