计算机辅助几何设计
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Implicit Surfaces

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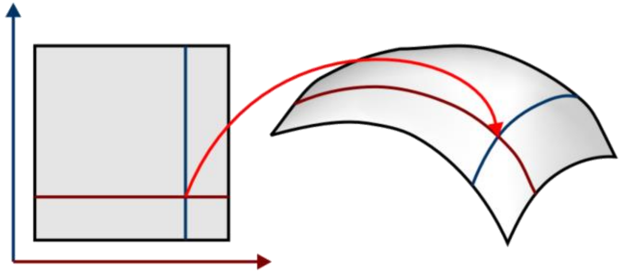
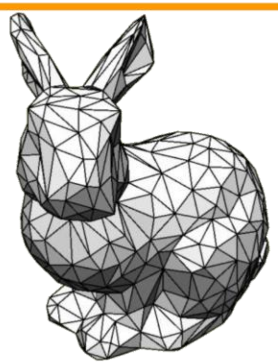
Implicit Surfaces

Introduction

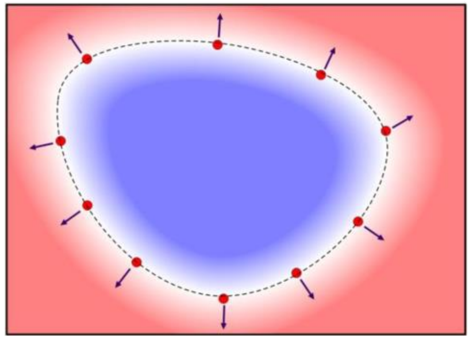
Modeling Zoo



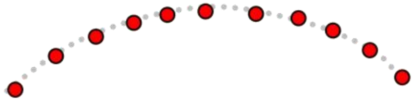
Parametric Models



Primitive Models



Implicit Models



Particle Models

Implicit Functions

Basic Idea:

- We describe an object $S \subseteq \mathbb{R}^d$ by an implicit equation:
  - $S = \{x \in \mathbb{R}^d | f(x) = 0\}$
  - The function f describes the shapes of the object.
- Applications:
  - In general, we could describe arbitrary objects
  - Most common case: surfaces in $\mathbb{R}^3$
  - This means, f is zero on an infinitesimally thin sheet only

The Implicit Function Theorem

Implicit Function Theorem:

- Given a *differentiable* function

  $f: \mathbb{R}^n \supseteq D \to \mathbb{R}$, $f\left(\boldsymbol{x}^{(0)}\right) = 0$, $\frac{\partial}{\partial x_n} f\left(\boldsymbol{x}^{(0)}\right) = \frac{\partial}{\partial x_n} f\left(x_1^{(0)}, \ldots, x_n^{(0)}\right) \neq 0$
- Within an ε-neighborhood of $\boldsymbol{x}^{(0)}$ we can represent the zero level set of f completely as a heightfield function g

  $g: \mathbb{R}^{n-1} \to \mathbb{R}$ such that for $\boldsymbol{x} - \boldsymbol{x}^{(0)} < \varepsilon$ we have:

  $f\left(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1})\right) = 0$ and

  $f(x_1, \ldots, x_n) \neq 0$ everywhere else
- The heightfield is a differentiable $(n-1)$-manifold and its surface normal is colinear to the gradient of f.

This means

If we want to model surfaces, we are on the safe side if:

- We use a smooth (differentiable) function f in $\mathbb{R}^3$
- The gradient of f does not vanish

This gives us the following guarantees:

- The zero-level set is actually a surface
    - We obtained a closed 2-manifold without boundary
    - We have a well defined interior / exterior.

Sufficient:

- We need smoothness / non-vanishing gradient only close to the zero-crossing.

Implicit Functions Types

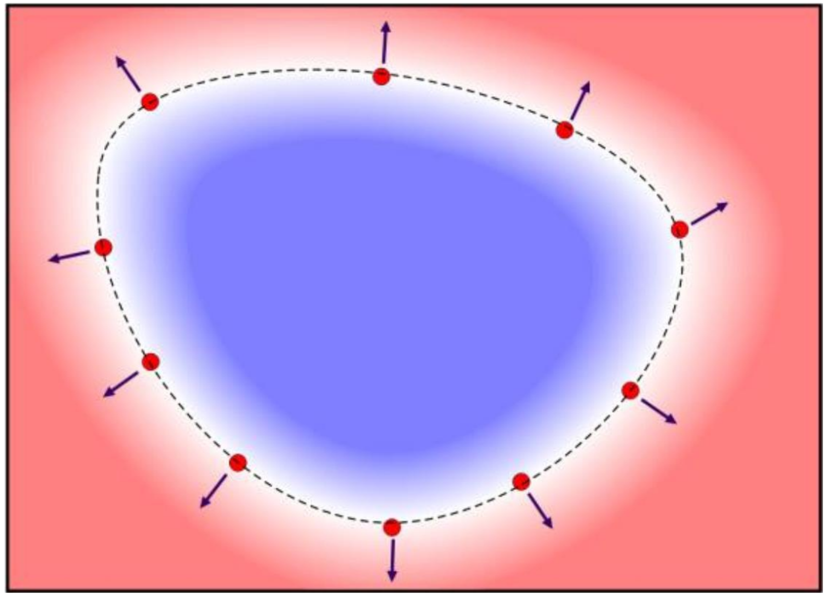
Function Types:

- General case
  - Non-zero gradient at zero crossings
  - Otherwise arbitrary
- Signed implicit function:
  - sign(f): negative inside and positive outside the object
    (or the other way round, but we assume this orientation here)
- Signed distance field (SDF)
  - $|f|$ = distance to the surface
  - sign(f): negative inside, positive outside
- Squared distance function
  - f = (distance to the surface)^2

Implicit Functions Types

Use depends on application:

- Signed implicit function
  - Solid modelling
  - Interior well defined
- Signed distance function (SDF)
  - Most frequently used representation
  - Constant gradient → numerically stable surface definition
  - Availability of distance value useful for many applications
- Squared distance function
  - This representation is useful for statistical optimization
  - Minimize sum of squared distances → least squares optimization
  - Useful for surface defined up to some insecurity / noise
  - Direct surface extraction more difficult (*gradient vanishes*!)

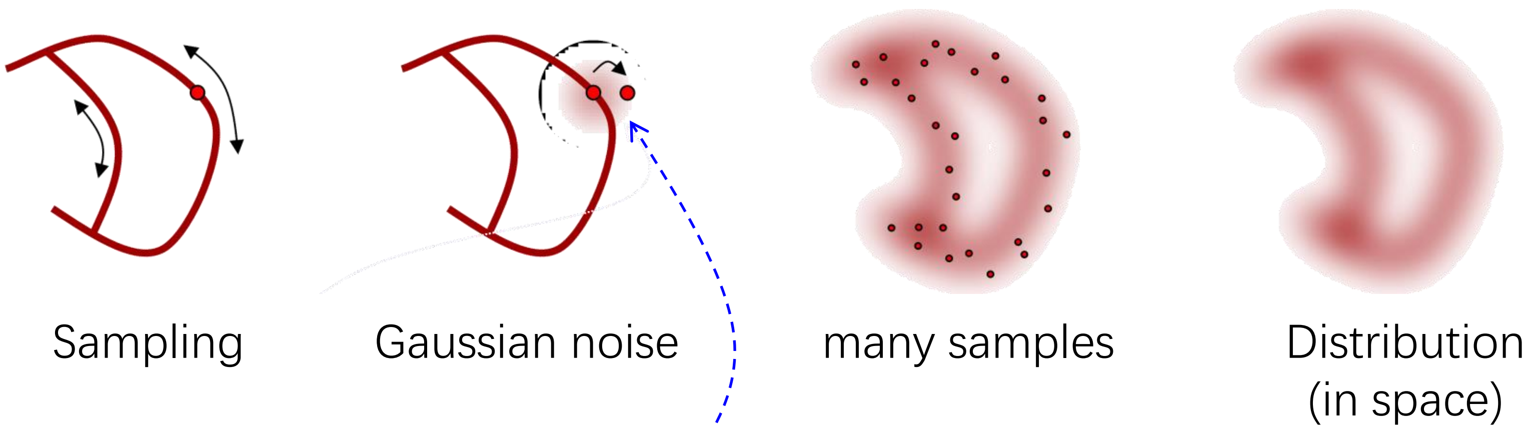


signed distance

Squared Distance Function

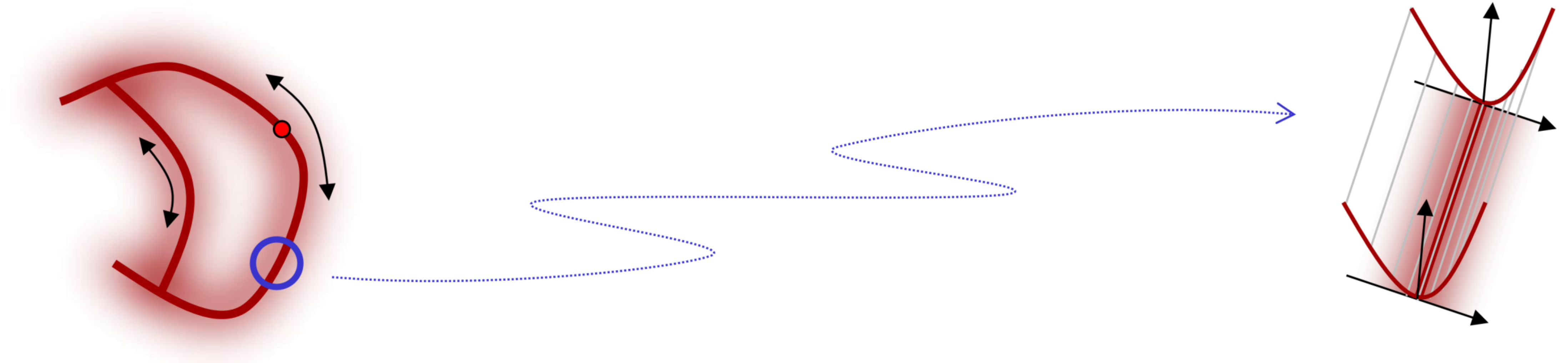
Example: Surface from random samples

1. Determine sample point (uniform)
2. Add noise (Gaussian)



$$p_{\mu,\Sigma}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{d}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

Squared Distance Function



Square Distance Function:

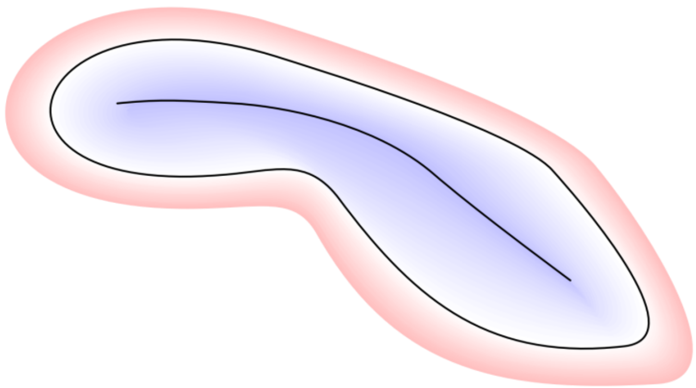
- Sampling a surface with uniform sampling and Gaussian noise:

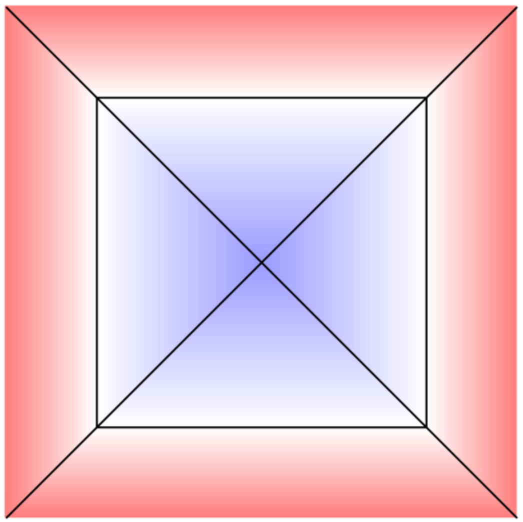
  ⇒ Probability density is a convolution of the object with a Gaussian kernel
- Smooth surfaces: The log-likelihood can be approximated by a squared distance function

Smoothness

Smoothness of signed distance function:

- Any distance function (signed, unsigned, squared) in general cannot be globally smooth
- The distance function is non-differentiable at the medial axis
  - Media axis = set of points that have the same distances to two or more different surface points
  - For sharp corners, the medial axis touches the surfaces
  - This means: f non-differentiable on the surface itself
  - Usually, this is no problem in practice





Differential Properties

Some useful differential properties:

- We look at a surface point $\boldsymbol{x}$, i.e. $f(\boldsymbol{x}) = 0$.
- We assume $\nabla f(\boldsymbol{x}) \neq 0$.
- The unit normal of the implicit surface is given by:

$$n(\boldsymbol{x}) = \frac{\nabla f(\boldsymbol{x})}{||\nabla f(\boldsymbol{x})||}$$

  - For signed functions, the normal is pointing outward
  - For signed distance functions, this simplifies to $\boldsymbol{n}(\boldsymbol{x}) = \nabla f(\boldsymbol{x})$

Differential Properties

Some useful differential properties:

- The mean curvature of the surface is proportional to the divergence of the unit normal:

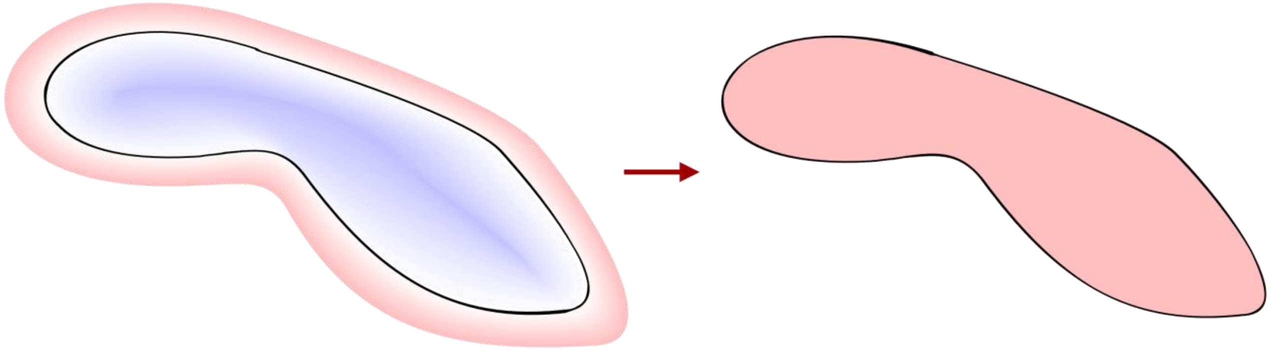
$$-2H(\boldsymbol{x}) = \nabla \cdot \boldsymbol{n}(\boldsymbol{x}) = \frac{\partial}{\partial x} n_x(\boldsymbol{x}) + \frac{\partial}{\partial y} n_y(\boldsymbol{x}) + \frac{\partial}{\partial z} n_z(\boldsymbol{x}) = \nabla \cdot \frac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$$

- For a signed distance function, the formula simplifies to:

$$-2H(\boldsymbol{x}) = \nabla \cdot \nabla f(x) = \frac{\partial^2}{\partial x^2} f(\boldsymbol{x}) + \frac{\partial^2}{\partial y^2} f(\boldsymbol{x}) + \frac{\partial^2}{\partial z^2} f(\boldsymbol{x}) = \Delta f(\boldsymbol{x})$$

Computing Volume Integrals

Computing volume integrals



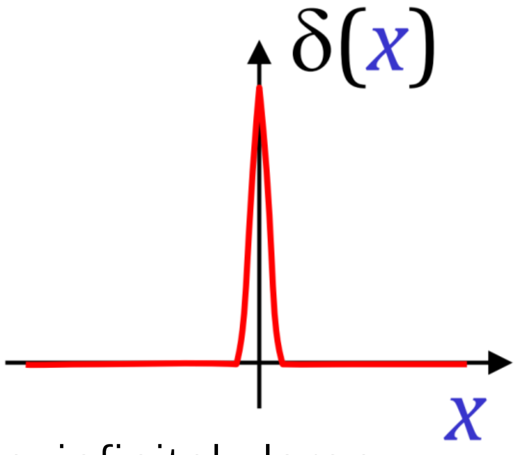
- Heavyside function

$$\text{step}(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

- Volume integral over interior volume Ω_f of some function $g(\boldsymbol{x})$ (assuming negative interior values):

$$\int_{\Omega_f} g(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{R}^3} g(\boldsymbol{x}) \left(1 - \text{step}\left(f(x)\right)\right) d\boldsymbol{x}$$

Computing Surface Integrals



Computing surface integrals:

- Dirac delta functions:
  - Idealized function (distribution)
  - Zero everywhere ($\delta(x) = 0$), except at $x = 0$, where it is positive, infinitely large
  - The integral of $\delta(x)$ over x is one
- Dirac delta function on the surface: directional derivative of $\mathrm{step}(x)$ in normal direction:

$$\hat{\delta} = \nabla\left[\mathrm{step}\left(f(\boldsymbol{x})\right)\right] \cdot n(\boldsymbol{x}) = [\nabla \mathrm{step}]\left(f(\boldsymbol{x})\right)\nabla f(\boldsymbol{x}) \cdot \frac{\nabla f(\boldsymbol{x})}{\|\nabla f(\boldsymbol{x})\|}$$

$$= \delta\left(f(\boldsymbol{x})\right) \cdot \|\nabla f(\boldsymbol{x})\|$$

Surface Integral

Computing surface integrals:

- Surface integral over the surface $\partial\Omega_f = \{\boldsymbol{x} | f(\boldsymbol{x}) = 0\}$ of some function $g(\boldsymbol{x})$:

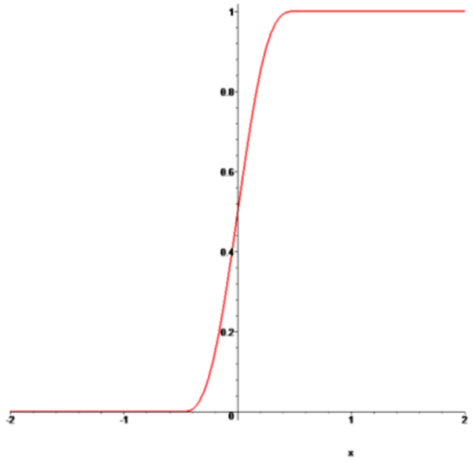
$$\int_{\Omega_f} g(\boldsymbol{x}) d\boldsymbol{x} = \int_{\mathbb{R}^3} \boldsymbol{g}(\boldsymbol{x}) \boldsymbol{\delta}(\boldsymbol{f}(\boldsymbol{x})) |\nabla \boldsymbol{f}(\boldsymbol{x})| \boldsymbol{dx}$$

- This looks nice, but is numerically intractable.
- We can fix this using smoothed out Dirac/Heavyside functions…

Smoothed Functions

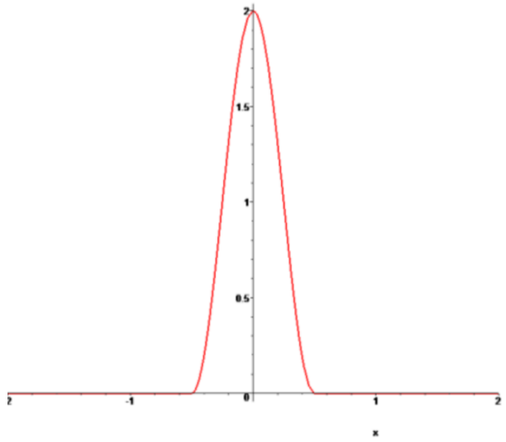
Smooth-step function

$$\text{smoothstep}(x) = \begin{cases} 0 & x < -\varepsilon \\ \dfrac{1}{2} + \dfrac{x}{2\varepsilon} + \dfrac{1}{2\pi}\sin\left(\dfrac{\pi x}{\varepsilon}\right) & -\varepsilon \le x \le \varepsilon \\ 1 & \varepsilon < x \end{cases}$$



Smoothed Dirac delta function

$$\text{smoothdelta}(x) = \begin{cases} 0 & x < -\varepsilon \\ \dfrac{1}{2\varepsilon} + \dfrac{1}{2\varepsilon}\cos\left(\dfrac{\pi x}{\varepsilon}\right) & -\varepsilon \le x \le \varepsilon \\ 1 & \varepsilon < x \end{cases}$$



Implicit Surfaces

Numerical Discretization

Representing Implicit Functions

Representation: Two basic techniques

- Discretization on grids
  - Simple finite differencing (FD) grids
  - Grids of basis functions (finite elements FE)
  - Hierarchical / adaptive grids (FE)
- Discretization with radial basis functions (particle FE methods)

Discretization

Discretization examples

- In the following, we will look at 2D examples
- The 3D (d-dimensional) case is similar

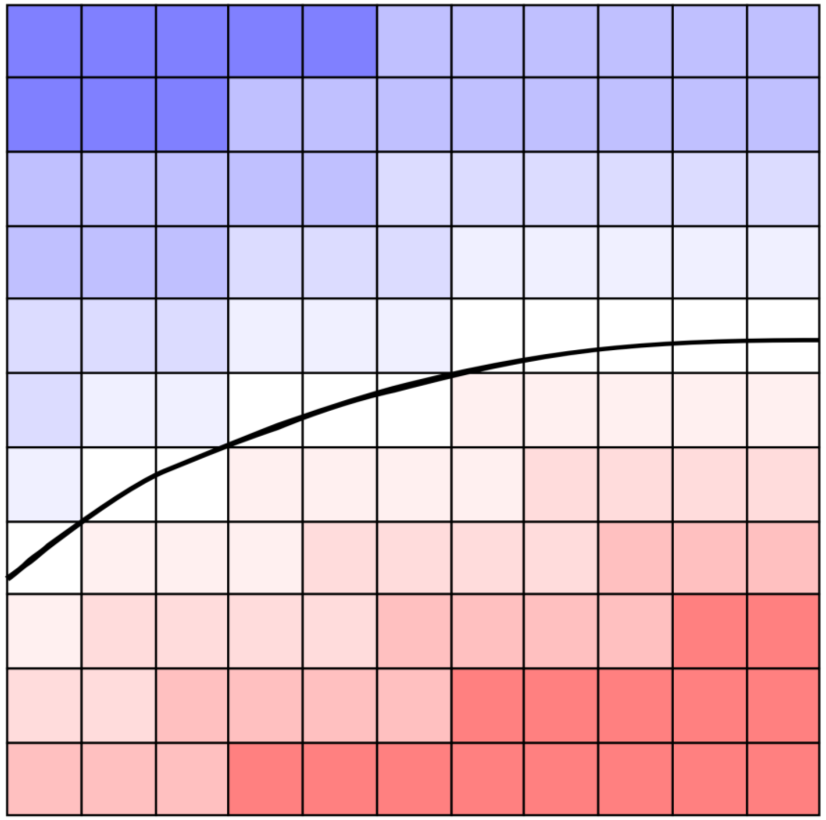
Regular Grids

Discretization:

- Regular grid of values $f_{i,j}$
- Grid spacing h
- Differential properties can be approximated by finite differences:
  - For example

$$\frac{\partial}{\partial x} f(\boldsymbol{x}) = \frac{1}{h}\left(f_{i(\boldsymbol{x}),j(\boldsymbol{x})} - f_{i(\boldsymbol{x})-1,j(\boldsymbol{x})}\right) + O(h)$$

$$\frac{\partial}{\partial x} f(\boldsymbol{x}) = \frac{1}{2h}\left(f_{i(\boldsymbol{x})+1,j(\boldsymbol{x})} - f_{i(\boldsymbol{x})-1,j(\boldsymbol{x})}\right) + O(h^2)$$

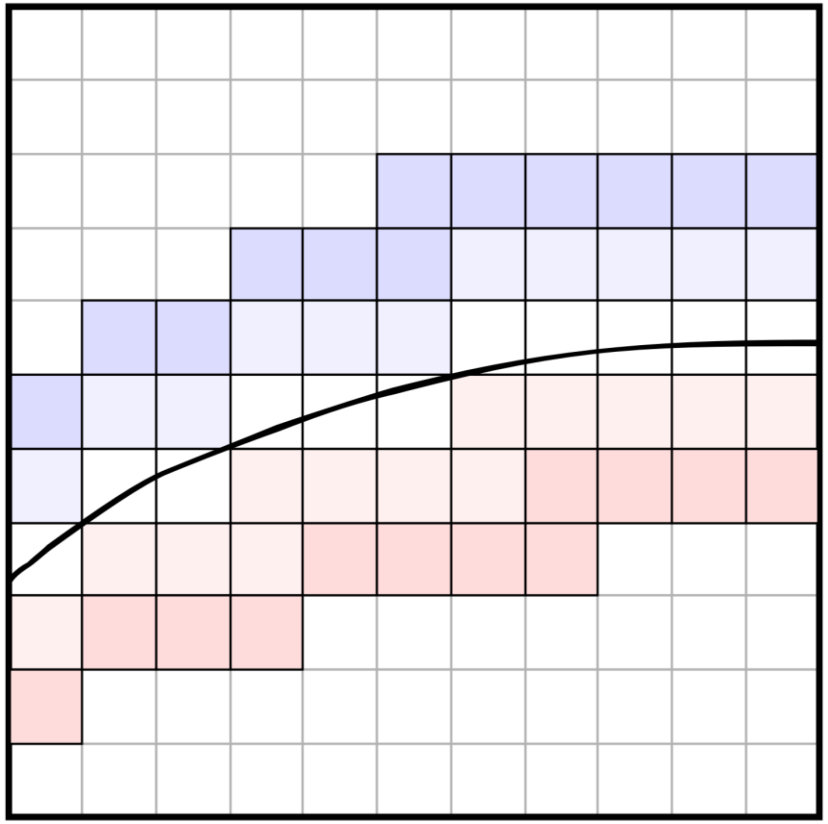


Regular Grids

Variant:

- Use only cells near the surface
- Saves storage & computation time
- However: we need to know an estimate on where the surface is located to setup the representation
- Propagate to the rest of the volume (if necessary):

  fast marching method



Fast Marching Method

Problem statement:

- Assume we are given the surface and signed distance value in a narrow band
- Now we want to compute distance values everywhere on the grid

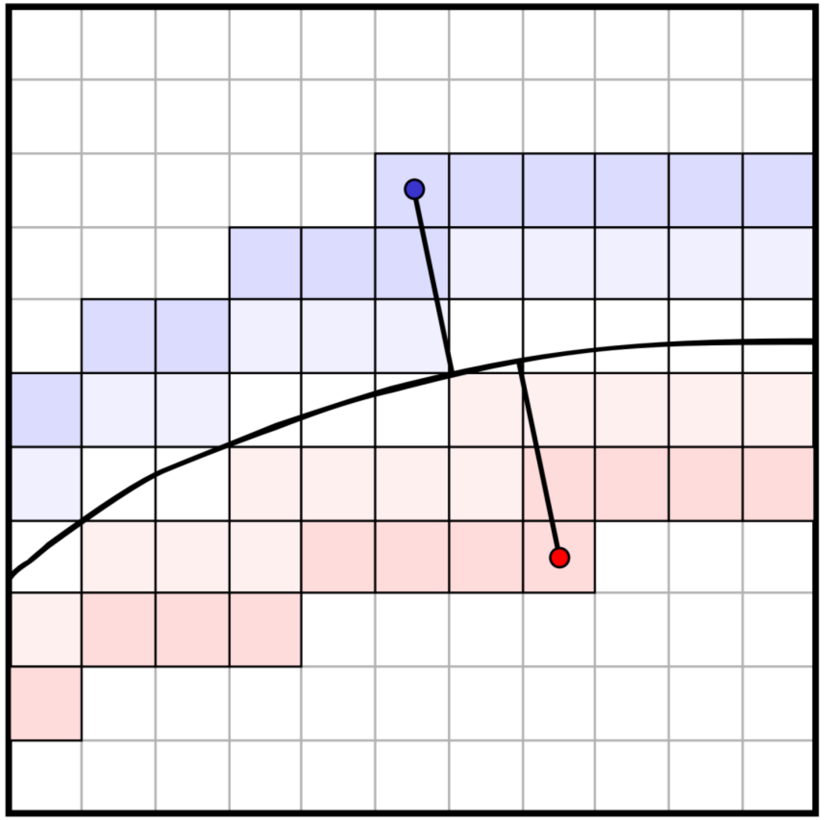
Three Solutions:

- Nearest neighbor queries
- Eikonal equation
- Fast marching

Nearest Neighbors

Algorithm:

- For each grid cell:
  - Compute nearest point on the surface
  - Enter distance
- Approximate nearest neighbor computation:
  - Look for nearest grid cell with zero crossing first
  - Then compute distance curve ↔ zero level set using a Newton-like algorithm (repeated point-to-plane distance)
- Costs: O(n) kNN queries (n empty cells)



Eikonal Equation

Eikonal Equation

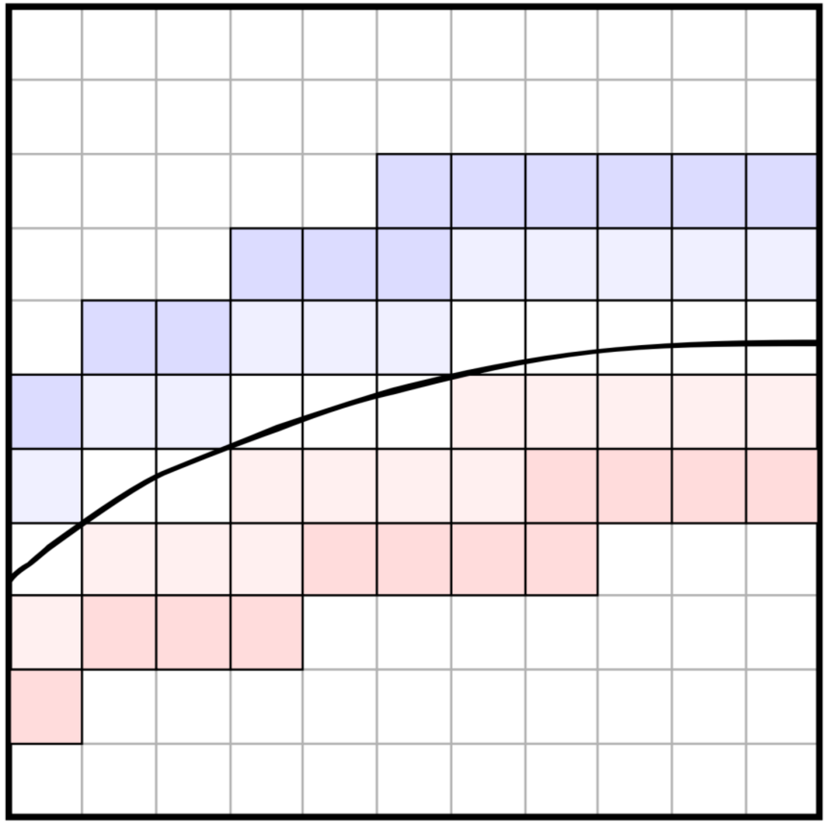
- Place variables in empty cells
- Fixed values in known cells
- Then solve the following PDE:

$$\|\nabla f\| = 1$$

subject to $f(\boldsymbol{x}) = f_{\text{known}}(\boldsymbol{x})$

on the known area $\boldsymbol{x} \in A_{\text{known}}$

- This is a (non-linear) boundary value problem

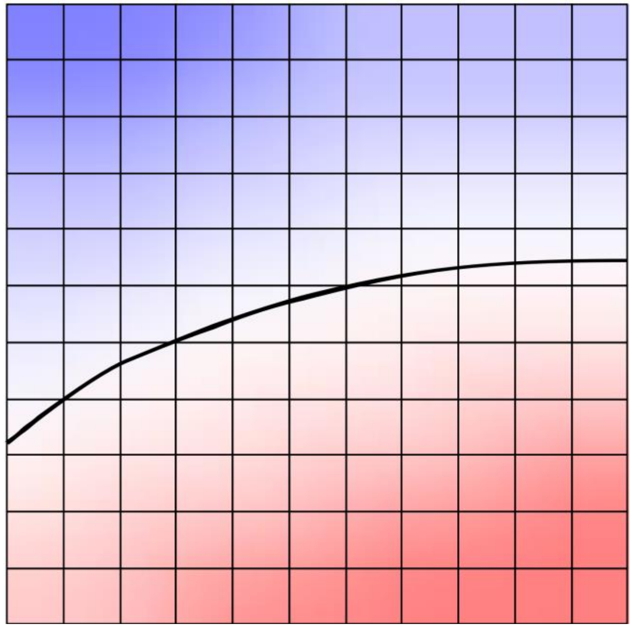


Fast Marching

Solving the Equation:

- The Eikonal equation can be solved efficiently by a region growing algorithm:
  - Start with the initial known values
  - Compute new distances at immediate neighbors solving a local Eikonal equation (*)
  - The smallest of these values must be correct (similar to Dijkstra’s algorithm)
  - Fix this value and update the neighbors again
  - Growing front, $O(n \log n)$ time

Regular Grids of Basis Functions



Discretization (2D):

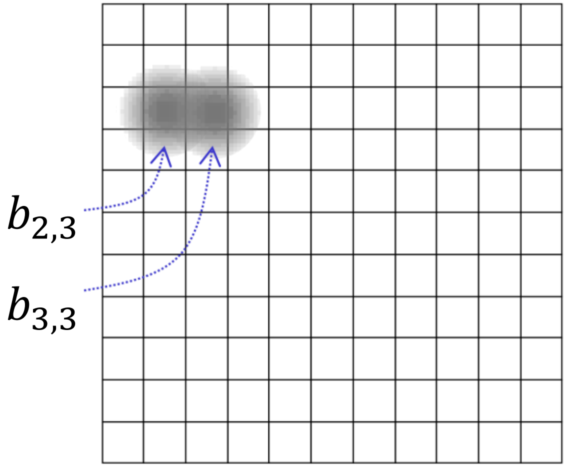
- Place a basis function in each grid cell:

$$b_{i,j} = b(x - i, y - j)$$

- Typical choices:
  - Bivariate uniform cubic B-splines (tensor product)
  - $b(x, y) = \exp[-\lambda(x^2 + y^2)]$
- The implicit function is then represented as

$$f(x, y) = \sum_{0}^{n_i} \sum_{0}^{n_j} \lambda_{i,j} b_{i,j}(x, y)$$

- The $\lambda_{i,j}$ describe different f



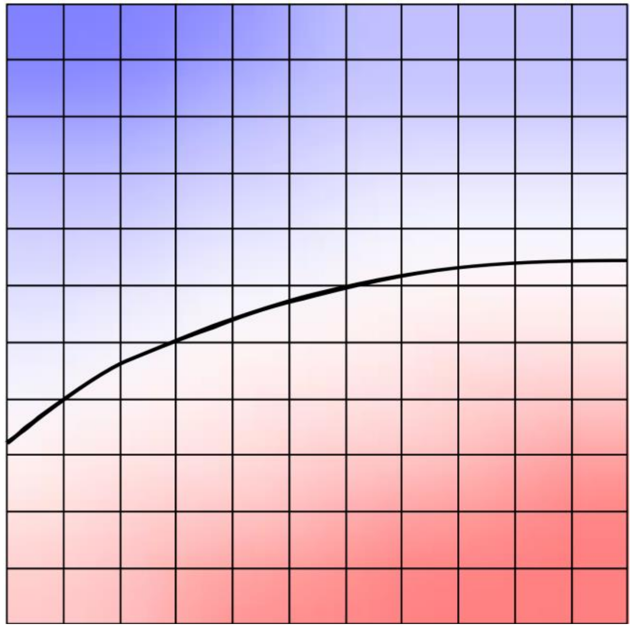
Regular Grids of Basis Functions

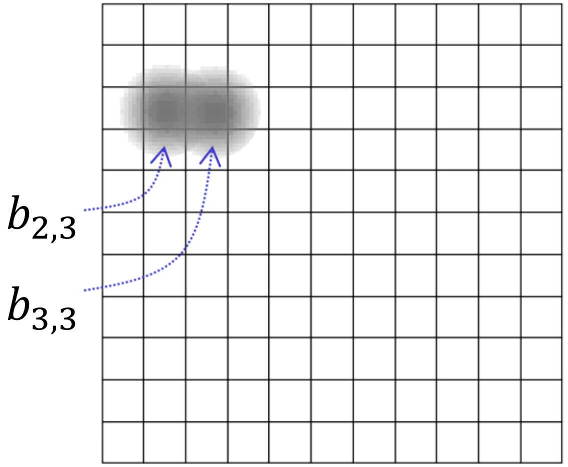
Differential Properties:

- Derivatives:

$$\frac{\partial}{\partial x_{k1} \dots \partial x_{km}} f(x,y) = \sum_{i=0}^{n_i} \sum_{j=0}^{n_j} \lambda_{i,j} \left(\frac{\partial}{\partial x_{k1} \dots \partial x_{km}} b \right)(x,y)$$

- Derivatives are linear combinations of the derivatives of the basis function
- In particular: we again get a linear expression in $\lambda_{i,j}$

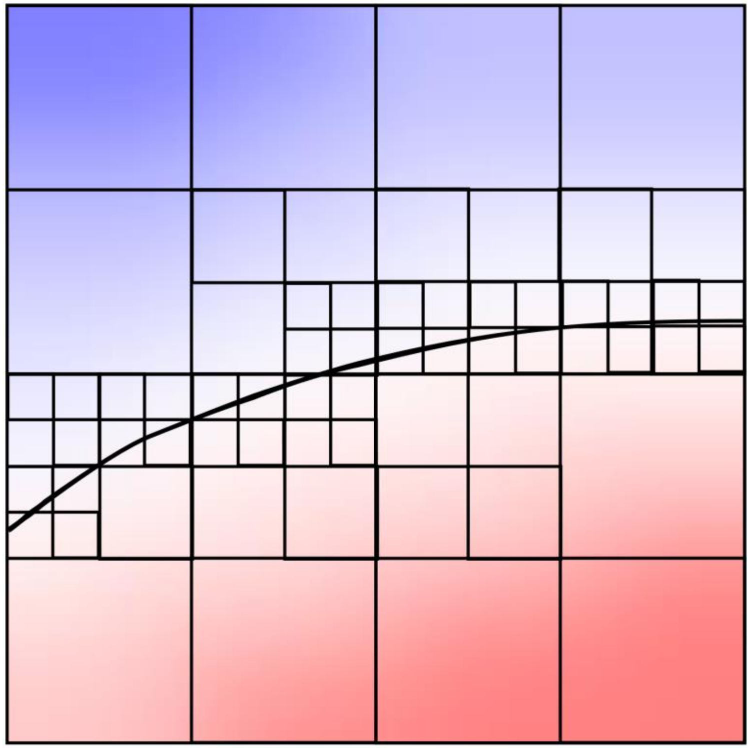




Adaptive Grids

Adaptive / hierarchical grids:

- Perform a quadtree / octree tessellation of the domain (or any other partition into elements)
- Refine where more precision is necessary (near surface, maybe curvature dependent)
- Associate basis functions with each cell (constant or higher order)



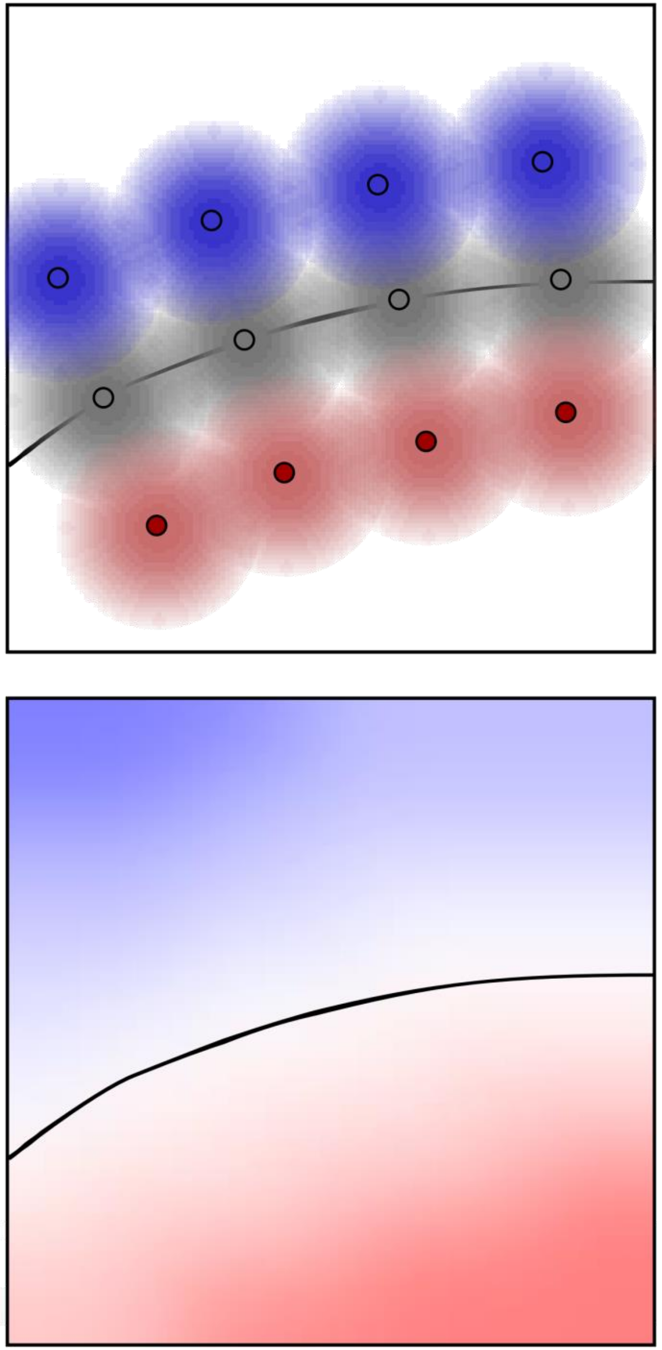
Particle Methods

Particle methods / radial basis function:

- Place a set of “particles” in space at positions $\boldsymbol{x}_i$
- Associate each with a radial basis function $b(\boldsymbol{x} - \boldsymbol{x}_i)$
- The discretization is then given by:

$$f(\boldsymbol{x}) = \sum_{i=0}^{n} \lambda_i b(\boldsymbol{x} - \boldsymbol{x}_i)$$

- The λ_i encode f.



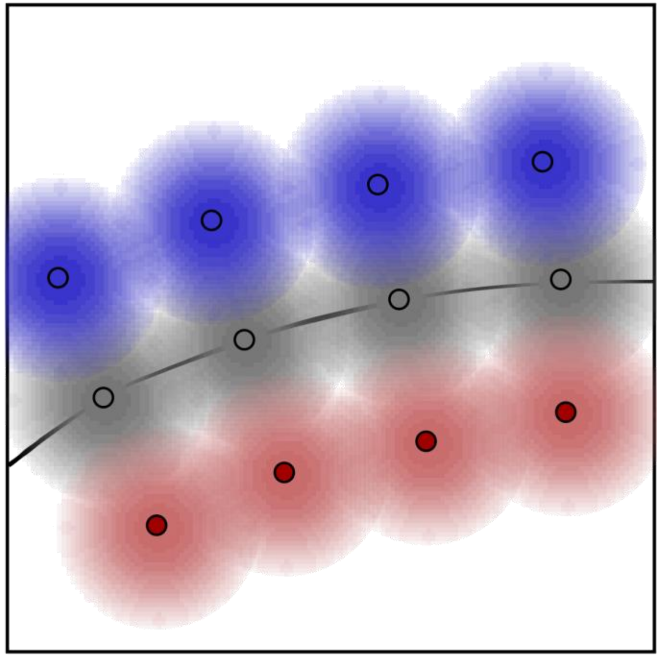
Particle Methods

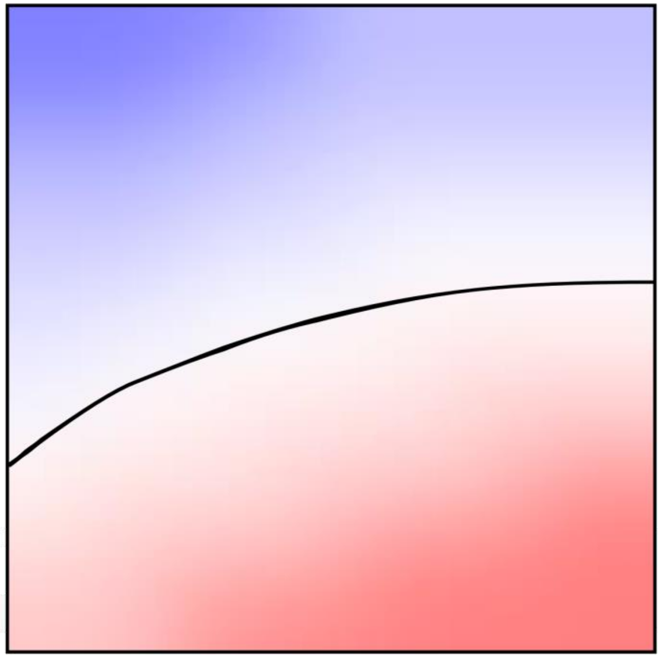
Particle methods / radial basis function:

- Obviously, derivatives are again linear in λ_i:

$$\frac{\partial}{\partial x_{k1} \ldots \partial x_{km}} f(\boldsymbol{x}) = \sum_{i=0}^{n_j} \lambda_{i,j} \left(\frac{\partial}{\partial x_{k1} \ldots \partial x_{km}} b \right) (\boldsymbol{x} - \boldsymbol{x_i})$$

- The radial basis functions can also have different size (support) for adaptive refinement
- Placement: near the expected surface

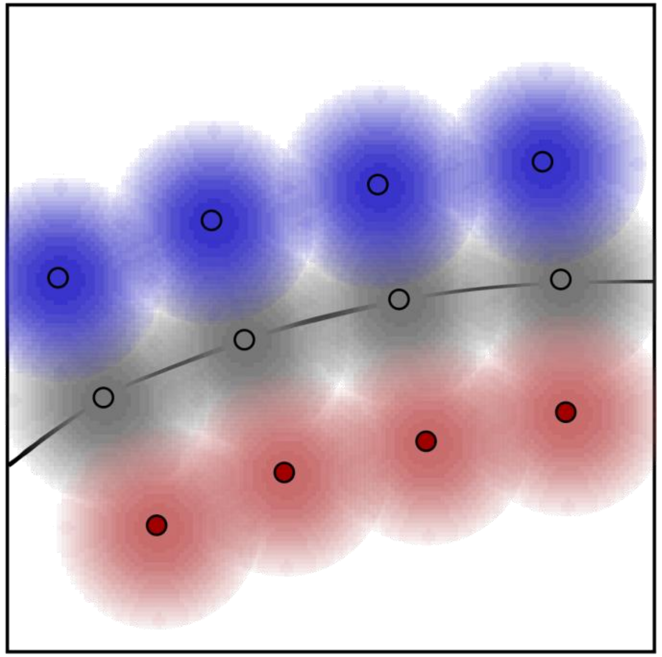


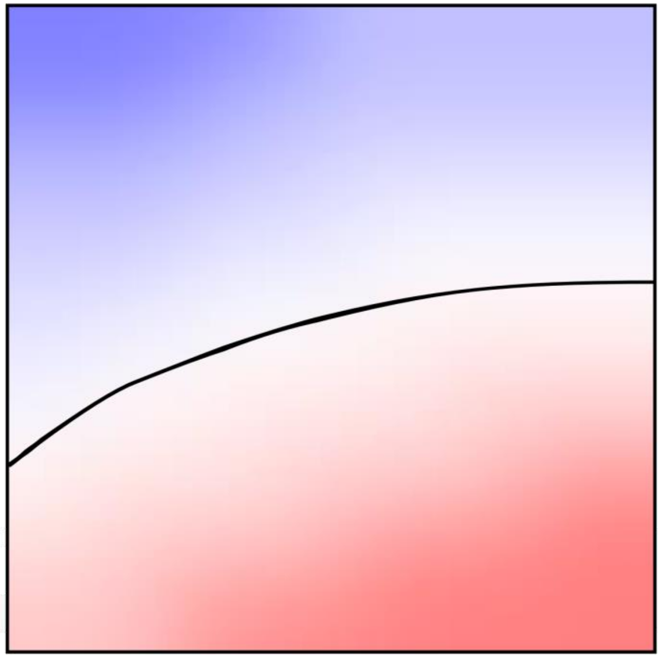


Particle Methods

Particle methods / radial basis function:

- Where should we place the radial basis functions?
  - If we have an initial guess for the surface shape:
    - Put some on the surface
    - And some in +/- normal direction
  - Otherwise:
    - Uniform placement in lowres
    - Solve for surface
    - Refine near lowres-surface, iterate





Types of Radial Basis Functions

Typical choices for radial basis functions:

- (Quasi-) compactly supported functions:
  - Exponentials / normal distribution densities: $\exp(-\lambda \boldsymbol{x}^2)$
  - Uniform (cubic) tensor product B-Splines
  - Moving-least squares finite element basis functions (will be discussed later)
- Globally supported functions:
  - Thin plate spline basis functions:
    $\|x - x_0\|^2 \ln\|x - x_0\|$ (2D), $\|x - x_0\|^3$ (3D)
  - These functions guarantee minimal integral second derivatives.

Pros & Cons

Why use globally supported basis functions?

- They come with smoothness guarantees
- However: computations might become expensive (we will see later how to device efficient algorithms for globally supported radial basis functions)

Locally supported functions:

- Easy to use
- Additional regularization might become necessary to compute a “nice” surface

Implicit Surfaces

Level Set Extraction

Iso-Surface Extraction

New task:

- Assume we have defined an implicit function
- Now we want to extract the surface
- I.e. convert it to an explicit, piecewise parametric representation, typically a triangle mesh
- For this we need an iso-surface extraction algorithm
  - a.k.a. level set extraction
  - a.k.a. contouring

Algorithms

Algorithms:

- Marching Cubes
  - This is the standard technique
  - We will also discuss some problems / modifications
- Particle methods
  - Just to show an alternative
  - Not used that frequently in practice

Marching Cubes

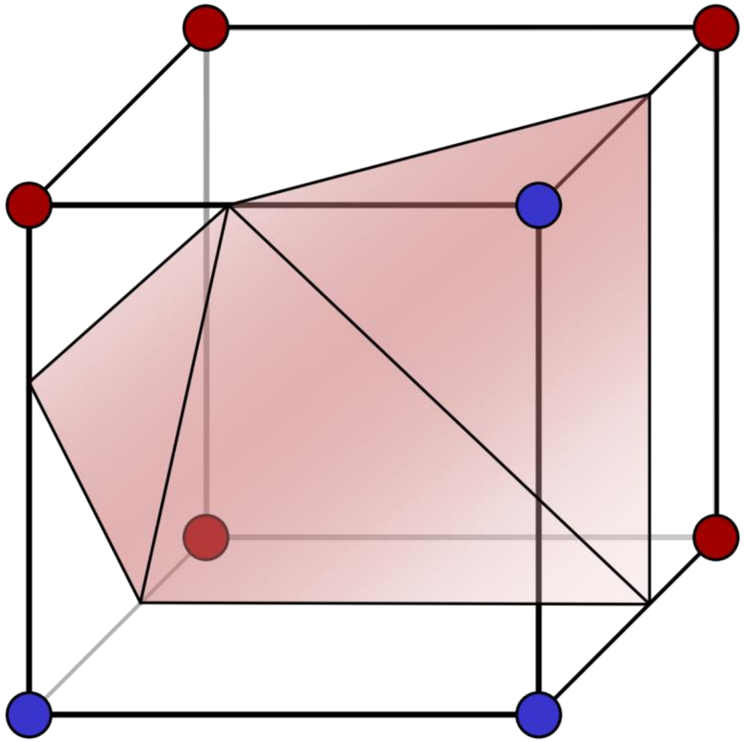
Marching Cubes:

- The most frequently used iso surface extraction algorithm
  - Creates a triangle mesh from an iso-value surface of a scalar volume
  - The algorithm is also used frequently to visualize CT scanner data and other volume data
- Simple idea:
  - Define and solve a fixed complexity, local problem
  - Compute a full solution by solving many such local problems incrementally

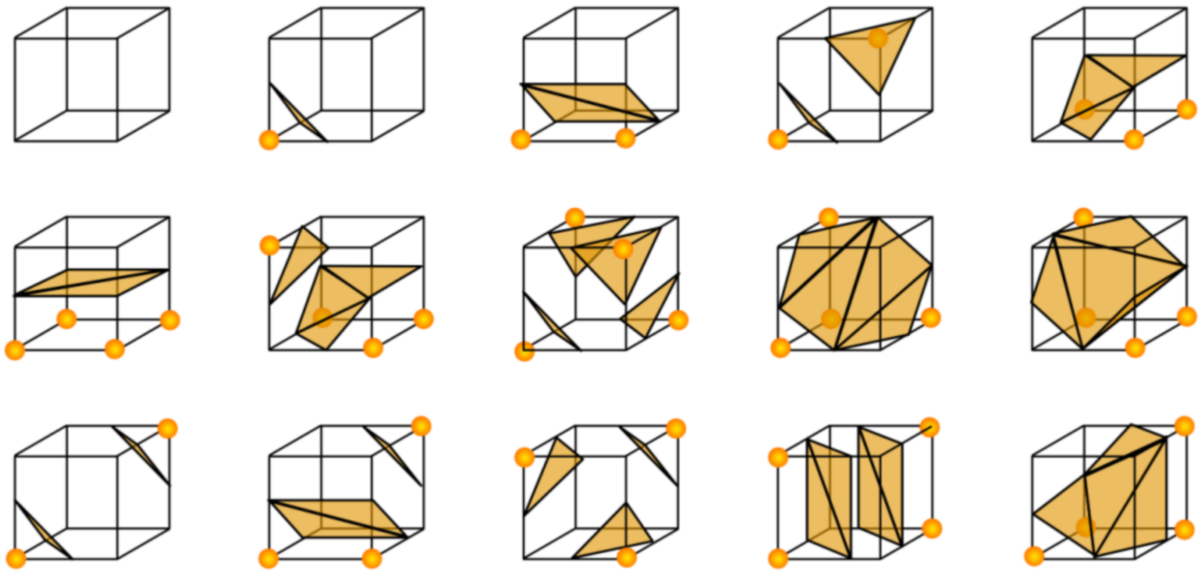
Marching Cubes

Marching Cubes:

- Here is the local problem:
  - We have a cube with 8 vertices
  - Each vertex is either inside or outside the volume (i.e. $f(\boldsymbol{x}) < 0$ or $f(\boldsymbol{x}) \geq 0$)
  - How should we triangulate this cube?
  - How should we place the vertices?



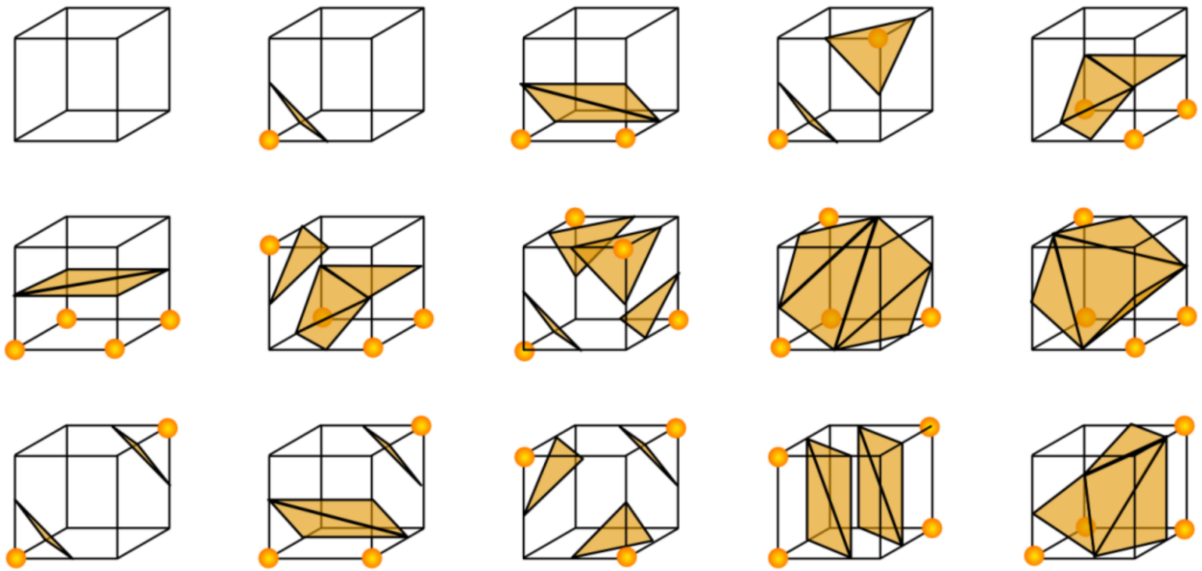
Triangulation



Triangulation:

- We have 256 different cases – each of the 8 vertices can be in or out
- By symmetry, this can be reduced to 15 cases
  - Symmetry: reflection, rotation, and bit inversion
- This means, we can compute the topology of the mesh

Vertex Placement



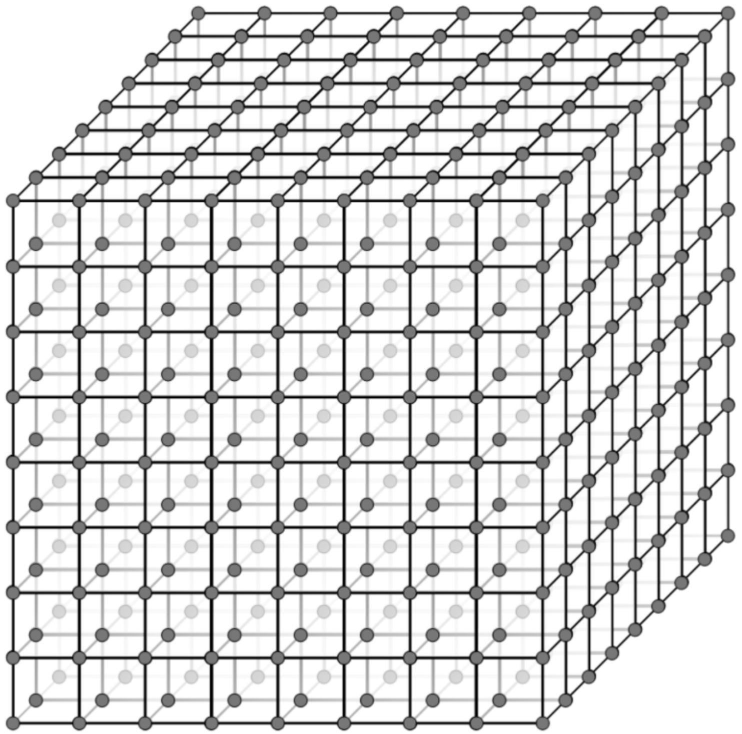
How to place the vertices?

- Zero-th order accuracy: Place vertices at edge midpoints
- First order accuracy: Linearly interpolate vertices along edges.
- Example: for scalar values $f(x) = -0.1$ and $f(y) = 0.2$, place the vertex at ratio $1:2$ between x and y

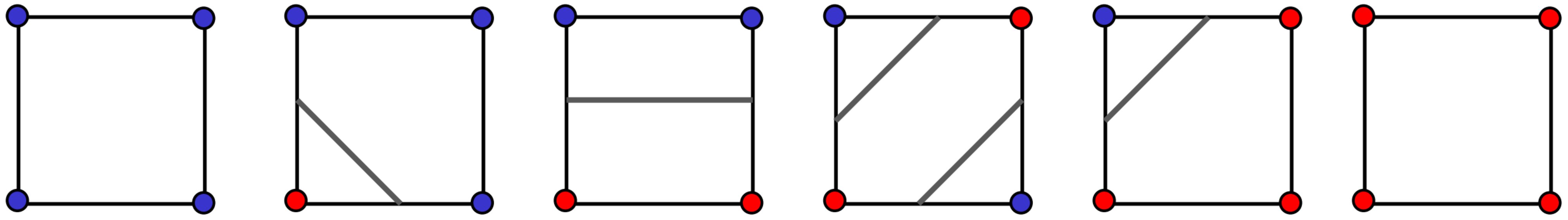
Outer Loop

Outer Loop:

- Compute a bounding box of the domain of the implicit function
- Divide it into cubes of the same size (regular cube grid)
- Execute “marching cube” algorithm in each subcube
- Output the union of all triangles generated
- Optionally: Use a vertex hash table to make the mesh consistent (remove double vertices)



Marching Squares



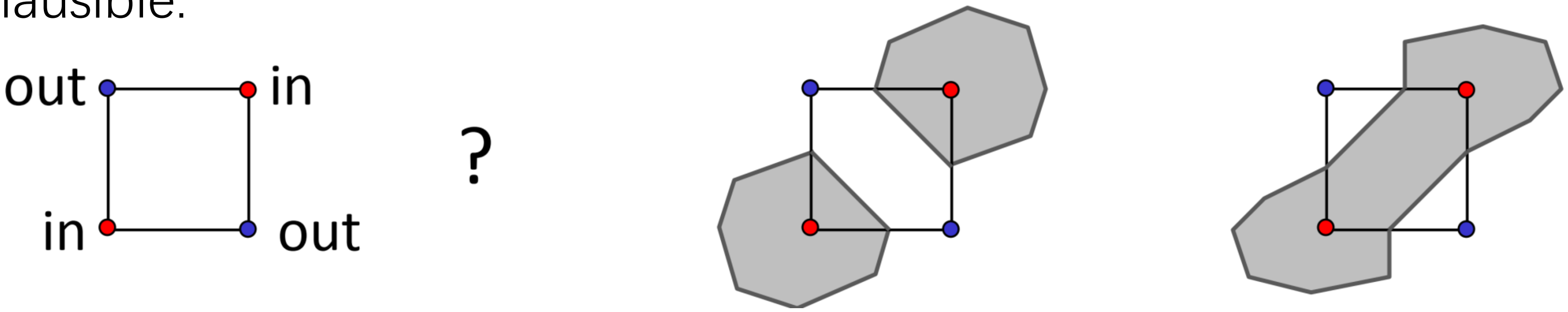
Marching Squares:

- There is also a 2D version of the algorithm, called marching squares
- Same idea, but fewer cases

Ambiguities

There is a (minor) technical problem remaining:

- The triangulation can be ambiguous
- In some cases, different topologies are possible which are all locally plausible:



- This is an *undersampling artifact*. At a sufficiently high resolution, this cannot occur.
- Problem: Inconsistent application can lead to holes in the surface (non-manifold solutions)

Ambiguities

Solution

- Always use the same solution pattern in ambiguous situations
- For example: Always *connect* diagonally
  - This might yield topologically wrong results.
  - But the surface is guaranteed to be a triangulated 2-manifold without holes and with well-defined interior / exterior
- Better solution:
  - Use higher resolution sampling (if possible)
- All of this (problem and solutions) also applies to the 3D case.

MC Variations

Empty space skipping:

- Marching cube uses an n^3 voxel grid, which can become pretty expensive
- The surface intersects typically only $O(n^2)$ voxels.
- If we roughly know where the surface might appear, we can restrict the execution of the algorithm (and the evaluations of f at the corners) to a narrow band around the surface.
- Example: Particle methods – only extract within the support of the radial basis functions.

MC Variations

Hierarchical marching cubes algorithm:

- One can use a hierarchical version of the marching cubes algorithms using a balanced octree instead of a regular grid
  - We need some refinement criterion to judge on where to subdivide
  - This is application dependent (depends on the definition of f).
- However, we obtain many more cases to consider (which is painful to derive).

Simple solution (common in practice):

- Extract high-resolution triangle mesh
- Then run mesh simplification (slower, but better quality).

Particle-Based Extraction

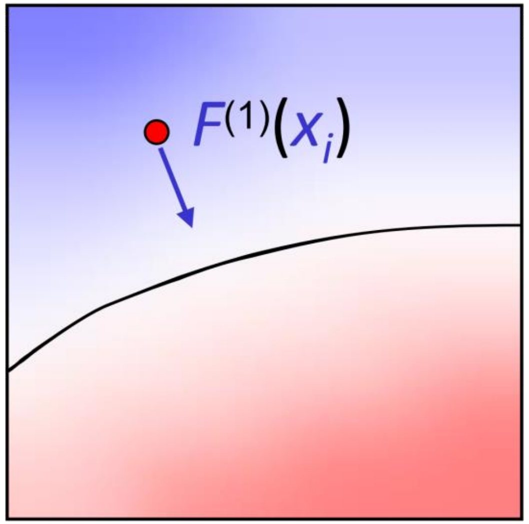
Particle-based method:

- This technique creates a set of points as output, which cover the iso-surface.
- Algorithm:
  - Start with a random point cloud (n points in a bounding volume)
  - Now define forces that attract particles to the zero-level set.
  - Also add some (weak) tangential repulsion to make them distribute uniformly

Forces

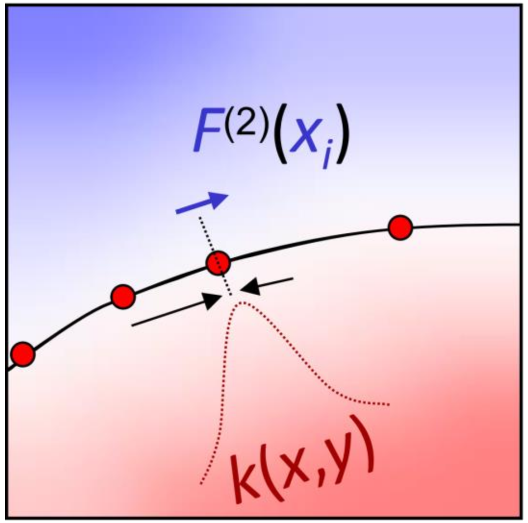
Attraction "force":

$$F^{(1)}(x_i) = m_i \|\nabla f(x_i)\|^2$$



Tangential repulsion force:

$$F^{(2)}(x_i) = \left(\sum_{j \neq i} k(x_i, x_j) \frac{x_i - x_j}{\left\|x_i - x_j\right\|^2}\right)\left(I - \left[\frac{\nabla f(x_i)}{\|\nabla f(x_i)\|}\right] \cdot \left[\frac{\nabla f(x_i)}{\|\nabla f(x_i)\|}\right]^T\right)$$



Solution

Solution:

- We obtain a system of ordinary differential equations
- The ODE can be solved numerically
- Simplest technique: gradient decent (explicit Euler)
  - Move every point by a fraction of the force vector
  - Recalculate forces
  - Iterate
- We have the solution if the system reaches a steady state (nothing moves anymore, numerically)

Implicit Surfaces

Solid Modeling

Solid Modeling

We want to:

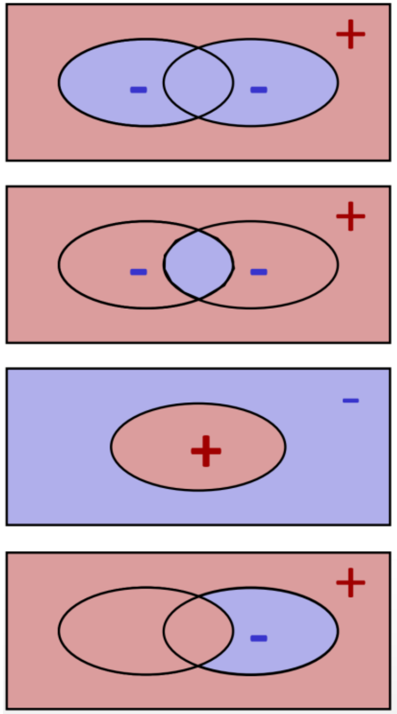
- Form basic volumetric primitives (spheres, cubes, cylinders) as implicit functions (this is easy, no details)
- Compute Boolean combinations of these primitives:

  Intersection, union, etc…
- Derive an implicit function from these operations

Boolean Operations

Actually, Boolean operations with implicit functions are simple:

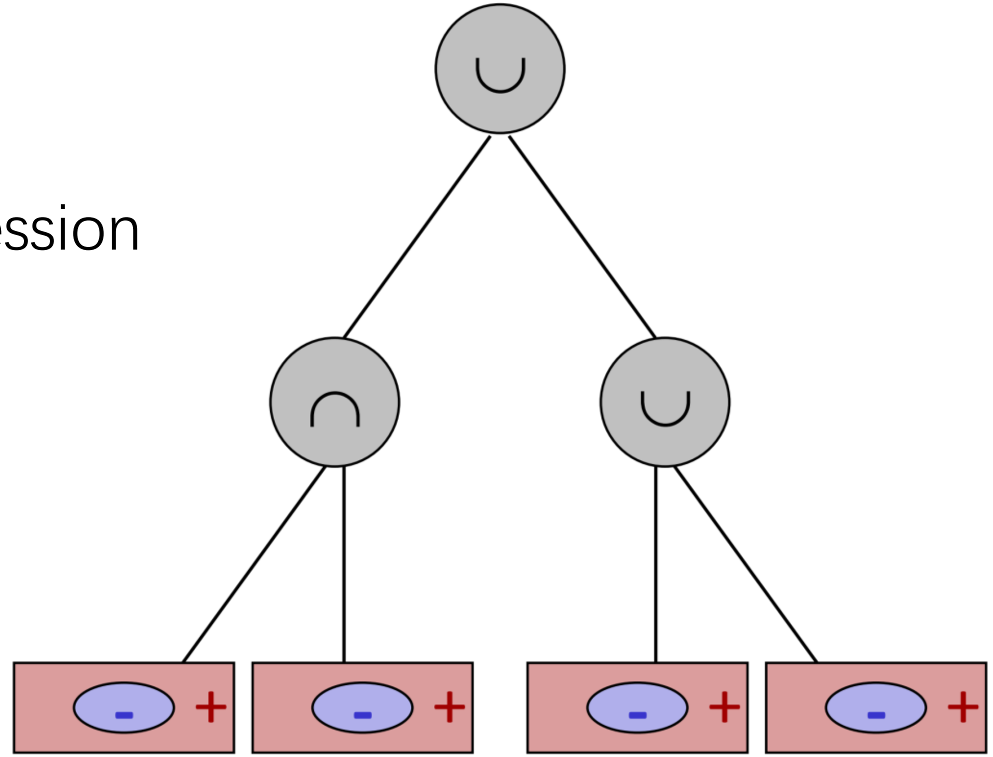
- Given two signed implicit functions (negative inside) f_A, f_B for objects A, B
- The Boolean combinations are given by:
  - Union $A \cup B$: $f_{A \cup B} = \min(f_A, f_B)$
  - Intersection $A \cap B$: $f_{A \cap B} = \max(f_A, f_B)$
  - Complement $\neg A$: $f_{\neg A} = -f_A$
  - Difference $A \setminus B$: $f_{A \setminus B} = \min(f_A, -f_B)$



Hierarchical Modeling

This can be models as a CSG tree (constructive solid geometry):

- Leaf nodes are signed distance functions
- Inner nodes are Boolean operations
- Evaluation translates to an arithmetic expression
- Other operations:
  - Deformation (apply vector field)
  - Blending (combine surface smoothly)

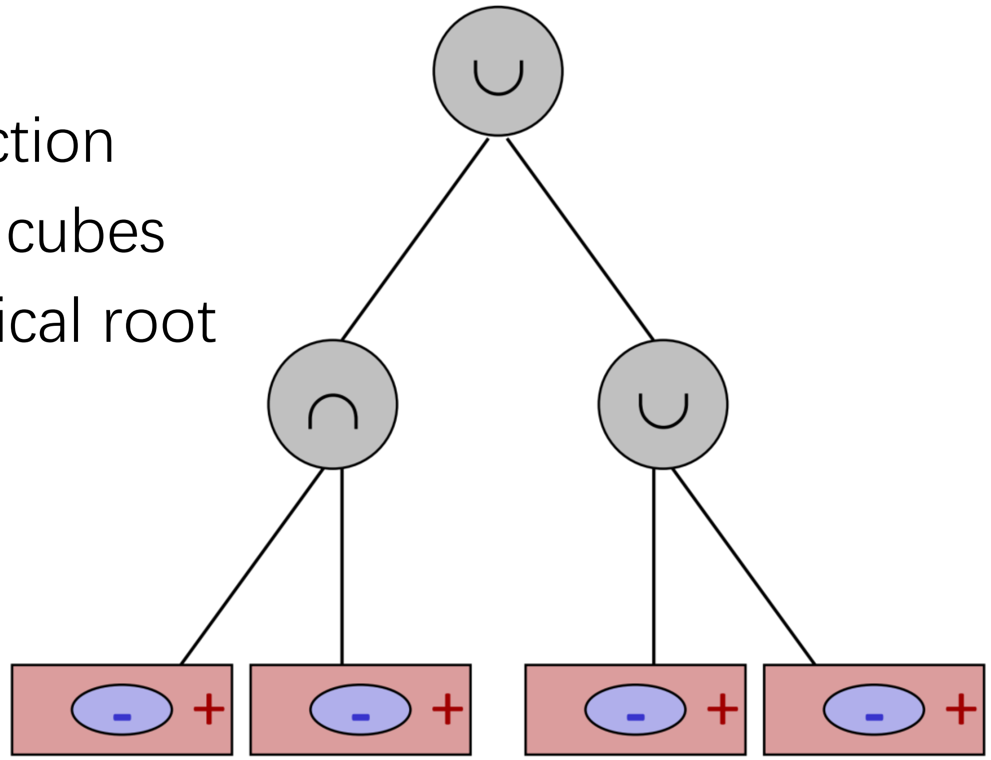


Hierarchical Modeling

Rendering CSG hierarchies:

- Rendering is simple
- We get one compound signed implicit function
- We can extract the surface using marching cubes
- We can raytrace the surface using a numerical root finding algorithm
  - For example:

    Newton scheme with voxel-based initialization



Implicit Surfaces

Data Fitting

Constructing Implicit Surfaces

Question: How to construct implicit surfaces?

- Basic primitives: Spheres, boxes etc … are (almost) trivial.
- We can construct implicit spline schemes by using 3D tensor product (or tetrahedral) constructions of 3D Bezier or B-Spline functions
- Another option: Variational modeling
- In this chapter of this lecture: Fitting to data

Data Fitting

Data Fitting Problem:

- We are given a set of points
- We want to find an implicit surface that interpolates or approximates these points
- This problem is ill-defined
- We need additional assumptions to make it well-defined
- We will look at three variants:
  - Hoppe’s method / plane blending
  - Thin-plate spline data matching
  - MPU Implicits (multi-level partition of unity implicits)

Plane Blending Method

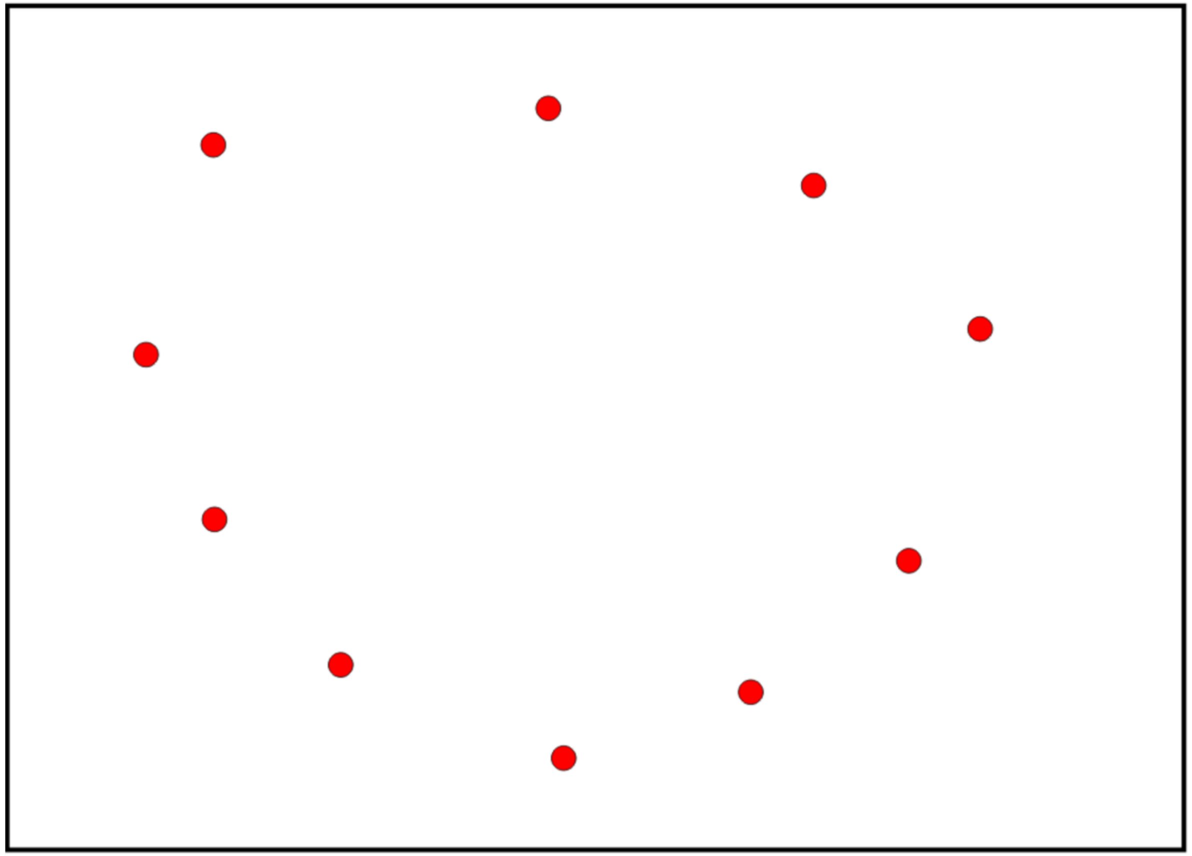
Initial data

Estimate normal

Signed distance func.

Marching cubes

Final mesh



Plane Blending Method

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Estimate normal

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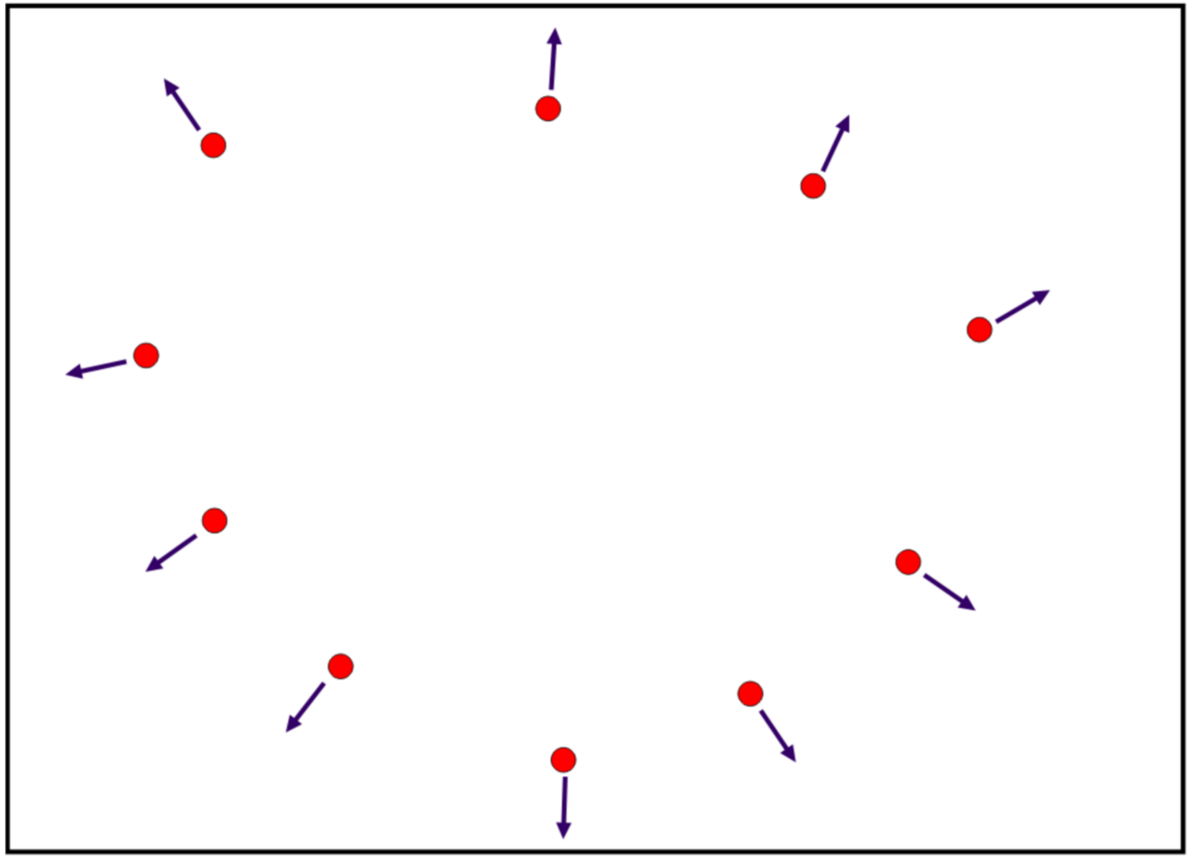
Marching cubes

Final mesh



unoriented normal:
total least squares plane fit (PCA)
in a k-nearest neighbors neighborhood

Plane Blending Method

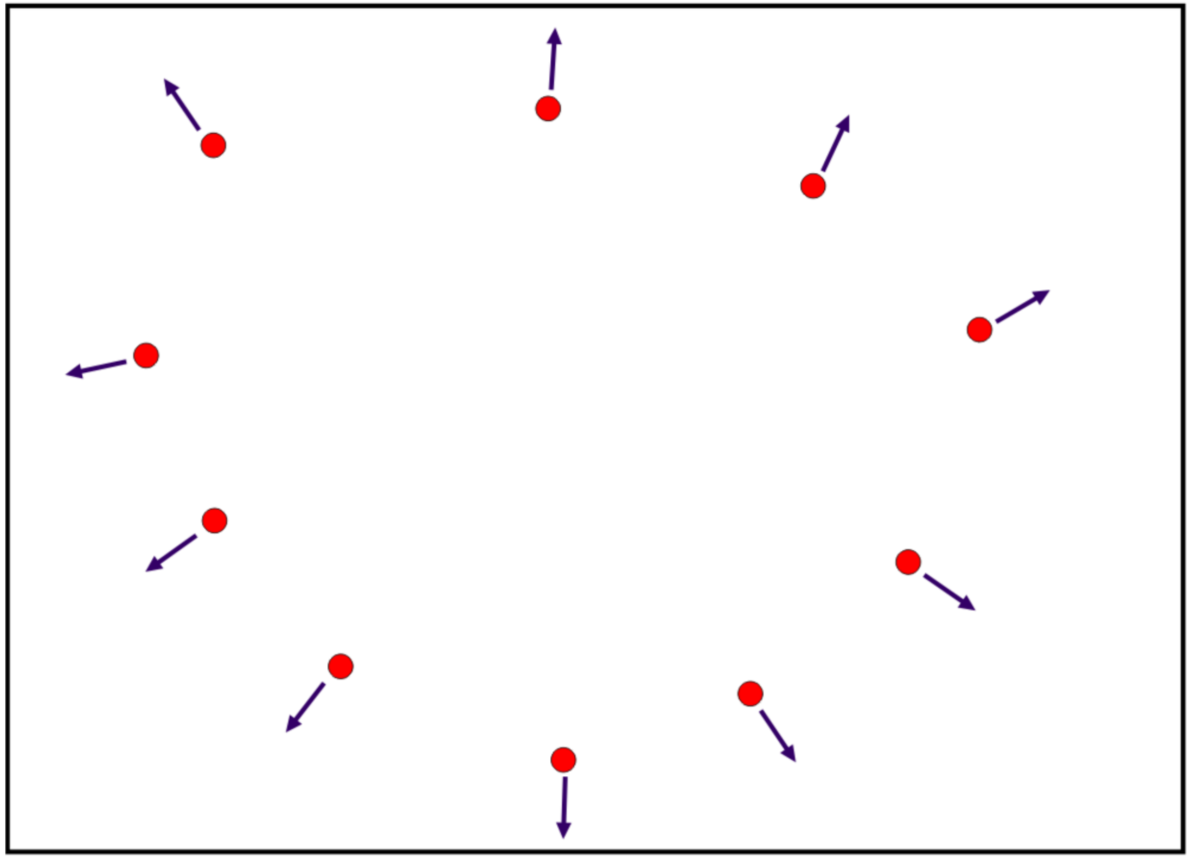
Initial data

Estimate normal

Signed distance func.

Marching cubes

Final mesh



consistent orientation:
region growing, flip normal if angle $> 180°$
pick most similar normal next in each step

Plane Blending Method

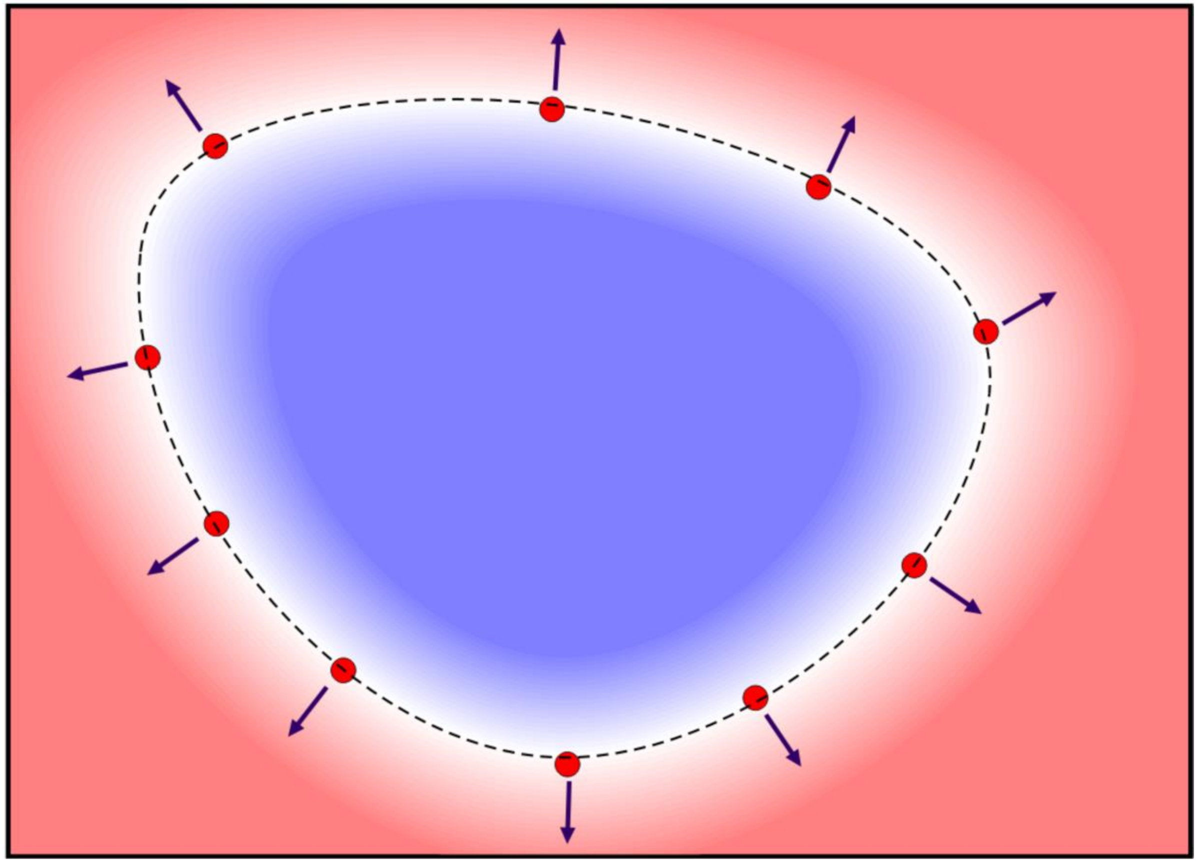
Initial data

Estimate normal

Signed distance func.

Marching cubes

Final mesh



consistent orientation:
blend between signed distance functions of planes associated with each point

Plane Blending Method

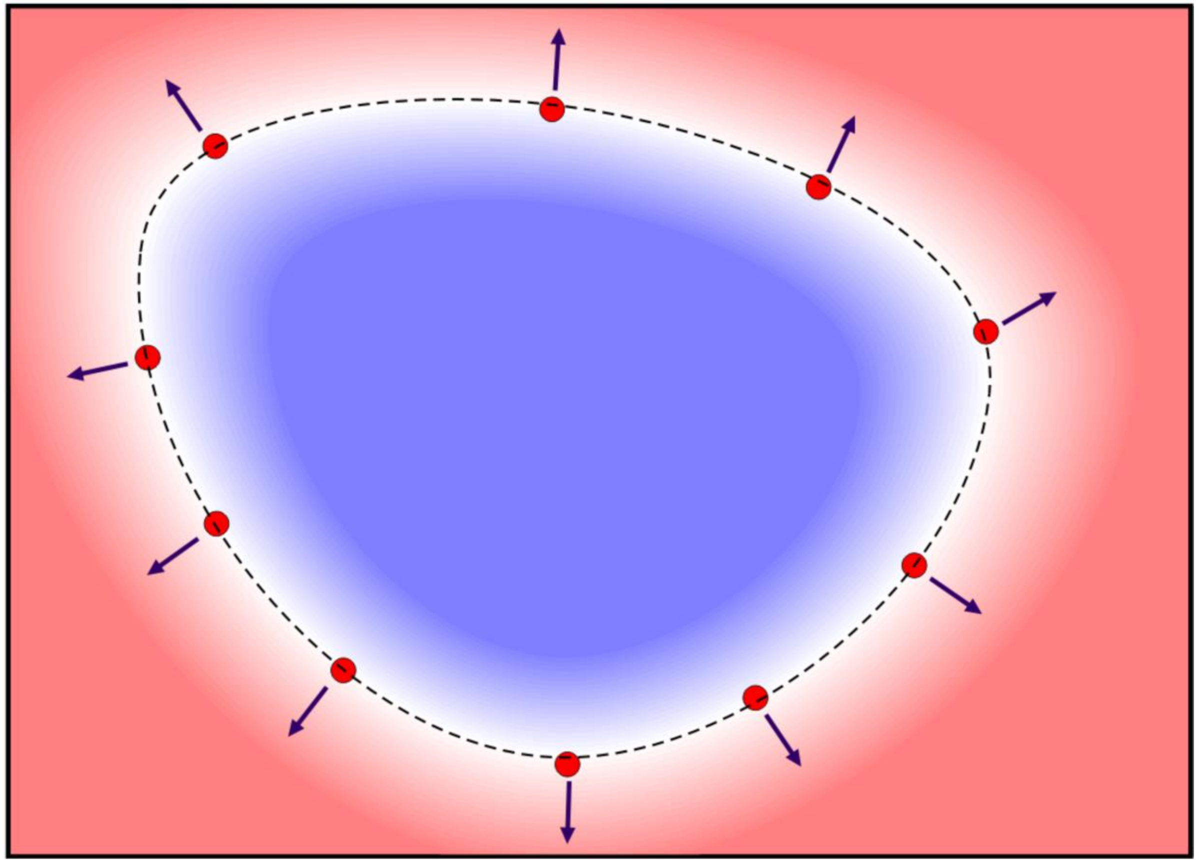
Initial data

Estimate normal

Signed distance func.

Marching cubes

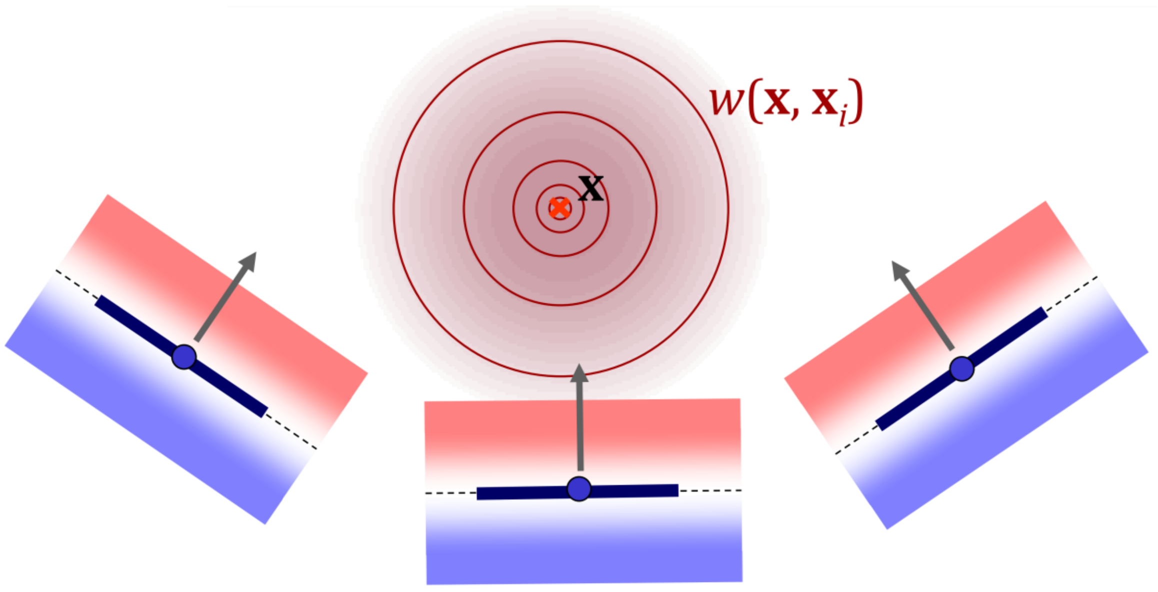
Final mesh



signed distance function:
plane blending (next slide)

Normal Constraints

Basic Idea:

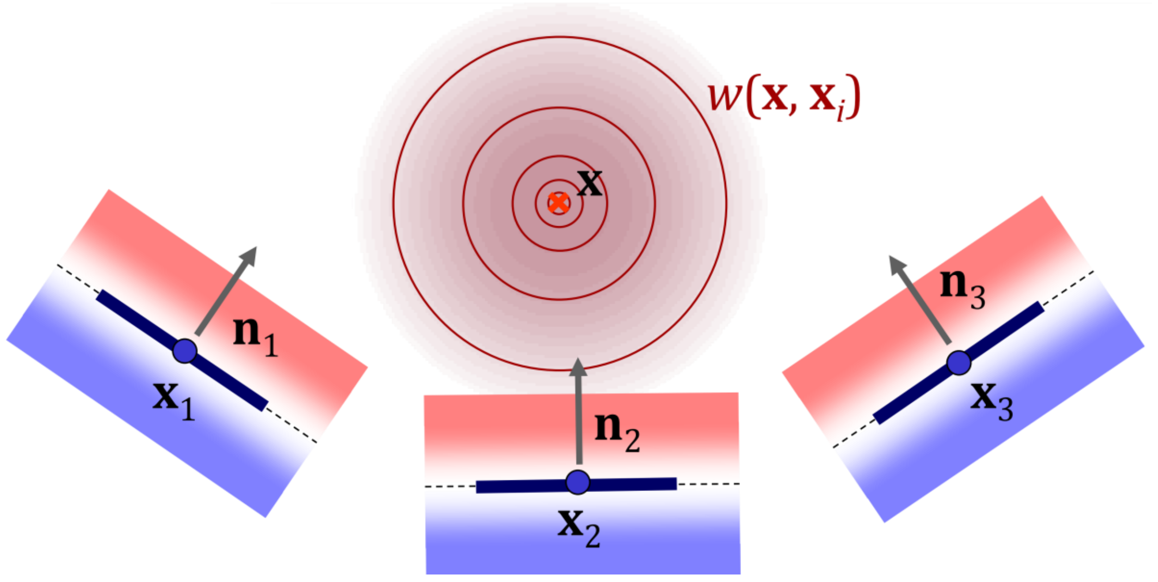


- Each point defines an oriented plane and a signed distance function
- To obtain a composite distance field in space:

  Blend these distance functions with weights from a kernel (Gaussian, or uniform B-Spline)

Normal Constraints

Basic Idea:



$$f(\boldsymbol{x}) = \frac{\sum_{i=1}^{n} \langle \boldsymbol{n}_i, \boldsymbol{x} - \boldsymbol{x}_i \rangle w(\|\boldsymbol{x} - \boldsymbol{x}_i\|)}{\sum_{i=1}^{n} w(\|\boldsymbol{x} - \boldsymbol{x}_i\|)}$$ (partition of unity weights)

Plane Blending Method

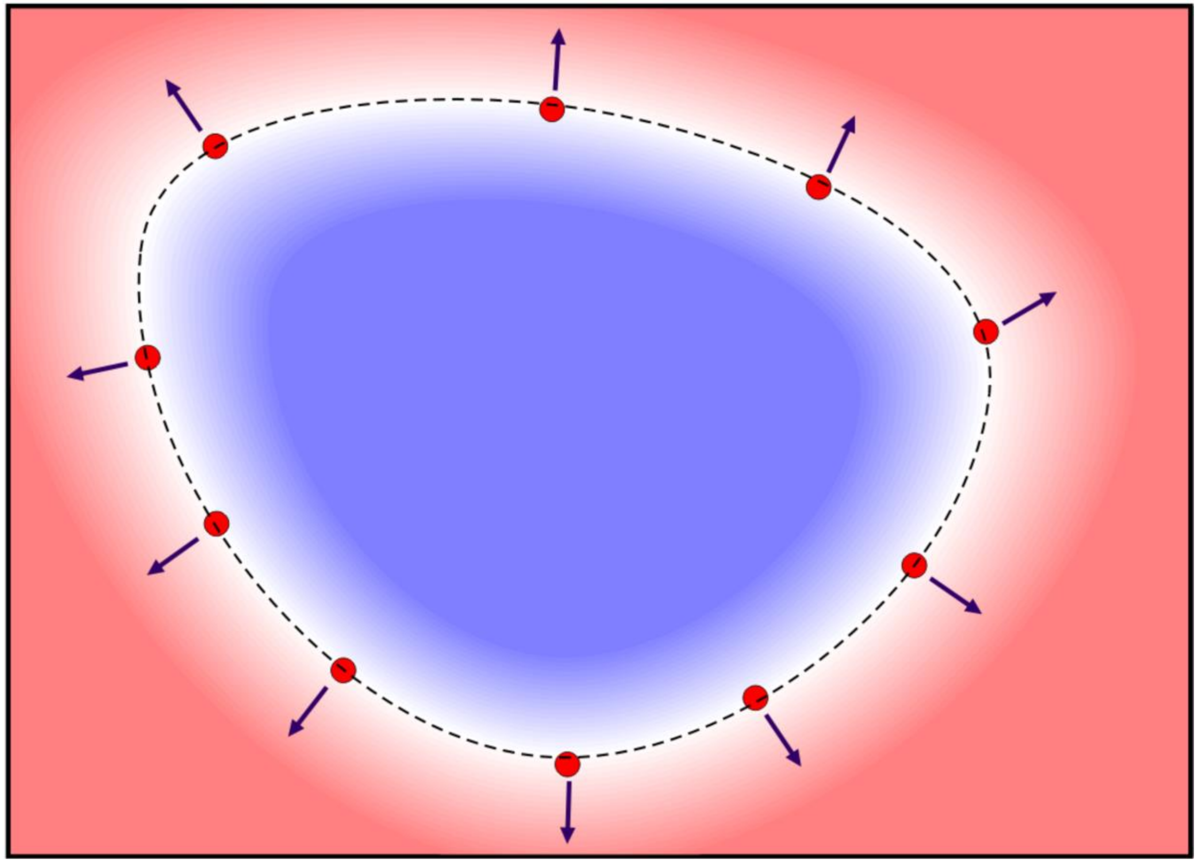
Initial data

Estimate normal

Signed distance func.

Marching cubes

Final mesh



Plane Blending Method

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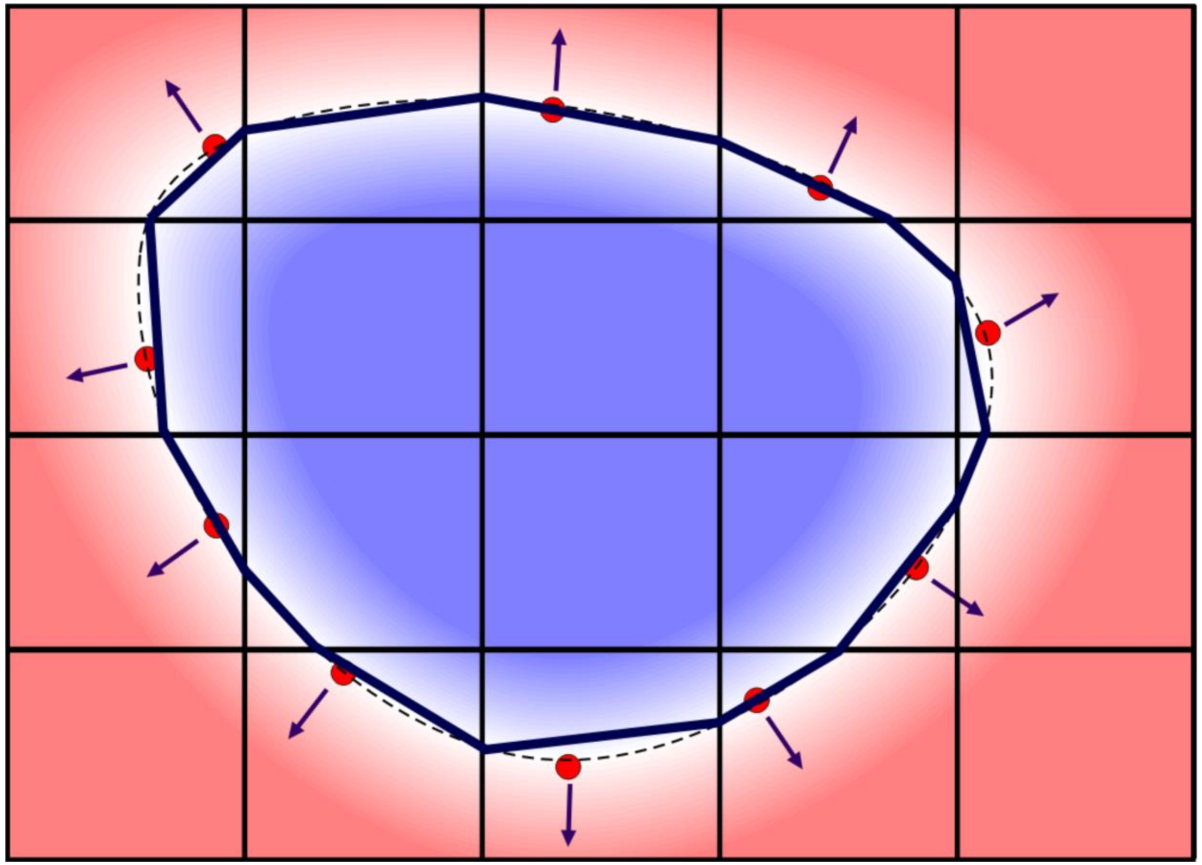

Plane Blending Method

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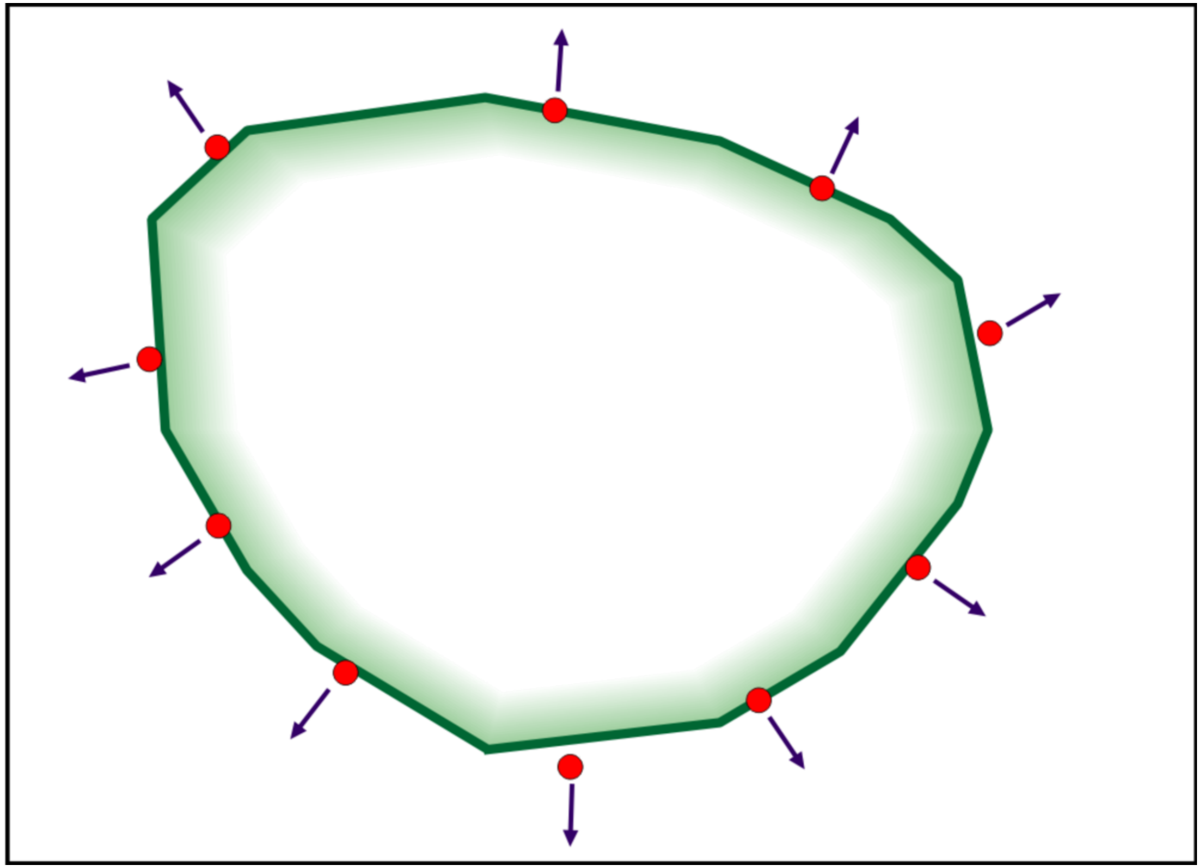
Estimate normal

Signed distance func.

Marching cubes

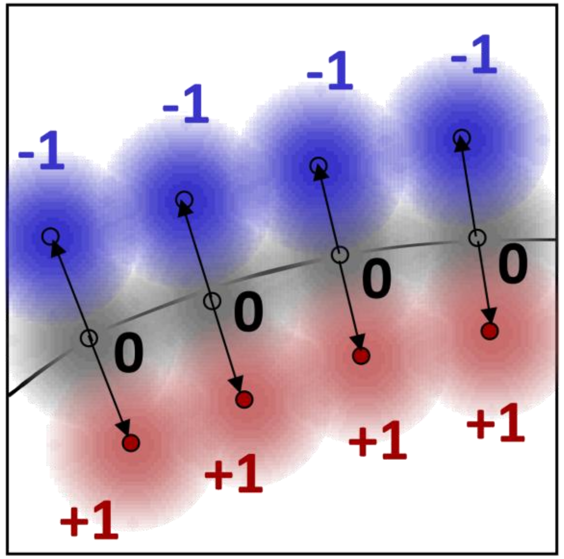
Final mesh

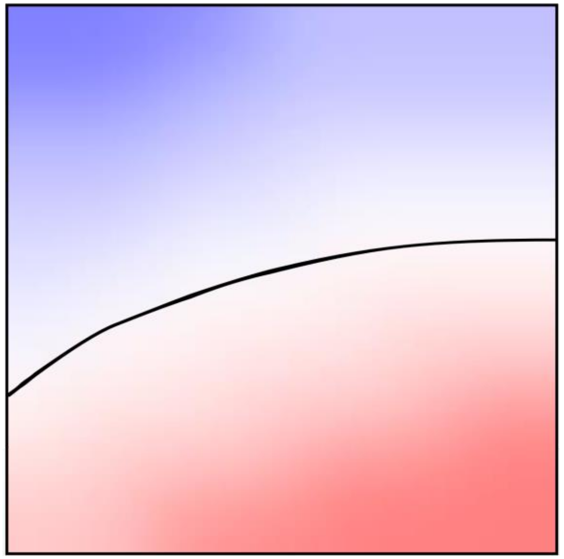


Thin-Plate Spline Data Matching

Agenda:

- Use radial basis functions
- Use a globally supported basis that guarantees smoothness
- Place radial basis functions at the input points
- Place two more in normal and negative normal direction
- Prescribe values +1,0,-1
- Solve a linear system to meet these constraints



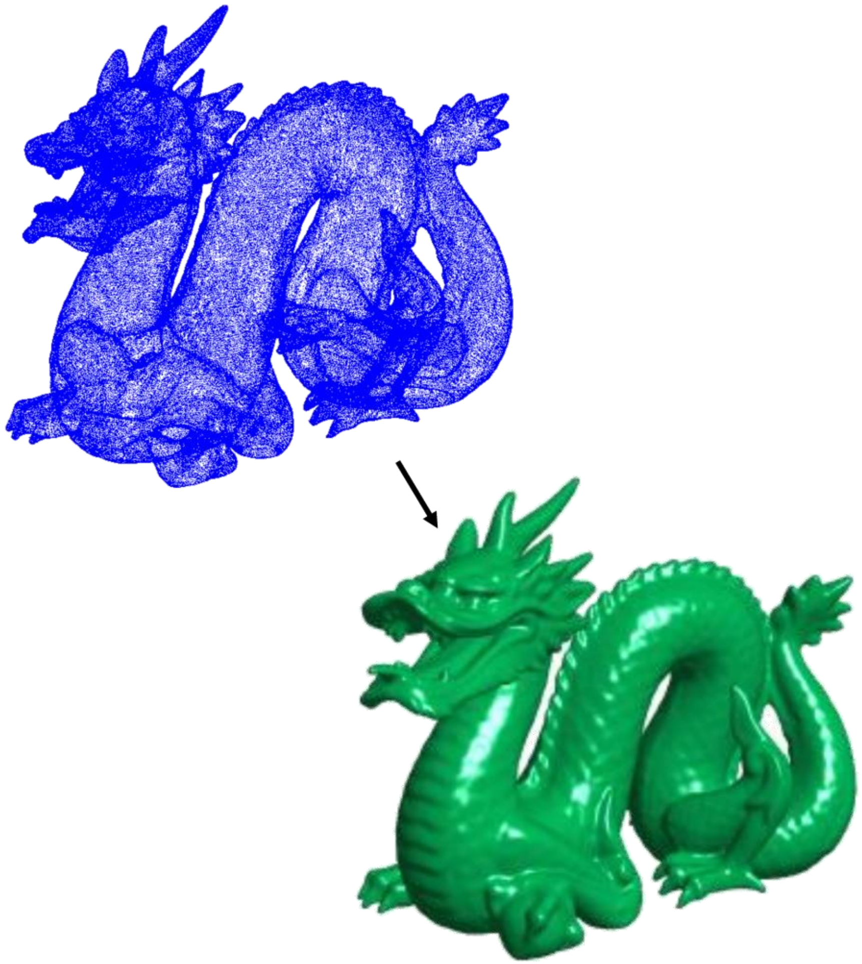


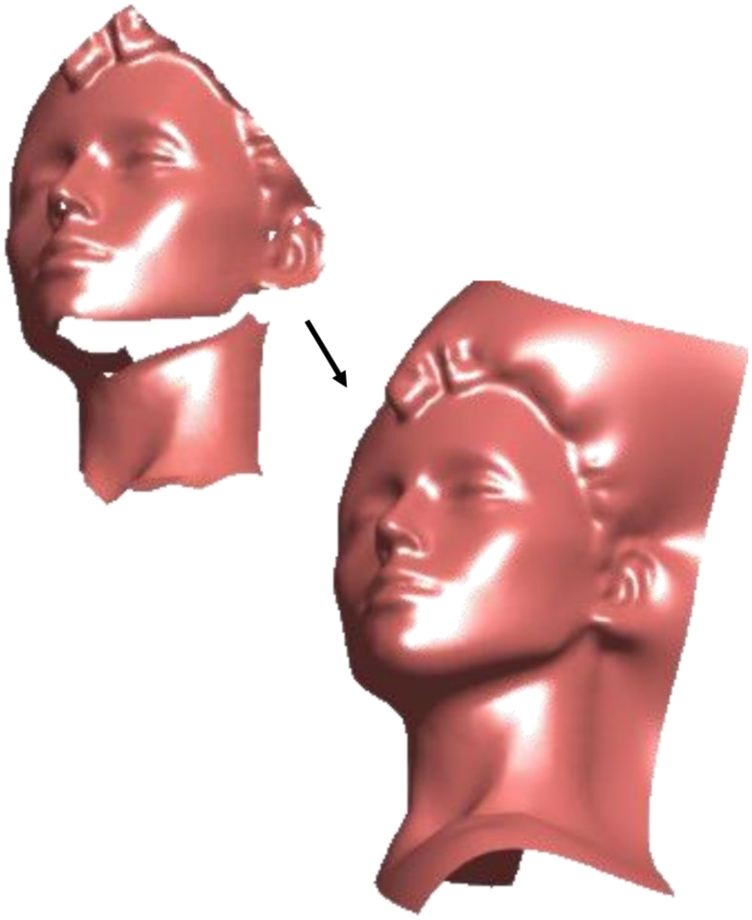
Types of Radial Basis Functions

Typical choices for radial basis functions:

- Globally supported functions:
  - Thin plate spline basis functions:
    $\|x - x_0\|^2 \ln\|x - x_0\|$ (2D), $\|x - x_0\|^3$ (3D)
  - These functions guarantee minimal integral second derivatives
- Problem: evaluation
  - Every basis function interacts with each other one
  - This creates a dense $n \times n$ linear system
  - One can use a fast multi pole method that clusters far away nodes in bigger octree boxes
  - This gives $O(\log n)$ interactions per particle, overall $O(n \log n)$ interactions

Examples





Carr et al. Reconstruction and representation of 3D objects with Radial Basis Functions, SIGGRAPH 2001

Alternative

Alternative:

- Use locally supported basis functions (e.g. B-Splines)
- Employ an additional regularization term to make the solution smooth.
- Optimize the energy function

$$E(\lambda) = \sum_{i=1}^{n} f(\boldsymbol{x}_i)^2 + \mu \int_{\Omega} \left(\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{2\partial^2}{\partial x \partial y} + \frac{2\partial^2}{\partial y \partial z} + \frac{2\partial^2}{\partial x \partial z} \right] f(\boldsymbol{x}) \right)^2 d\boldsymbol{x}$$

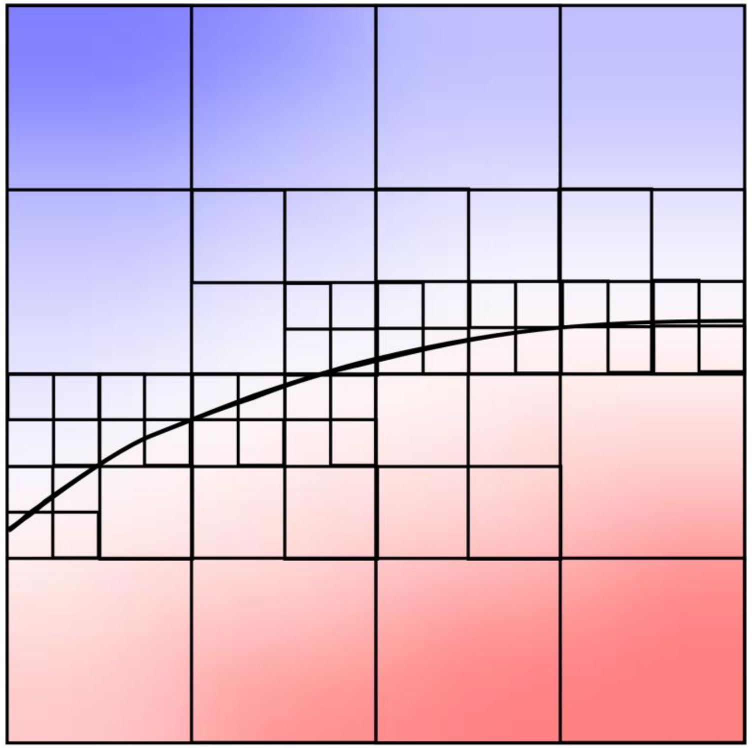
with $f(\boldsymbol{x}) = \sum_{i=1}^{m} \lambda_i b\left(\boldsymbol{x} - \boldsymbol{x}_j\right)$

- The critical point is the solution to a linear system

MPU Implicits

Multi-level partition of unity implicits:

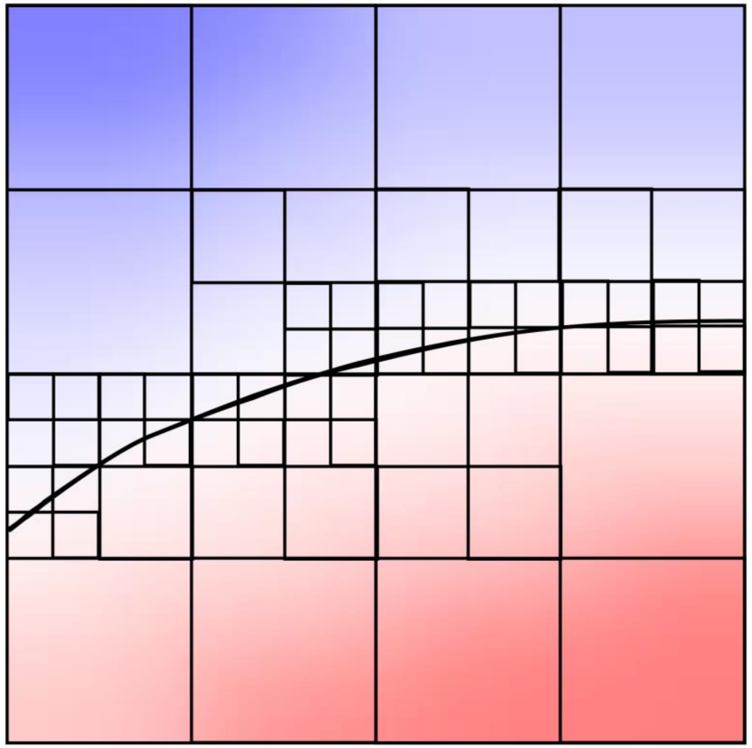
- Hierarchical implicit function approximation
  - Given: data points with normal
  - Computes: hierarchical approximation of the signed distance function



MPU Implicits

Multi-level partition of unity implicits:

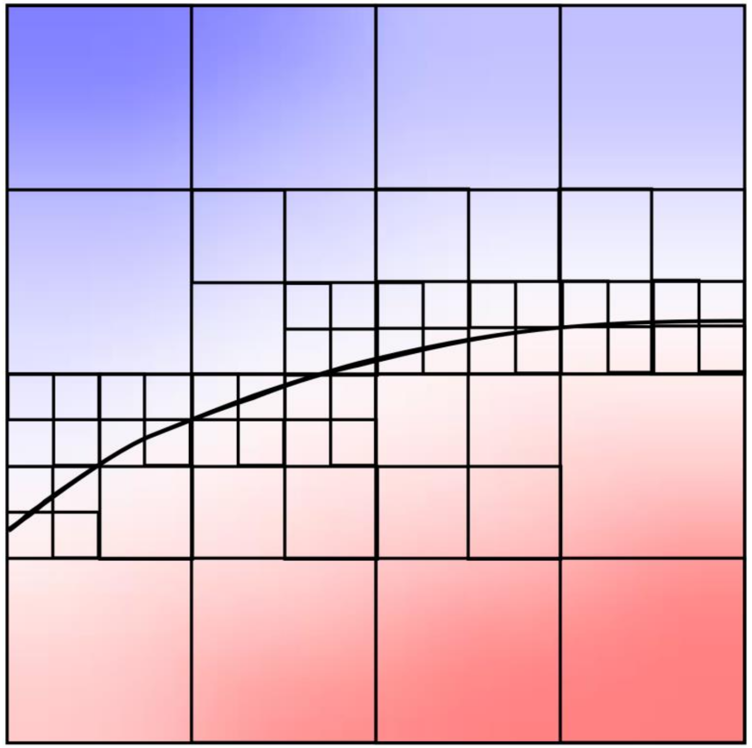
- Octree decomposition of space
- In each octree cell, fit an implicit quadratic function to points
  - $f(\boldsymbol{x}_i) = 0$ at data points
  - Additional normal constraints
- Stopping criterion:
  - Sufficient approximation accuracy (evaluate f at data points to calculate distance)
  - At least 15 points per cell.



MPU Implicits

Multi-level partition of unity implicits:

- This gives an adaptive grid of local implicit function approximations
- Problem: How to define a global implicit function?
- Idea: Just blend between local approximants using a windowing function

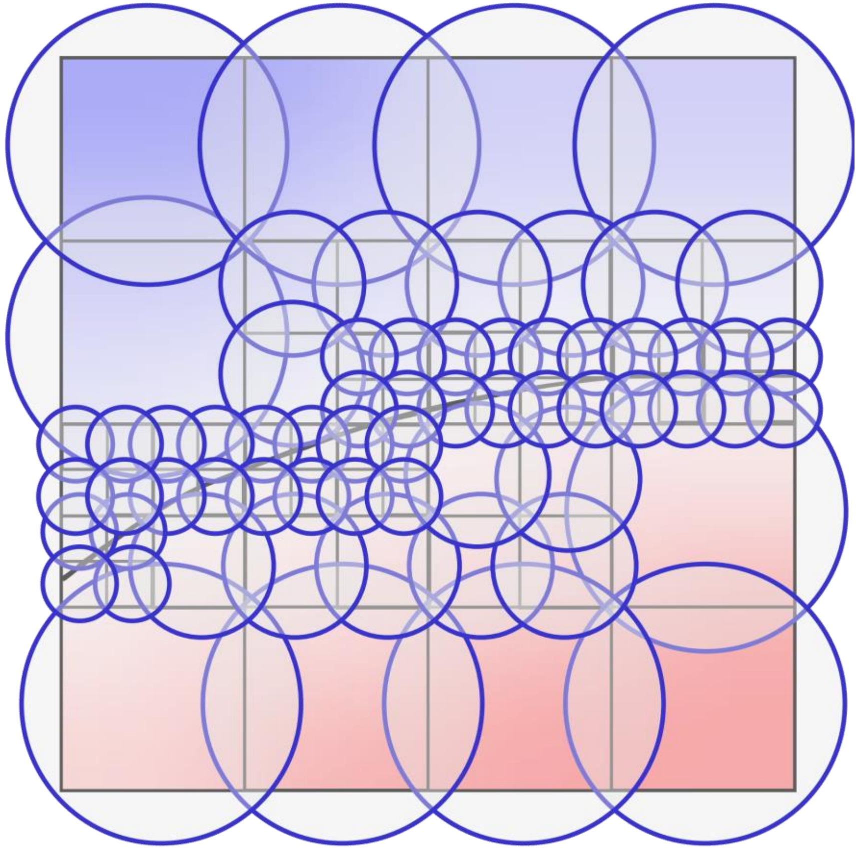


MPU Implicits

Multi-level partition of unity implicits:

- Windowing function:
  - Use smooth windowing function w
    - B-splines / normal distribution
    - Original formulation: quadratic tensor product B-spline function, support = $1.5 \times$ cell diagonal
  - Renormalize to form partition of unity:

$$f(\boldsymbol{x}) = \frac{\sum_{i=1}^{n} w(\boldsymbol{x} - \boldsymbol{x}_i) f_i(\boldsymbol{x})}{\sum_{i=1}^{n} w(\boldsymbol{x} - \boldsymbol{x}_i)}$$

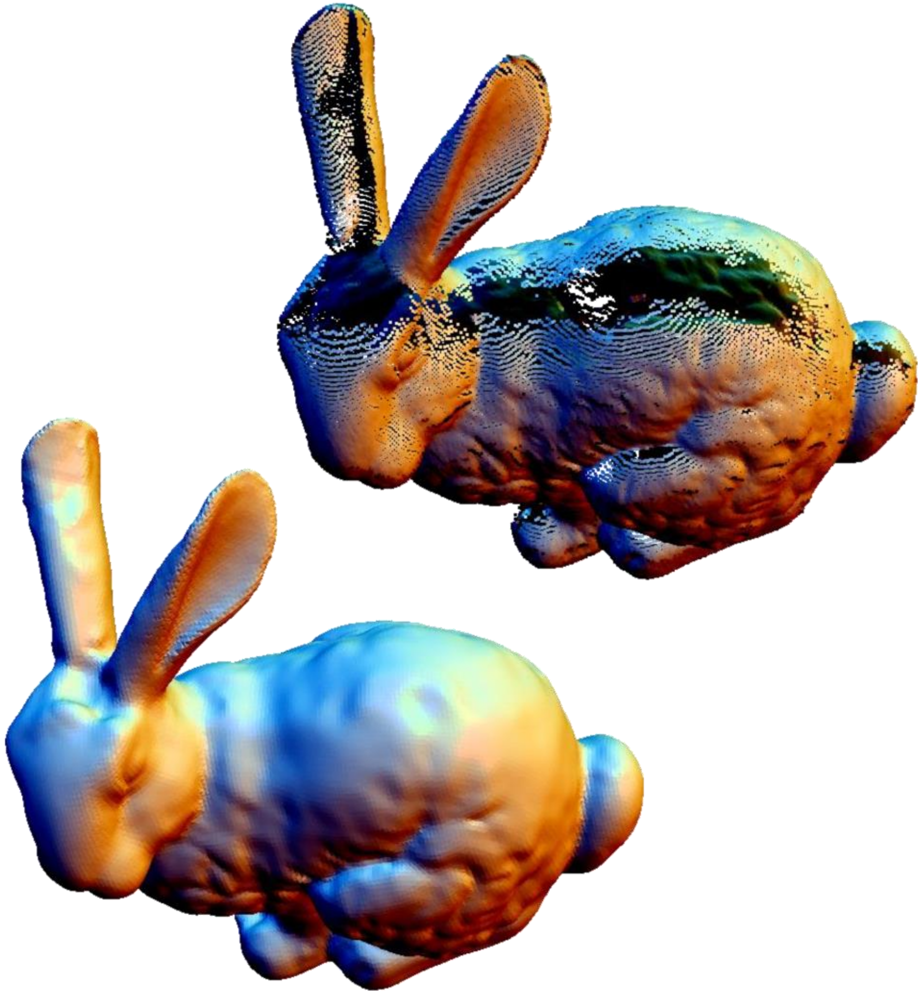


MPU Implicits

Multi-level partition of unity implicits:

- Sharp features:
  - If a leaf cell with a few points has strongly varying normal, this might be a sharp feature.
  - Multiple functions can be fitted to parts of the data
  - Boolean operations to obtain composite distance field

Examples



Ohtake et al. Multi-level Partition of Unity Implicits, SIGGRAPH 2003