Computer Aided Geometric Design
Fall Semester 2024

Interpolation & Approximation

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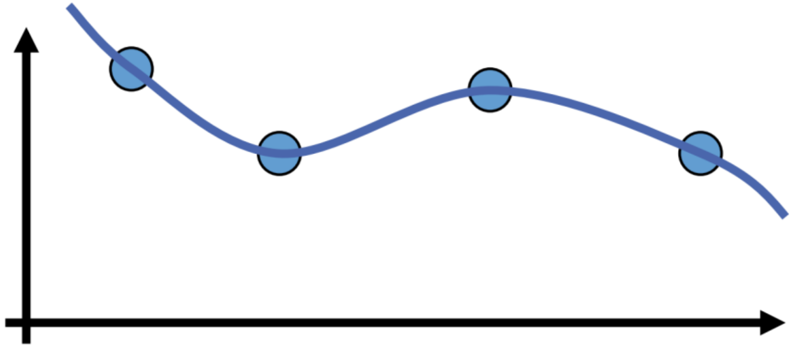
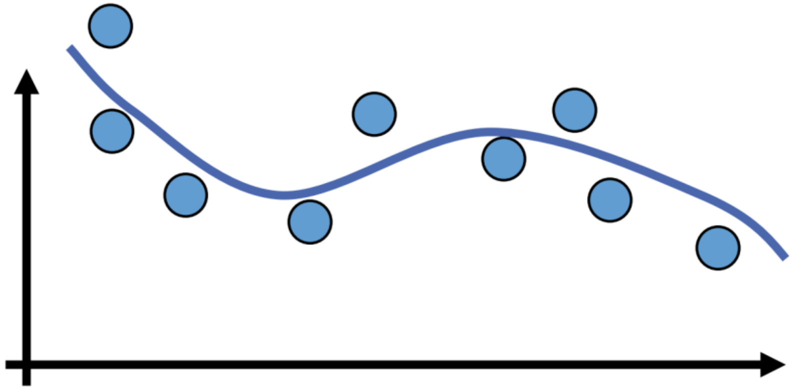


Interpolation



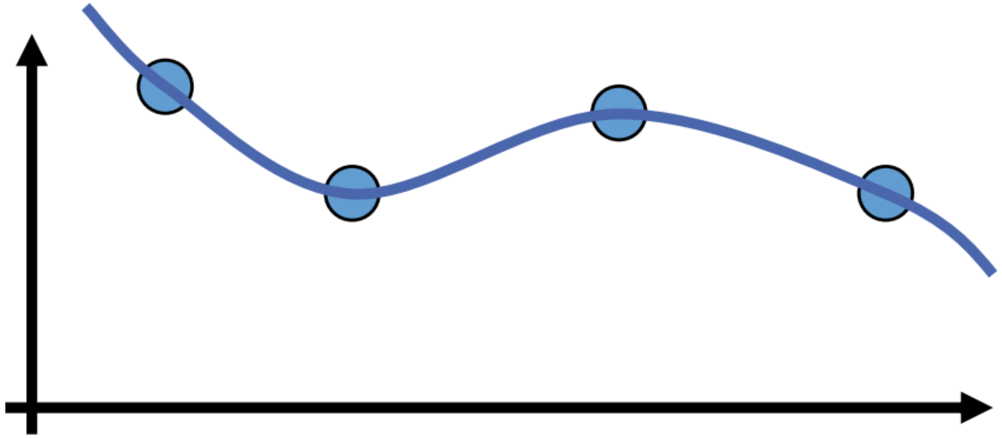
Approximation

Interpolation

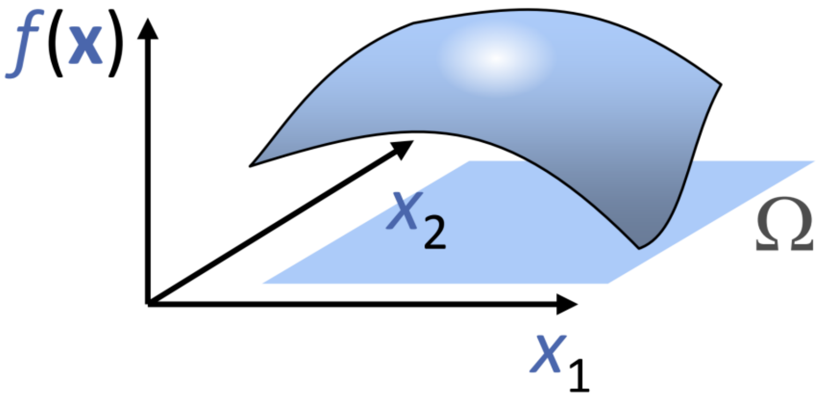
General interpolation and polynomial interpolation

Interpolation Problem

- Our first attempt at modeling smooth objects:
  - Given a set of points along a curve or surface
  - Choose basis functions that span a suitable function space
    - Smooth basis functions
    - Any linear combination will be smooth, too
  - Find a linear combination such that the curve/surface interpolates the given points



General Formulation



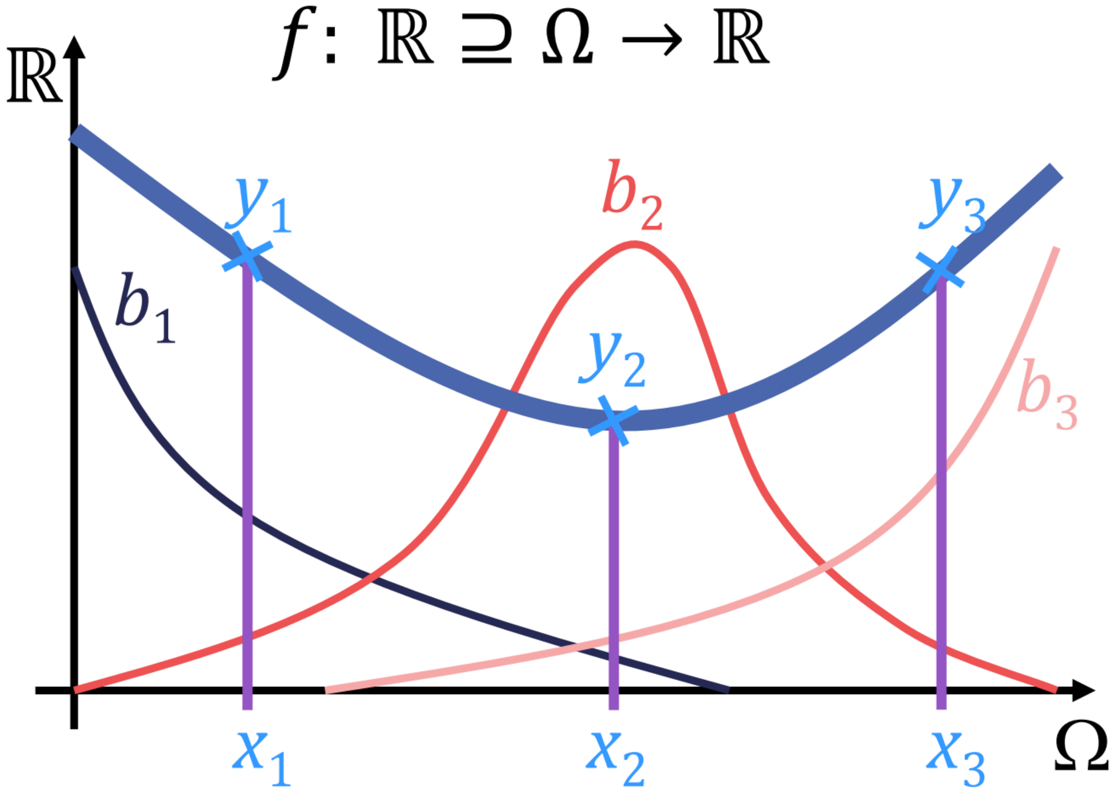
- Settings
  - Domain $\Omega \subseteq \mathbb{R}^d$, mapping to $\mathbb{R}$
  - Looking for a function $f : \Omega \to \mathbb{R}$
  - Basis set： $B = \{b_1, \dots, b_n\}, b_i : \Omega \to \mathbb{R}$
  - Represent f as linear combination$n$of basis functions

$$f_\lambda(x) = \sum_{k=0}^{n} \lambda_i b_i(x)$$

i.e. f is just determined by $\lambda = \begin{pmatrix} \lambda_1 \\ \dots \\ \lambda_n \end{pmatrix}$

  - Function values: $\{(x_1, y_1), \dots, (x_n, y_n)\}, (x_i, y_i) \in \mathbb{R}^d \times \mathbb{R}$
  - We want to find λ such that: $f_\lambda(x_i) = y_i$ for all i

Illustration



$$f\colon \mathbb{R} \supseteq \Omega \to \mathbb{R}$$

1D Example

Solving the Interpolation Problem

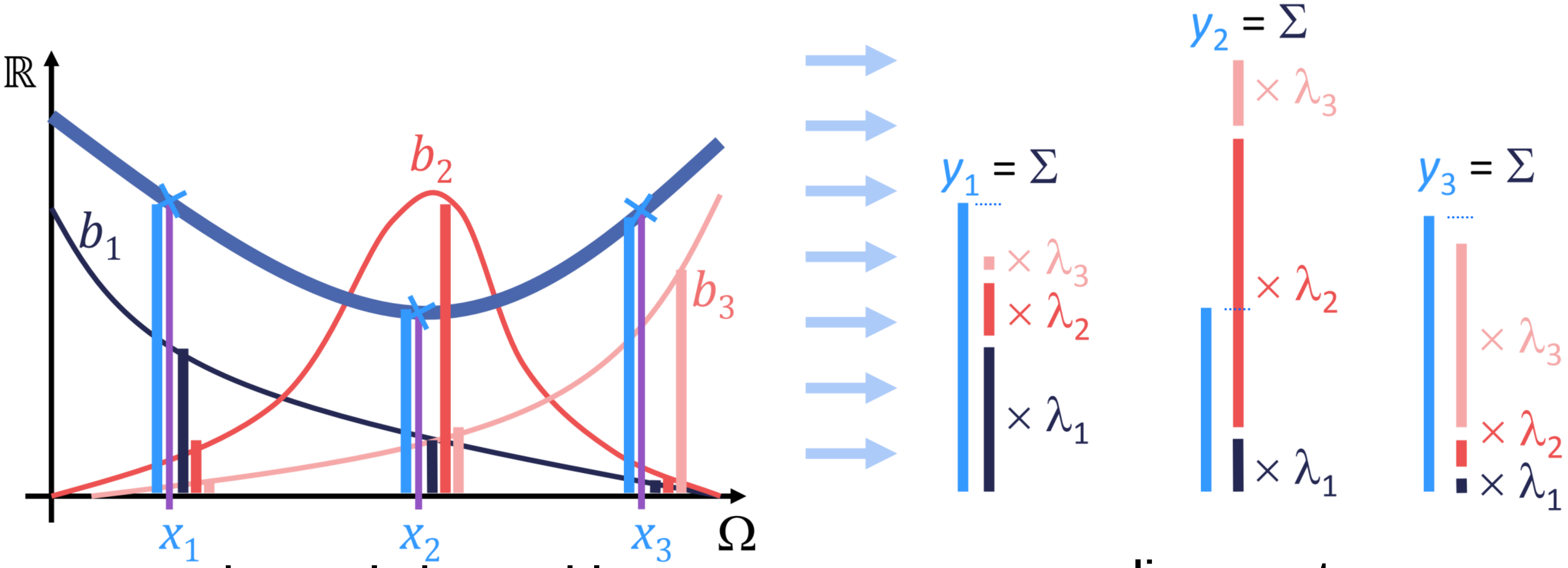
- Solution: linear system of equations
  - Evaluate basis functions at points x_i:

$$\forall i \in \{1, \dots, n\}: \sum_{i=1}^{n} \lambda_i b_i(x_i) = y_i$$

  - Matrix form:

$$\begin{pmatrix} b_1(x_1) & \cdots & b_n(x_1) \\ \vdots & \ddots & \vdots \\ b_1(x_n) & \cdots & b_n(x_n) \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

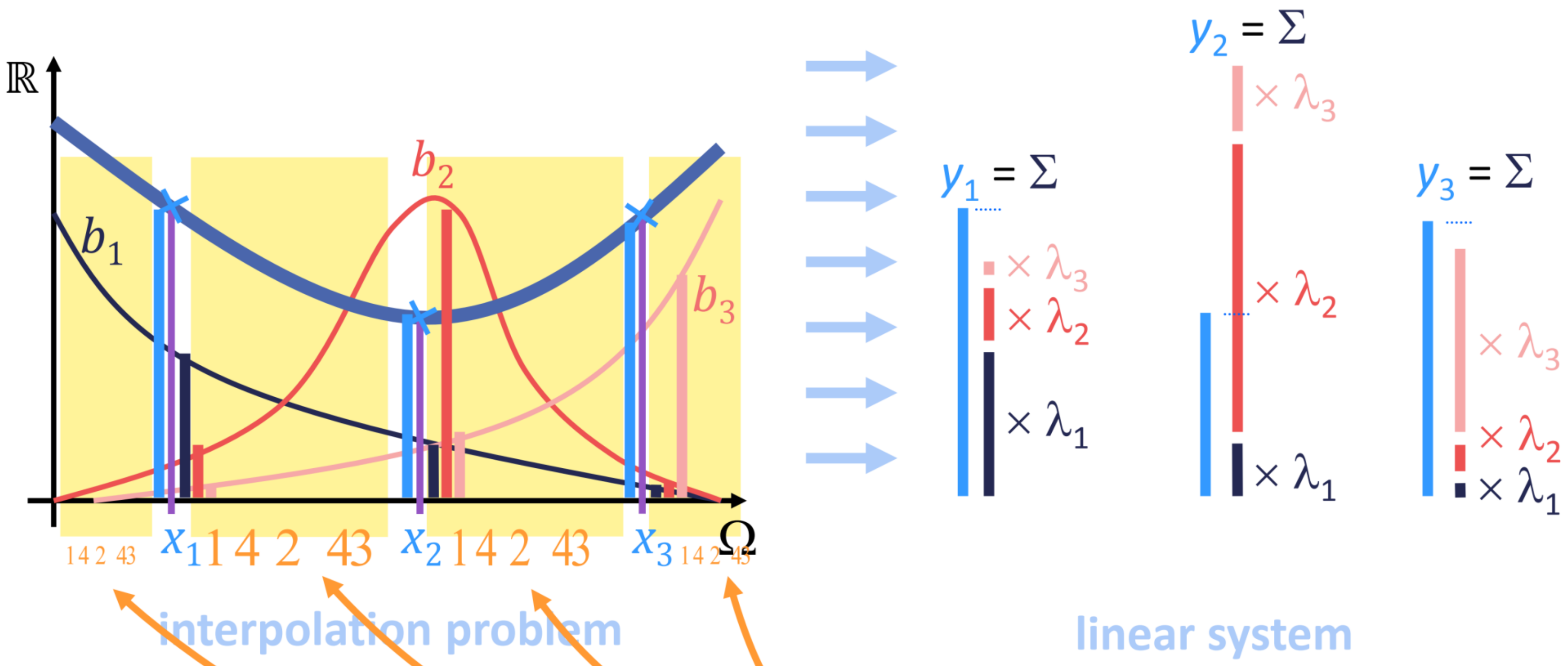
Illustration



interpolation problem

linear system

Illustration



anything in between does not matter
(determined by basis only)

Example

Polynomial Interpolation

- Monomial basis $B = \{1, x, x^2, x^3, \dots, x^{n-1}\}$
- Linear system to solve

$$\begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \dots \\ \lambda_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{pmatrix}$$

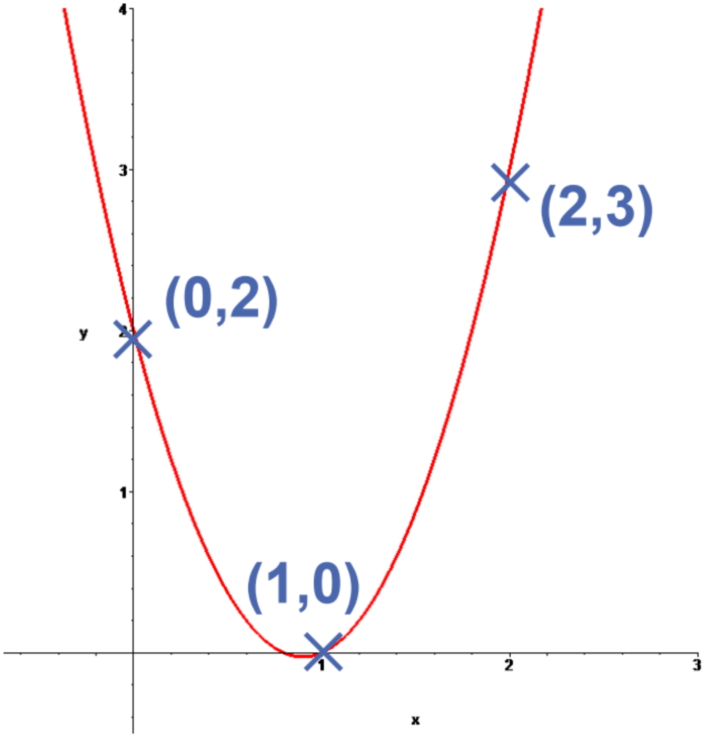
Vandermonde Matrix

Example with Numbers

- Quadratic monomial basis $B = \{1, x, x^2\}$
- Function values: $\{(0,2),(1,0),(2,3)\}\quad[(x,y)]$
- Linear system to solve:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

- Result： $\lambda_1 = 2,\ \lambda_2 = -\frac{9}{2},\ \lambda_3 = \frac{5}{2}$



Problems with interpolation

- The arising system matrix is generally dense

- Depending on the choice of the basis, the matrix can be ill-conditioned (difficult to invert/solve)

ill-conditioning example

- Consider the system
  - Clearly (1,1) is a solution

$$\begin{aligned} x_1 + 0.5x_2 &= 1.5 \\ 0.667x_1 + 0.333x_2 &= 1 \end{aligned}$$

- Now perturb the right hand side of the second equation by 0.001 (order 10−3)
  - The solution is then (0.000,3.000) (order 1)

$$\begin{aligned} x_1 + 0.5x_2 &= 1.5 \\ 0.667x_1 + 0.333x_2 &= \underset{\uparrow}{0.999} \end{aligned}$$

- Now consider perturbing the coefficient
  - The solution (2.000, -1.000)

$$\begin{aligned} x_1 + 0.5x_2 &= 1.5 \\ 0.667x_1 + \underset{\uparrow}{0.334}x_2 &= 1 \end{aligned}$$

ill-conditioning

- Small change in the input data induces relatively large change in the output (solution)

- Thinking of equations as lines (hyperplanes), when the system is ill-conditioned the lines become almost parallel
  - Obtaining a solution (intersection) becomes difficult and imprecise

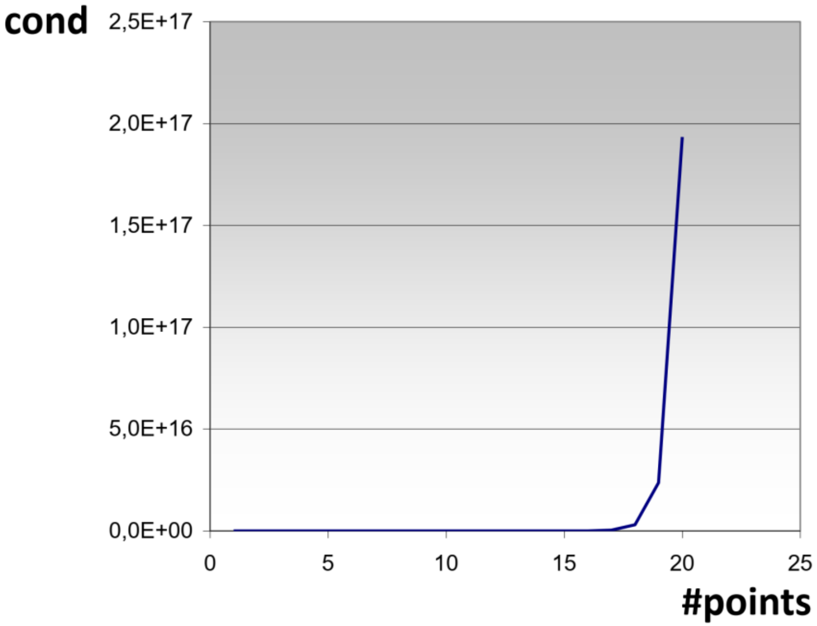
Condition number

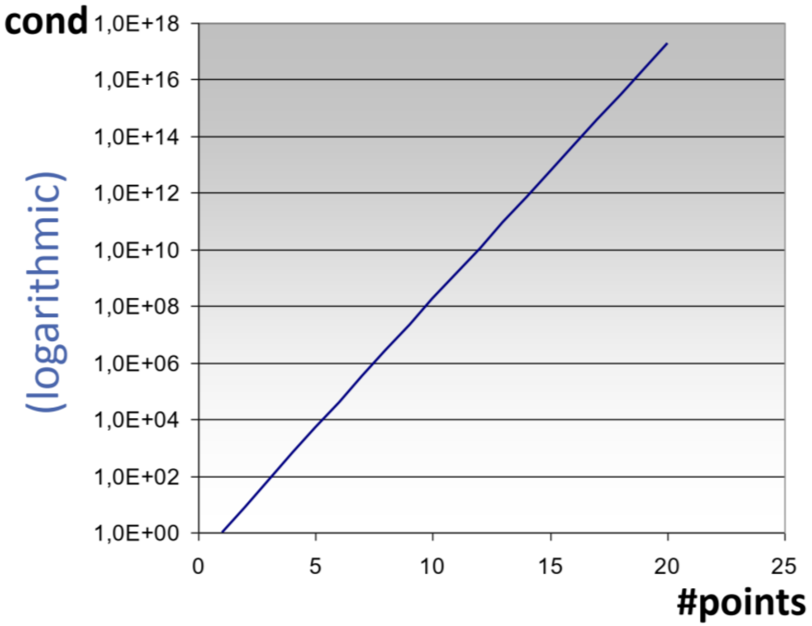
$$\kappa_2(A) = \frac{\max\limits_{x\neq 0} \frac{||Ax||}{||x||}}{\min\limits_{x\neq 0} \frac{||Ax||}{||x||}}$$

- Can be regarded as the ratio of highest eigenvalues / lowest eigenvalue
- When the condition number is high it reflects there is too much interdependence between the elements of the basis

Condition Number…

- The interpolation problem is ill conditioned:
- For equidistant x_i, the condition number of the Vandermode matrix grows exponentially with n
  - (maximum degree+1 = number of points to interpolate)

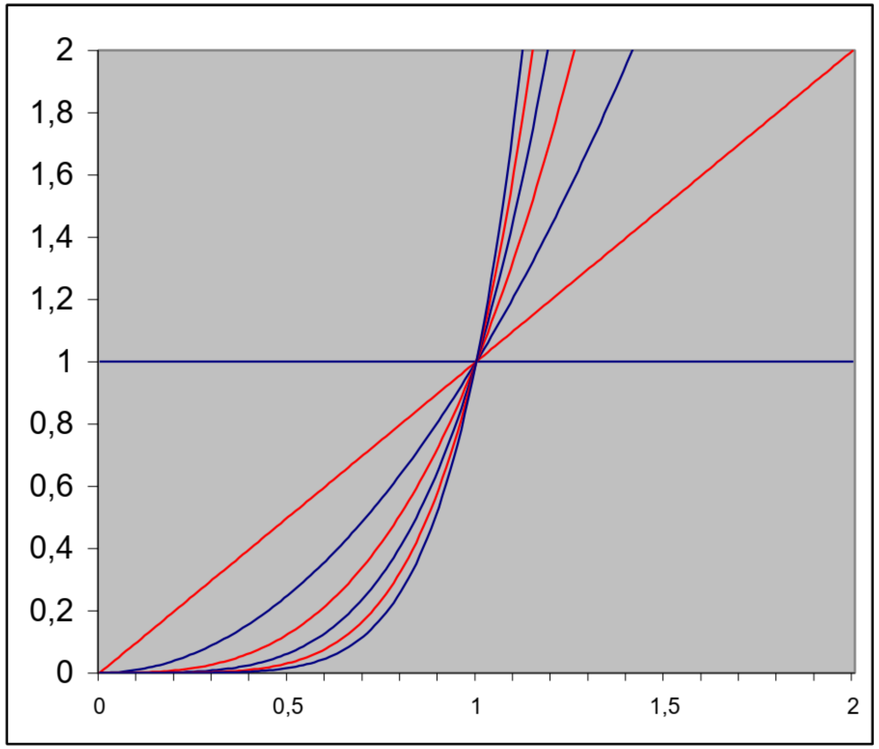




Why is that??

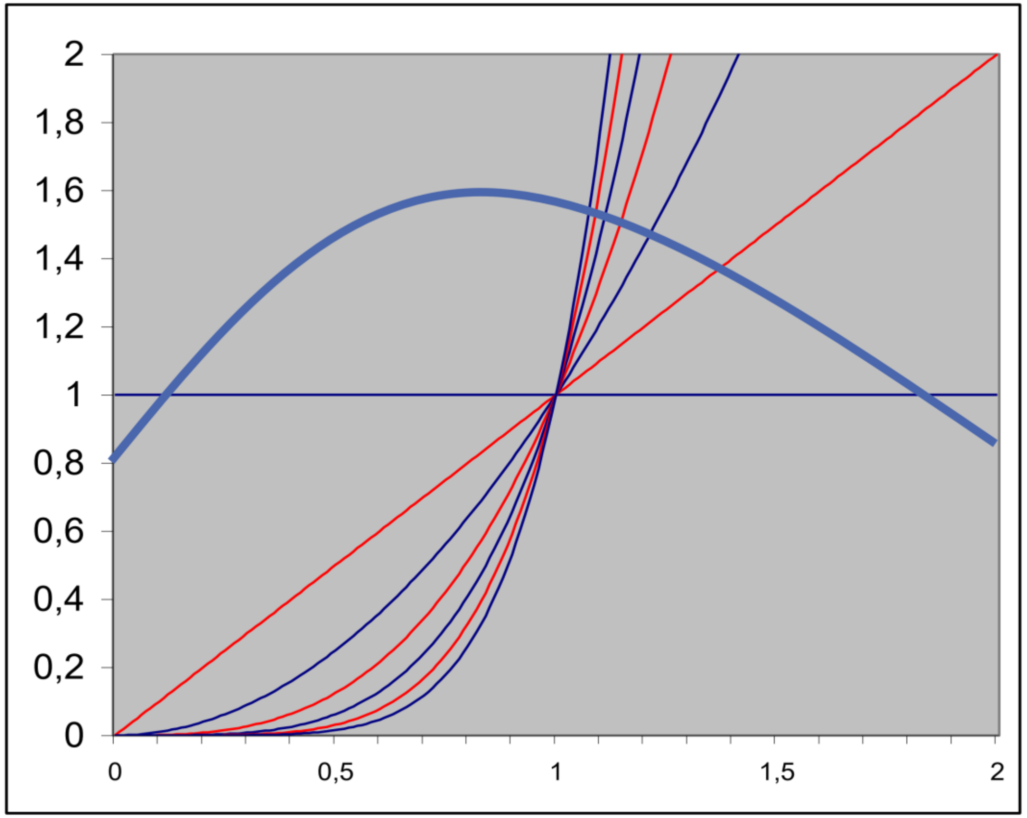
Monomial Basis:

- Functions become increasingly indistinguishable with degree
- Only differ in growing rate
  - x^i grows faster than x^{i-1}



Monomial basis

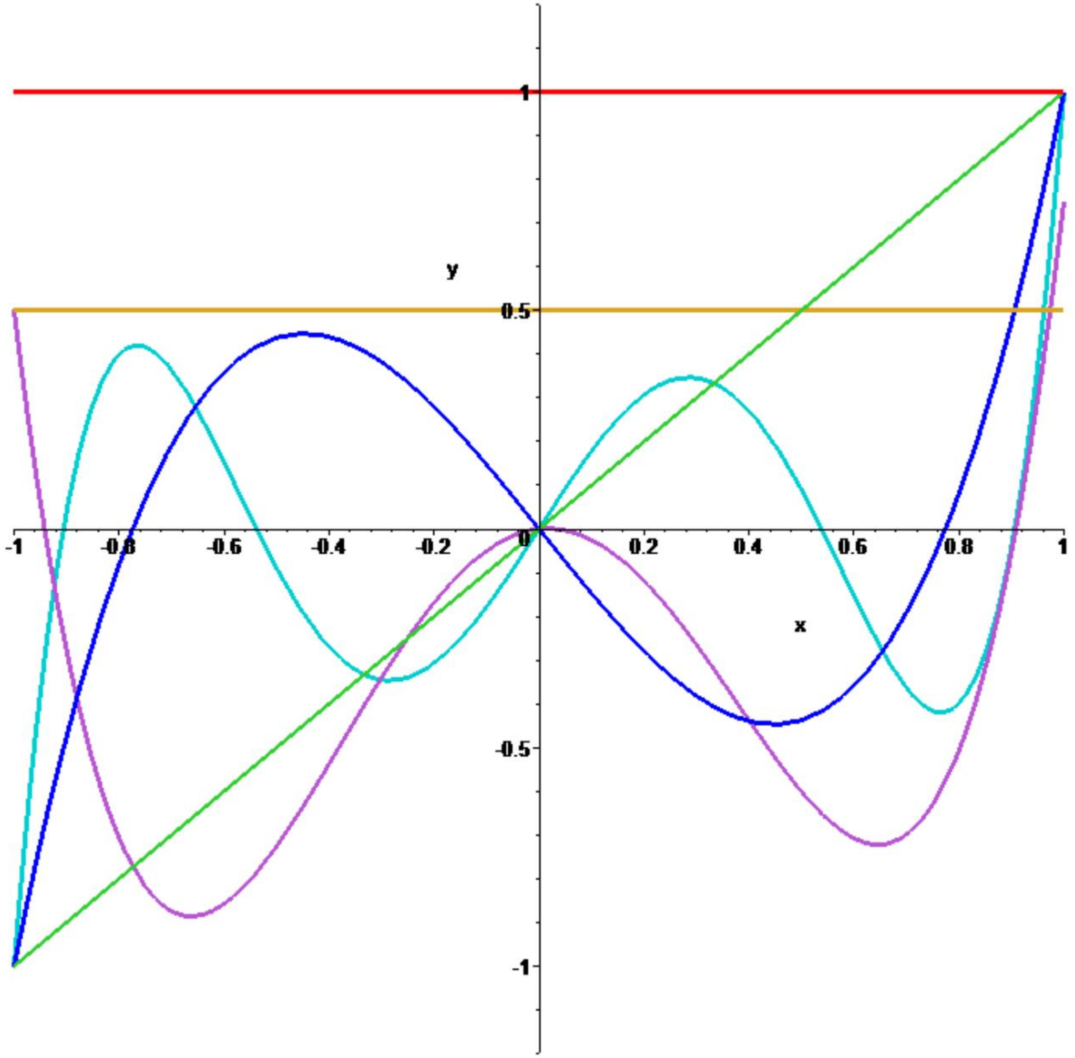
Cancellation



- Monomials:
- From left to right in x- direction…
  - First 1 dominates
  - Then x grows faster
  - Then x^2 grows faster
  - Then x^3 grows faster
  - …
- Tendency:
  - Well behaved functions often require alternating sequence of coefficients (left turn, right turn, left turn,…)
  - *Cancellation* problems

The Cure…

- This problem can be fixed:
  - Use orthogonal polynomial basis
  - How to get one? → e.g. Gram-Schmidt orthogonalization



Alternative approach

- Can we avoid solving a system in the first place?

Alternative approach

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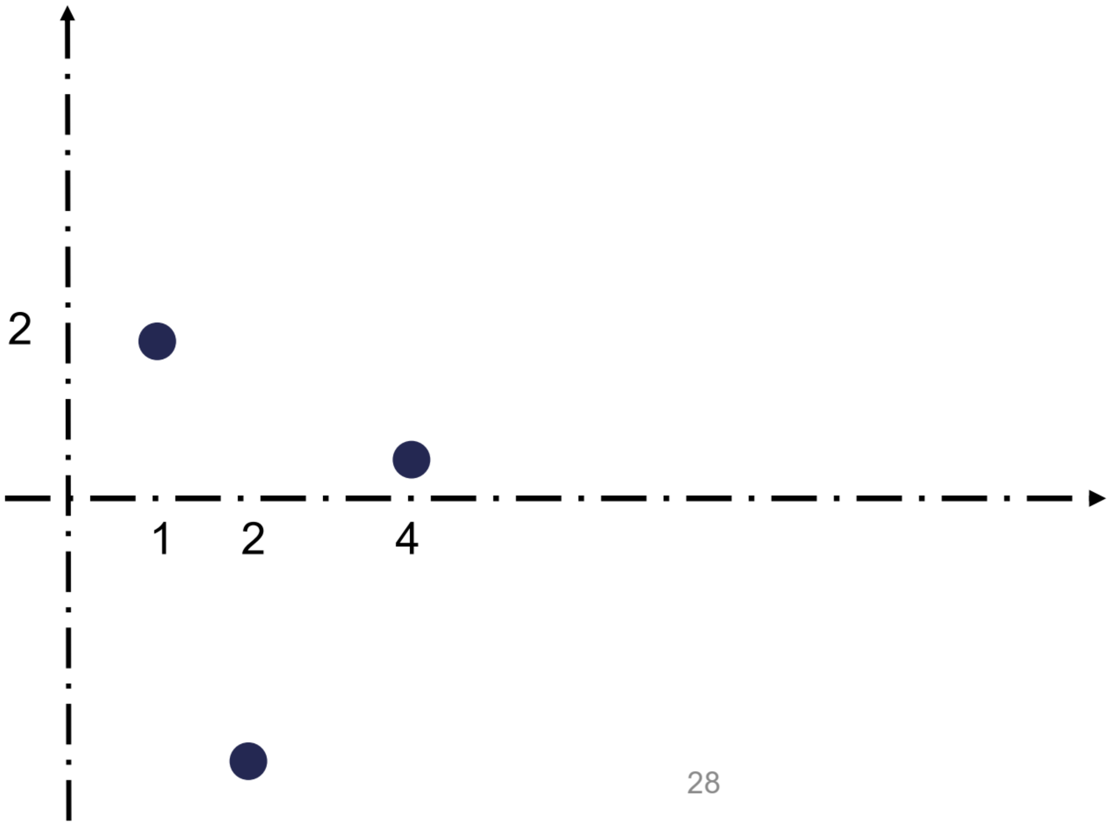
Think of a different basis!

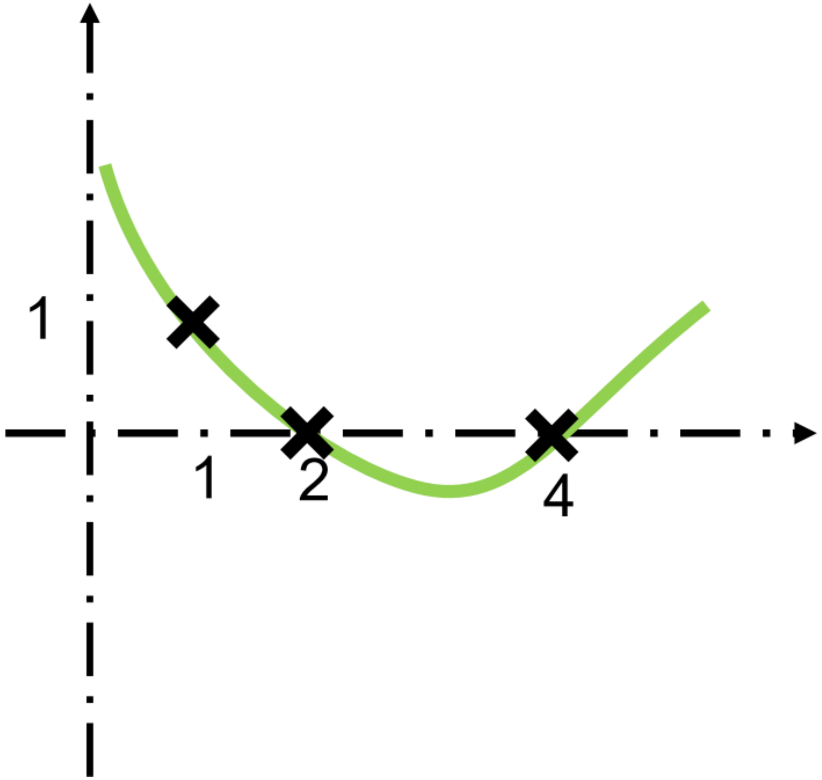
Alternative approach: Example

- Pass a quadratic polynomial through (1, 2), (2,−3), (4, 0.5)



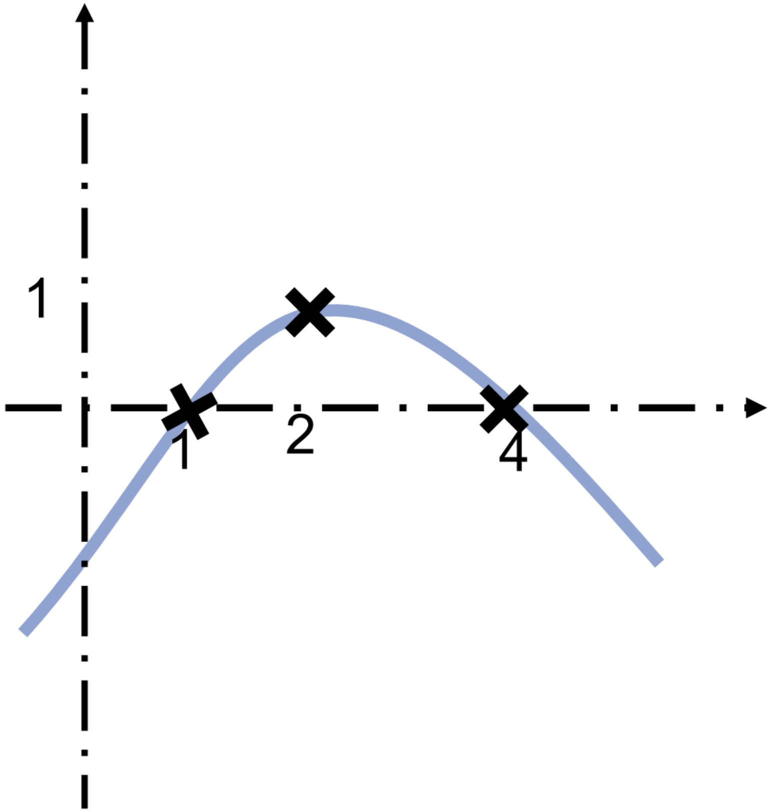
Alternative approach: Example

- Assume we can construct a quadratic polynomial $P_0(x)$ such that it is equal to 1 at x_0, and equals zero at the other two points x_1, x_2 :



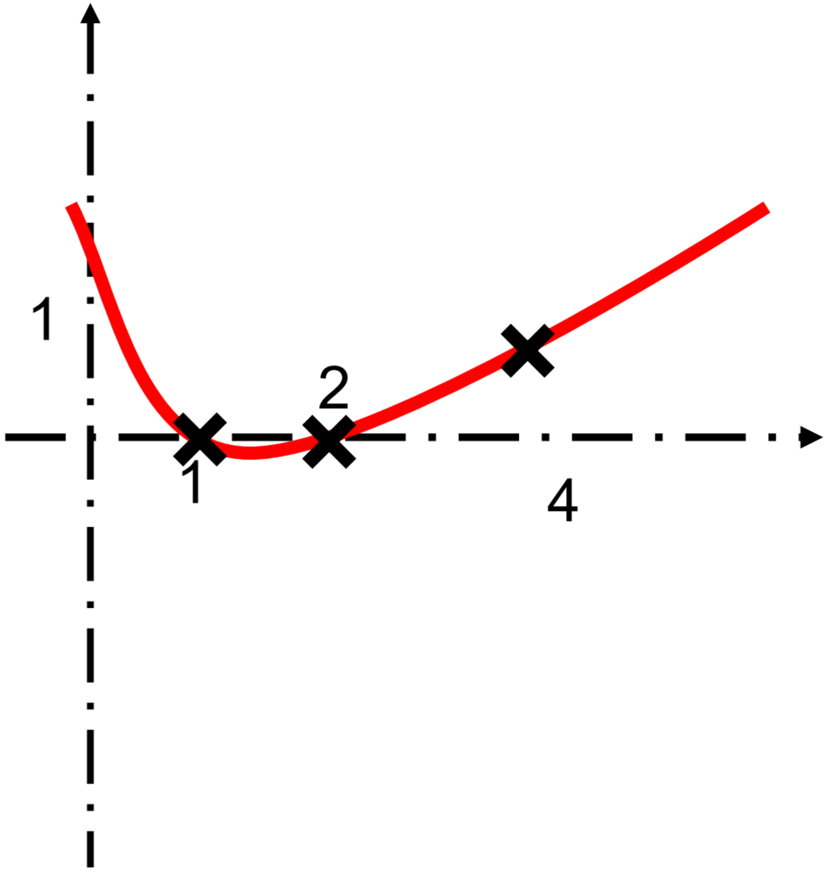
Alternative approach: Example

- $P_1(x)$), is constructed similarly and set equal to 1 at location x_1, and to zero at x_0, x_2 :



Alternative approach: Example

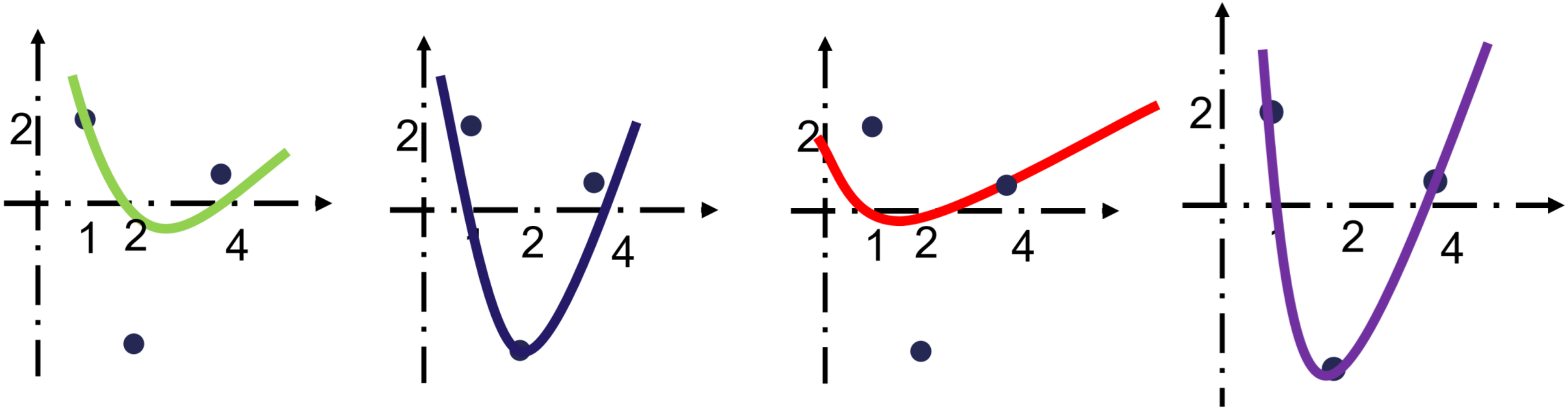
- $P_2(x)$ is set equal to 1 at location x_2, and to zero at x_0, x_1



Alternative approach: Example

- Now, the idea is to scale each $P_i(x)$ such that $P_i(x_i) = y_i$ and add them all together:

$$P(x) = y_0P_0(x) + y_1P_1(x) + y_2P_2(x)$$



$$2P_0(x) \quad + \quad -3P_1(x) \quad + \quad 0.5P_2(x) \quad = \quad P(x)$$

Alternative approach: general case

- Construction of general solution to the interpolation problem:
  - For a set of $n+1$ points $\{(x_0, y_0), \dots, (x_n, y_n)\}$, we seek a basis of polynomials l_i of degree n such that

$$l_i(x_j) = \begin{cases} 1, 若 i = j \\ 0, 若 i \neq j \end{cases}$$

  - The solution to the interpolation problem is then given as

$$P(x) = y_0 l_0(x) + y_1 l_1(x) + \cdots + y_n l_n(x) = \sum_{i=0}^{n} y_i l_i(x)$$

Alternative approach: general case

- How can we find the polynomials $l_i(x)$?
- They are polynomials of degree n and have the following n roots

$$x_0, x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n$$

  - They can be expressed as

$$l_i(x) = C_i(x - x_0)(x - x_1)\dots(x - x_{i-1})(x - x_{i+1})\dots(x - x_n)$$
$$= C_i \prod_{j \neq i} (x - x_j)$$

  - Since $l_i(x_i) = 1$

$$1 = C_i \prod_{j \neq i} (x_i - x_j) \Rightarrow C_i = \frac{1}{\prod_{j \neq i}(x_i - x_j)}$$

Alternative approach: general case

- Finally we have

$$l_i(x) = \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}$$

- The polynomials $l_i(x)$ are called Lagrange polynomials

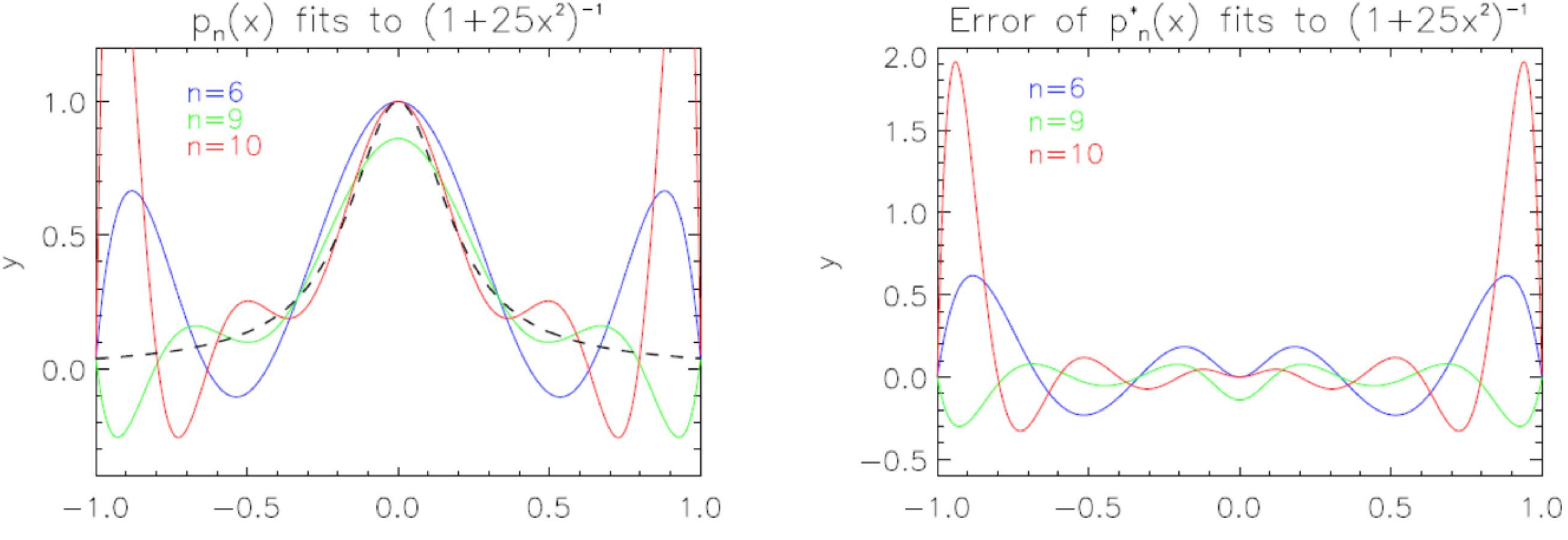
Question

- Is the solution to the interpolation problem obtained using the Lagrange polynomials different from the solution obtained using the Vandermonde matrix (monomial basis)?

Question

- Is the solution to the interpolation problem obtained using the Lagrange polynomials different from the solution obtained using the Vandermonde matrix (monomial basis)?
- Answer: they are the same!
  - Assume they are different. Let's denote R_n the polynomial defined by their difference. R_n has a degree of at most n.
  - We have $R_n(x_i) = 0, i = 0 \dots n,$ where x_i are the distinct interpolation points. So R_n has a degree of at most n and has $n + 1$ roots $\Rightarrow R_n = 0$
- Of course there are many other ways of representing the same polynomial!

How good is our interpolation?



$p_n(x)$ fits to $(1+25x^2)^{-1}$

Error of $p'_n(x)$ fits to $(1+25x^2)^{-1}$

Wiggling (Runge's Phenomenon) and high sensitivity to the change of number of interpolation points.

Observe the difference between $n = 9$ (10 data points) and $n = 10$ (11 data points)

Conclusion

- Polynomial interpolation is instable
- Small changes in control points can lead to very different result. x_i sequence is important.

- “Runge’s phenomenon”: Oscillating behavior
- Wiggling of the polynomial as the number of fitting points increases (even slightly).

- We need better basis functions for interpolation
  - For example, piecewise polynomials will work much better

Approximation

Polynomial and least squares approximation

Motivation

- Why do we need approximation:
  - Noise in the data (sample points)
  - Compact representation
  - Simpler evaluations
- Common approximating functions
  - **Polynomials**
  - Rational functions (quotient of polynomials)
  - Trigonometric functions

Why use polynomials?

- Easy to evaluate, well behaved, smooth,⋯
- Can be justified analytically:
  - Weierstrass' theorem: Let f be any continuous function on a closed interval $[a, b]$, then for any ε, there exist an n and polynomial P_n s.t.

    $$|f(x) - P_n(x)| < \varepsilon, \forall x \in [a, b]$$

  - Weierstrass only proved existence without generating the polynomials

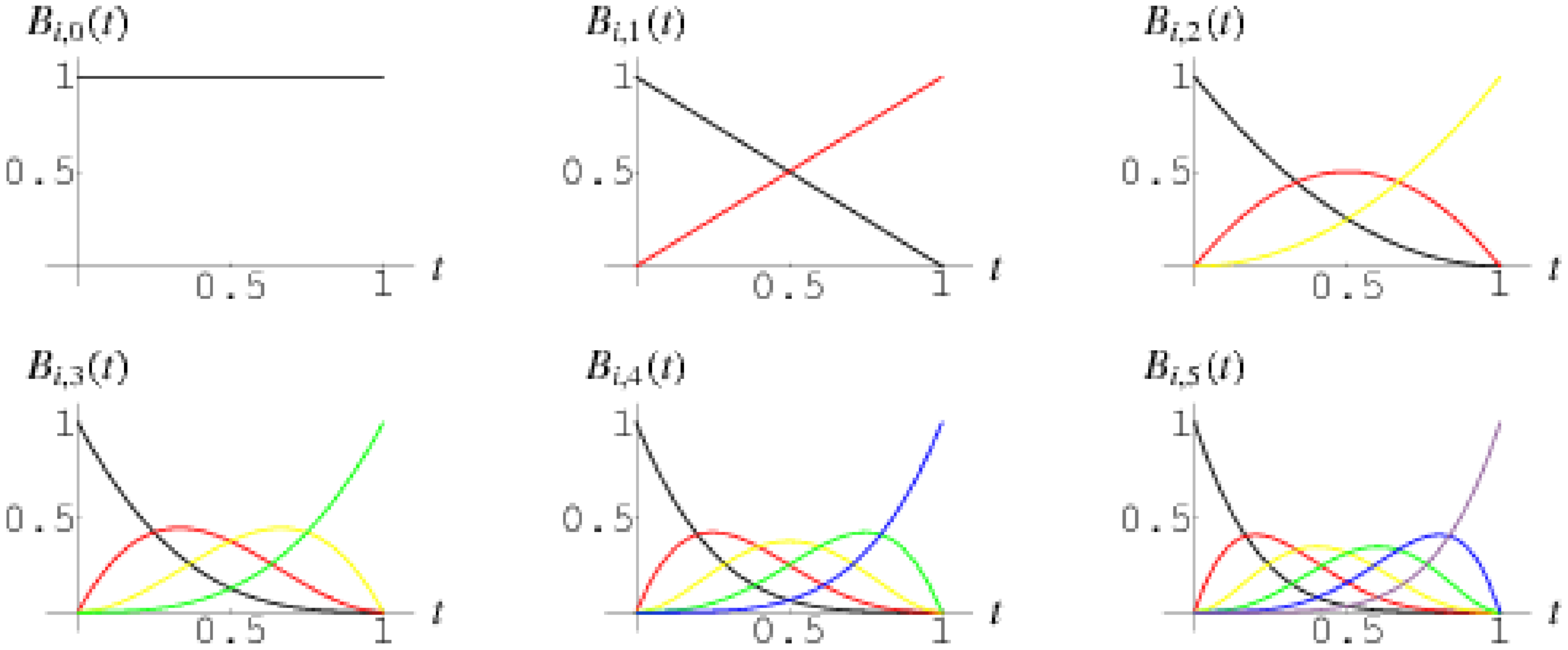
Approximation with Bernstein Polynomials

- Bernstein gave a constructive proof (Powerful!)
  - For any continuous function on $[0, 1]$ and any positive integer n, we have for all x in $[0, 1]$

$$|f(x) - B_n(f, x)| < \frac{9}{4} m_{f,n}$$

  - $m_{f,n} = \text{lower upper bound}_{y_1,y_2 \in [0,1] \& |y_1 - y_2| < \frac{1}{\sqrt{n}}} |f(y_1) - f(y_2)|$
  - $B_n(f, x) = \sum_{j=0}^{n} f\left(x_j\right) b_{n,j}(x)$, where x_j are equally spaced sampling points on $[0,1]$
  - $b_{n,j} = \binom{n}{j} x^j (1 - x)^{n-j}$called Bernstein polynomials

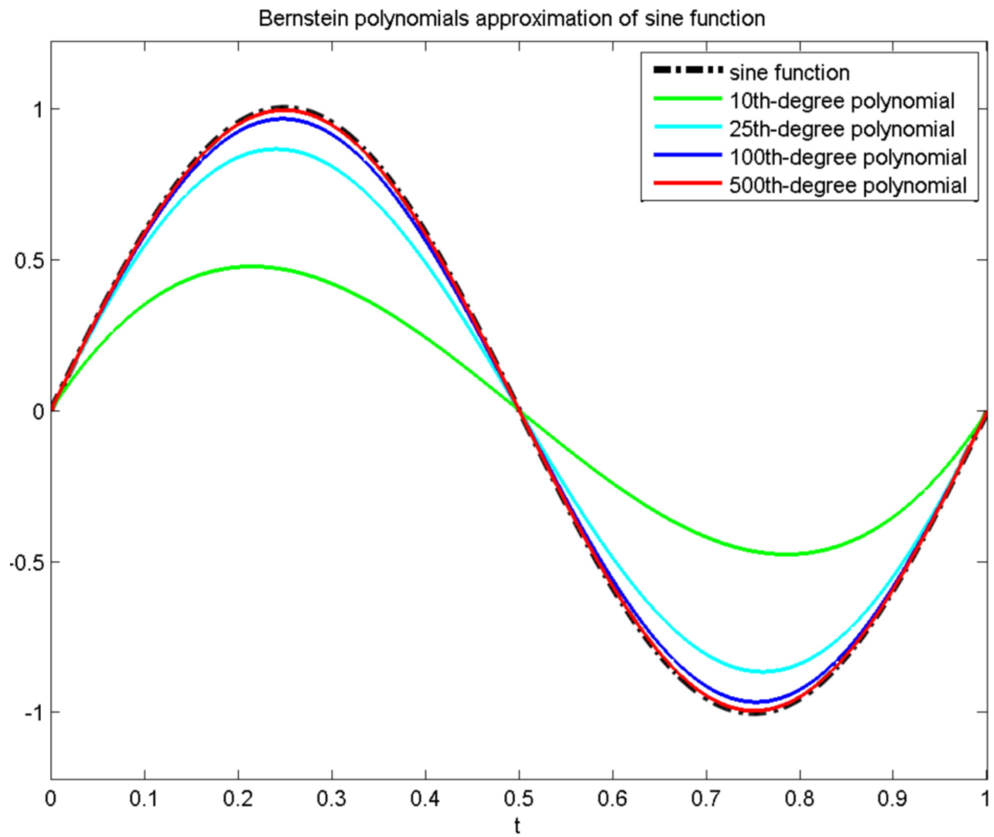
Bernstein Polynomials



- $b_{0,0}(x) = 1$
- $b_{0,1}(x) = 1 - x, \qquad b_{1,1} = x$
- $b_{0,2}(x) = (1-x)^2, \;\; b_{1,2} = 2x(1-x), \quad b_{2,2} = x^2$
- $b_{0,3}(x) = (1-x)^3, \;\; b_{1,3} = 3x(1-x)^2, \;\; b_{2,3} = 3x^2(1-x), \quad b_{3,3} = x^3$
- $b_{0,4}(x) = (1-x)^4, \;\; b_{1,4} = 4x(1-x)^3, \;\; b_{2,4} = 6x^2(1-x)^2, \;\; b_{3,4} = 4x^3(1-x), \quad b_{4,4} = x^4$

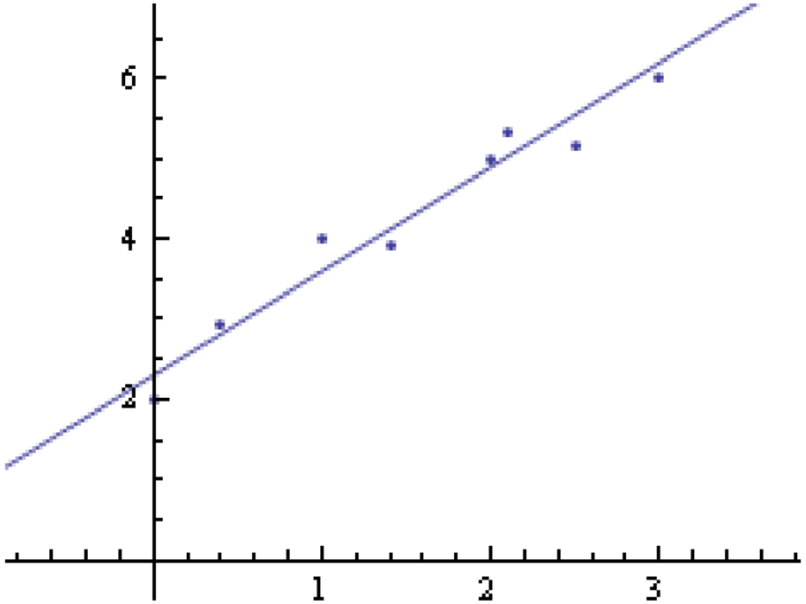
Approximation with Bernstein polynomials

- Example: approximation with Bernstein polynomials
  - Produces excellent approximation but requires a high order
  - Expensive evaluations
  - Can be prone to errors



Least-squares approximation

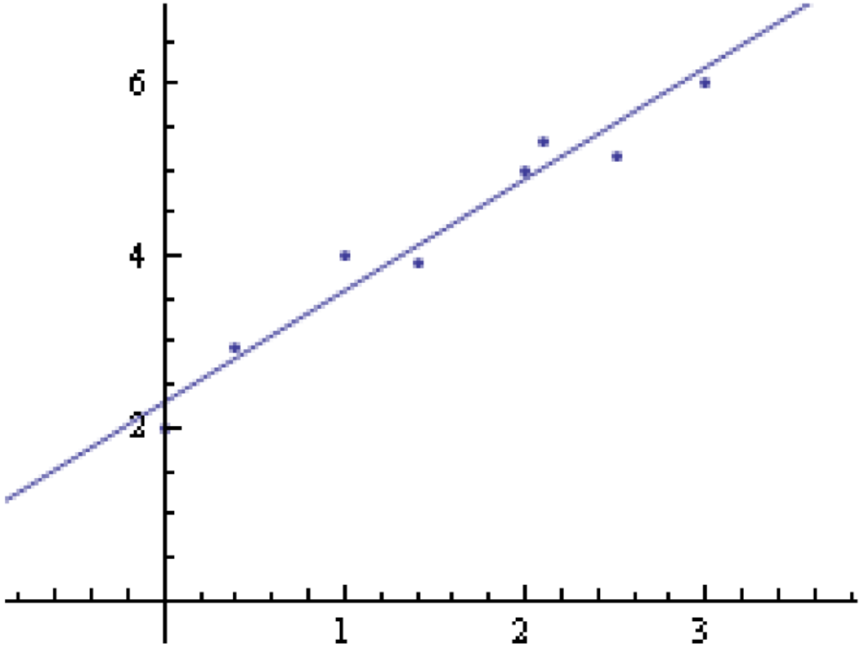
- Approximation Problem
  - Given a linearly independent set $B = \{b_1, \dots b_n\}$ of continuous functions and nodes $\{(x_1, y_1), \dots, (x_m, y_m)\}$ with $m > n$.
  - What function $f \in \mathrm{span}(B)$ best *approximates* the nodes?
  - Example: Best approximating linear function for a set of nodes
  - How do we define "*best approximating*"?



What is meant by *best approximating*?

- Least-Squares Approximation

$$\underset{f\in\mathrm{span}(B)}{\operatorname{argmin}} \sum_{j=1}^{m} \left(f(x_j) - y_j\right)^2$$



$$\sum_{j=1}^{m} \left(f(x_j) - y_j\right)^2 = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} \lambda_i b_i(x_j) - y_j\right)^2$$

$$= (M\lambda - y)^T (M\lambda - y)$$

$$= \lambda^T M^T M\lambda - y^T M\lambda - \lambda^T M^T y + y^T y$$
$$= \lambda^T M^T M\lambda - 2y^T M\lambda + y^T y$$

$$M = \begin{pmatrix} b_1(x_1) & \dots & b_n(x_1) \\ \dots & \dots & \dots \\ b_1(x_m) & \dots & b_n(x_m) \end{pmatrix}$$

Solving the Problem

- This is a quadratic polynomial in λ

$$\lambda^T M^T M\lambda - 2y^T M\lambda + y^T y$$

- Normal equation
  - The minimizer satisfies

$$M^T M\lambda = M^T \mathrm{y}$$

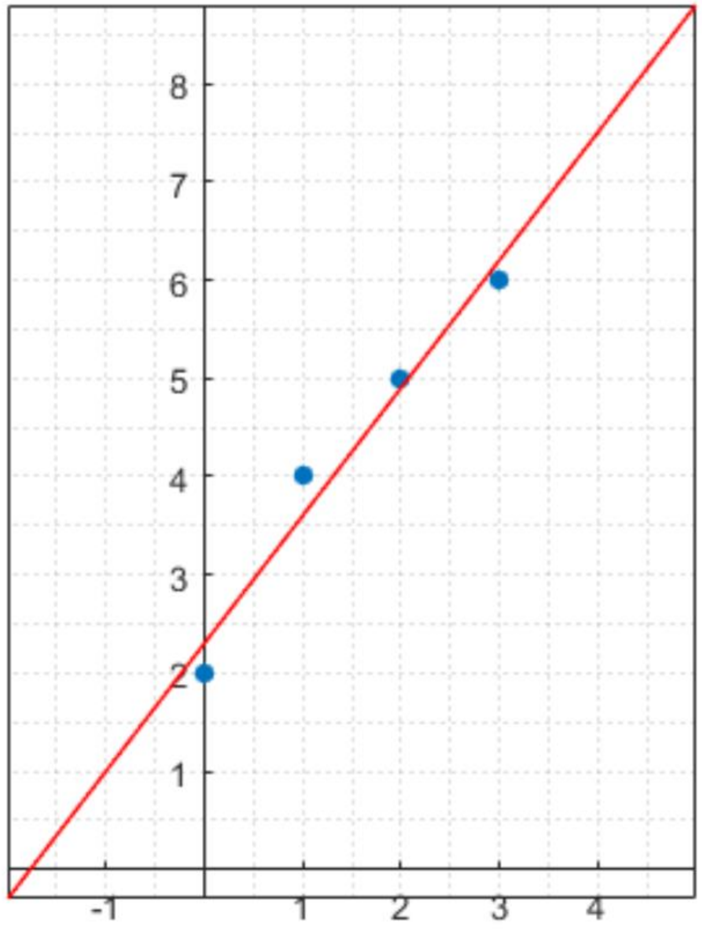
- Reminder
  - Minimize quadratic objective function $x^T A x + b^T x + c$
  - Necessary and sufficient condition: $2Ax = -b$

Example: linear approximation

```
%input
x=[0 1 2 3]’;
y=[2 4 5 6]’;

%setup the matrix
M=[ones(4,1) x];

%solve the least square
c=(M’*M)\(M’*y);
```

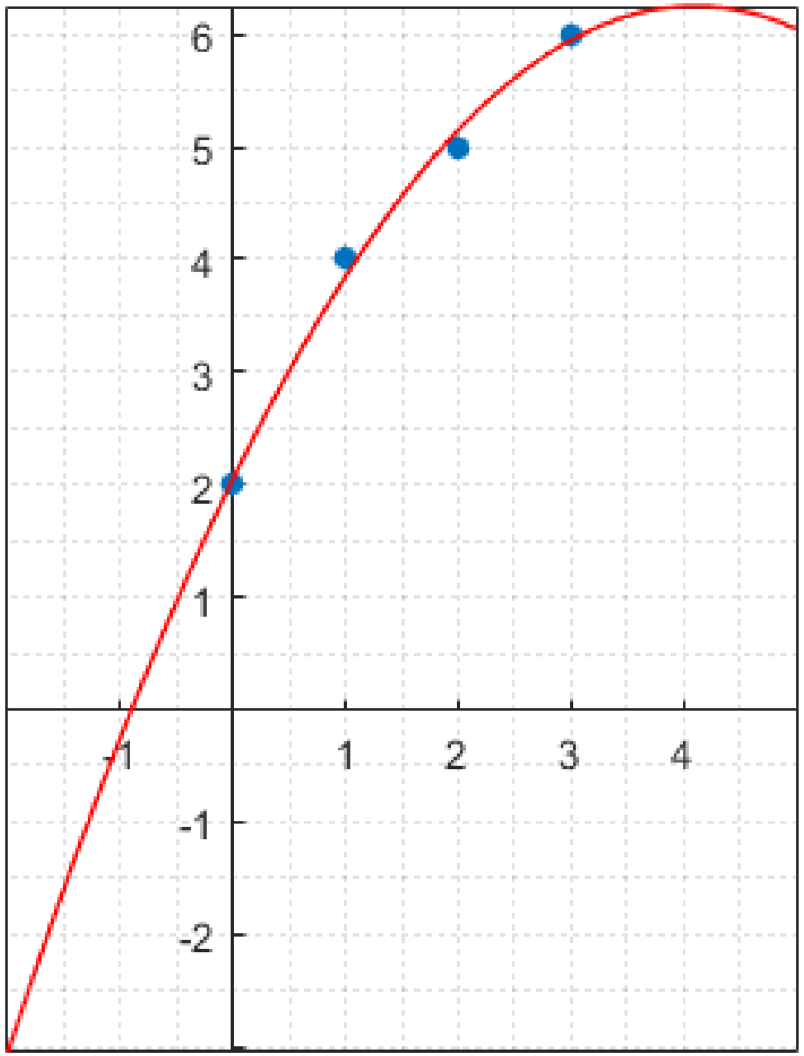


Example: Quadratic approximation

```
%input
x=[0 1 2 3]’;
y=[2 4 5 6]’;

%setup the matrix
M=[ones(4,1) x x.^2];

%solve the least square
c=(M’*M)\(M’*y);
```



Questions?