中国科学技术大学
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计算机图形学
Computer Graphics

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四元数与三维旋转

Rotations

- Very important in computer animation and robotics
- Joint angles, rigid body orientations, camera parameters
- 2D or 3D

Rotations in Three Dimensions

- Orthogonal matrices:

$$RR^{\mathrm{T}} = R^{\mathrm{T}}R = I$$
$$\det(R) = 1$$

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

Representing Rotations in 3D

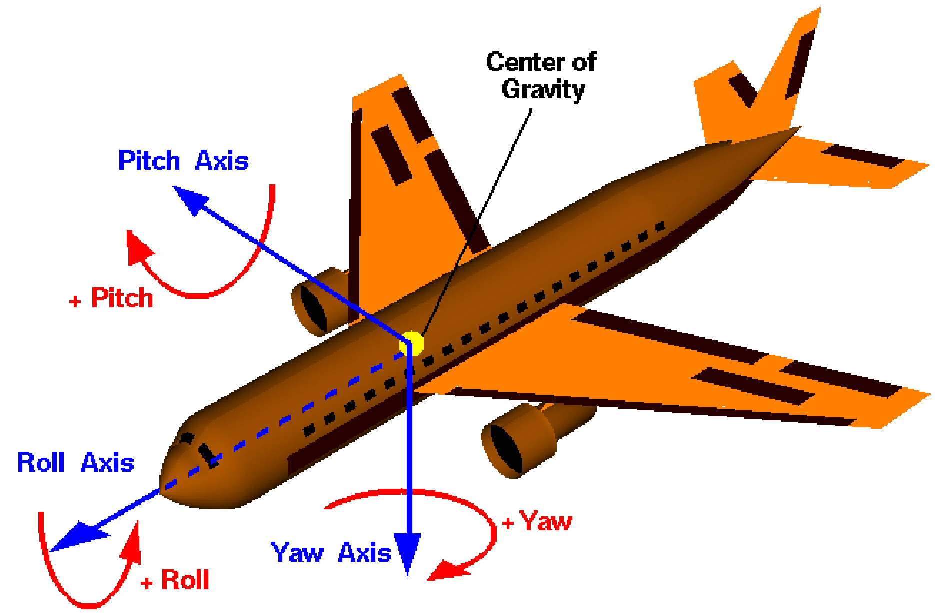
- Rotations in 3D have essentially three parameters

- Axis + angle (2 DOFs + 1DOFs)
  – How to represent the axis?
  Longitude / lattitude have singularities

- 3x3 matrix
  – 9 entries (redundant)

Representing Rotations in 3D

- Euler angles
  - roll, pitch, yaw
  - no redundancy (good)
  - gimbal lock singularities



Source: Wikipedia

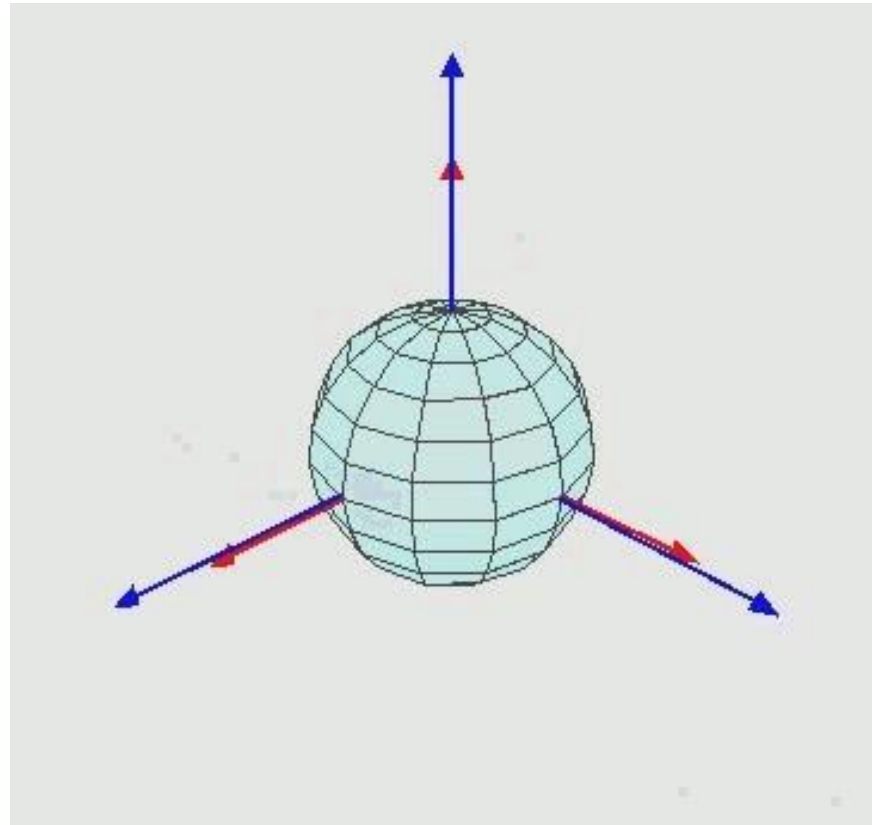
- Quaternions
  - generally considered the “best” representation
  - redundant (4 values), but only by one DOF (not severe)
  - stable interpolations of rotations possible

Euler Angles

1. Yaw
   rotate around y-axis

2. Pitch
   rotate around (rotated) x-axis

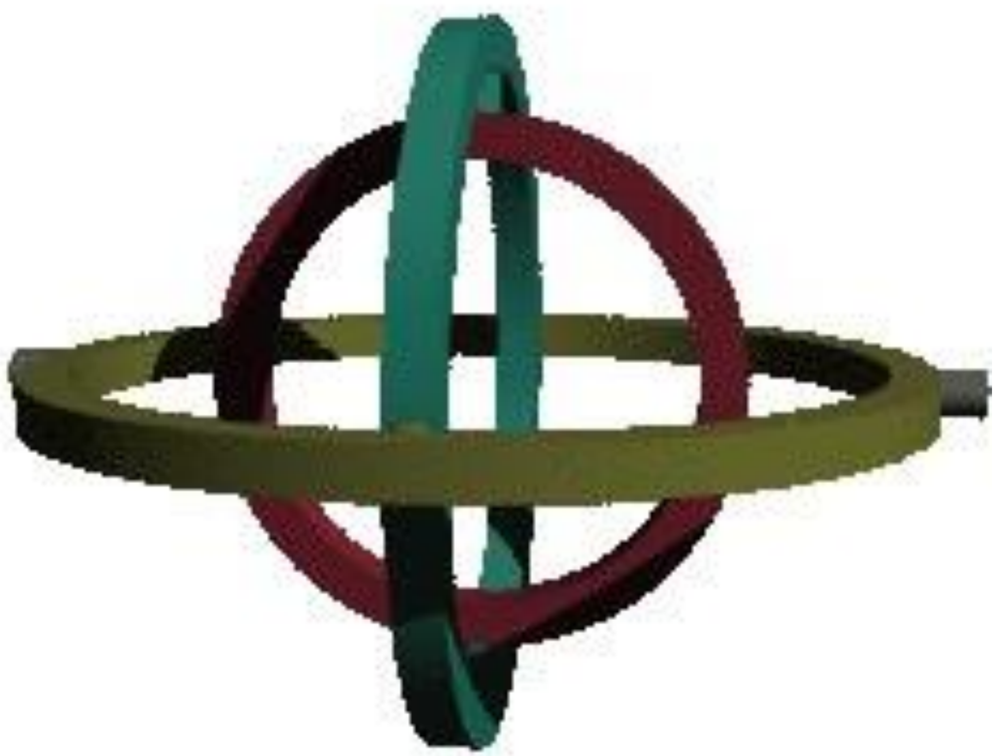
3. Roll
   rotate around (rotated) y-axis



Source:
Wikipedia

Gimbal Lock

When all three gimbals are lined up (in the same plane), the system can only move in two dimensions from this configuration, not three, and is in *gimbal lock*.



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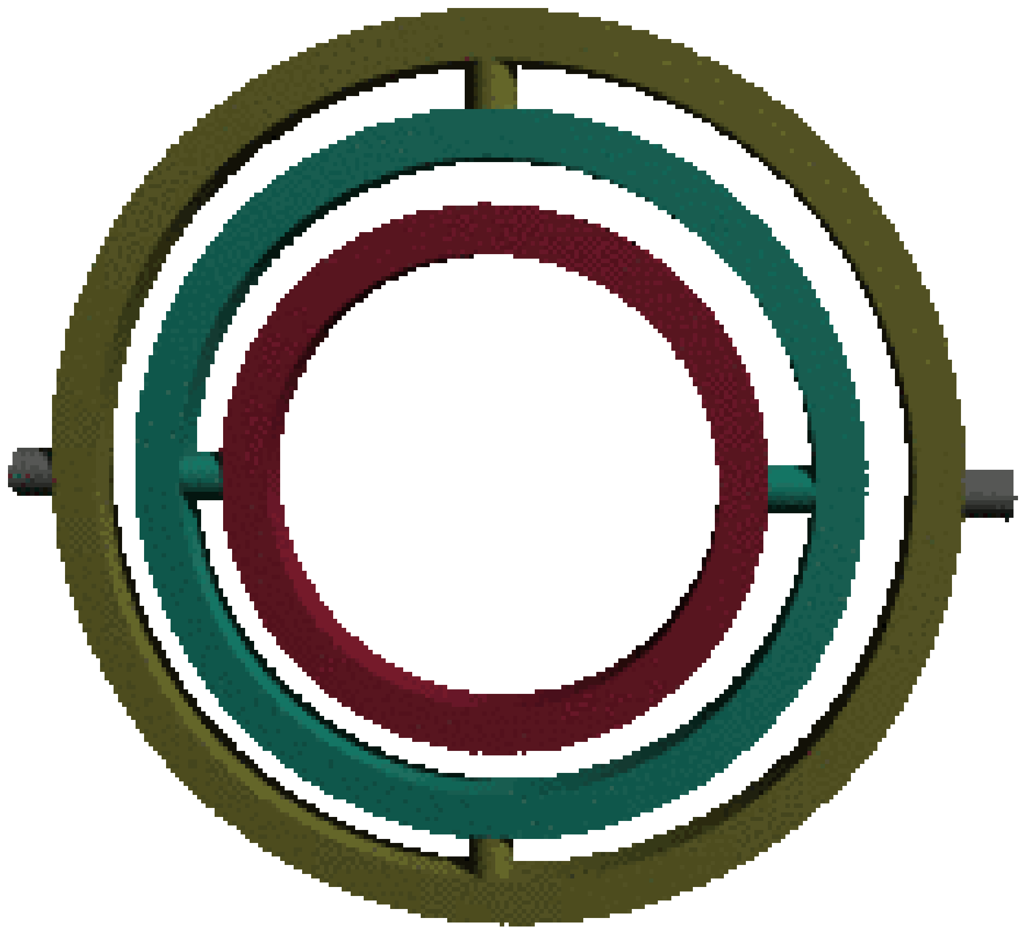
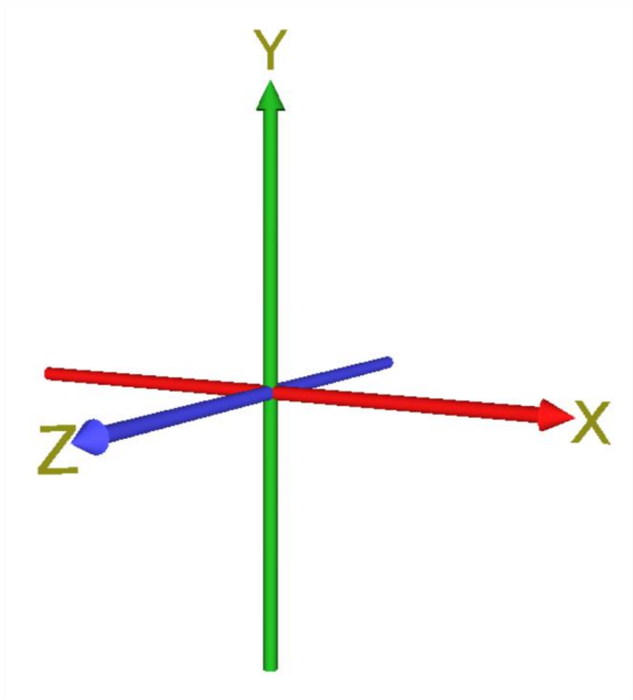


Gimbal Lock (Apollo Systems)

Red-painted area = Danger of real Gimbal Lock

Choice of rotation axis sequence for Euler Angles

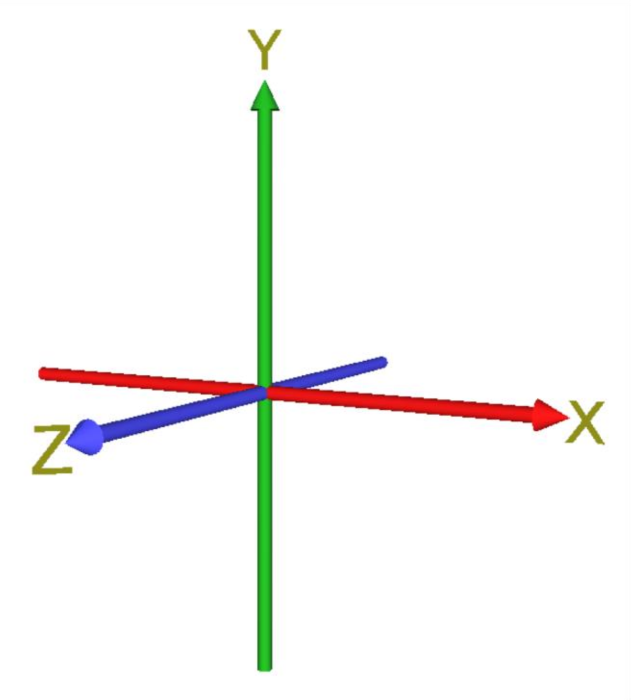
- 12 choices:
  XYX, XYZ, XZX,
  XZY, YXY, YXZ, YZX,
  YZY, ZXY, ZXZ, ZYX,
  ZYZ



- Each choice can use static axes, or rotated axes, so we have a total of 24 Euler Angle versions!

Example: XYZ Euler Angles

- First rotate around X by angle θ_1, then around Y by angle θ_2, then around Z by angle θ_3 .



- Used in CMU Motion Capture Database AMC files

- Rotation matrix is:

$$R = \begin{bmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) \\ 0 & 1 & 0 \\ \sin(\theta_2) & 0 & \cos(\theta_2) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_1) & -\sin(\theta_1) \\ 0 & \sin(\theta_1) & \cos(\theta_1) \end{bmatrix}$$

Outline

- Rotations
- Quaternions
- Quaternion Interpolation

Quaternions

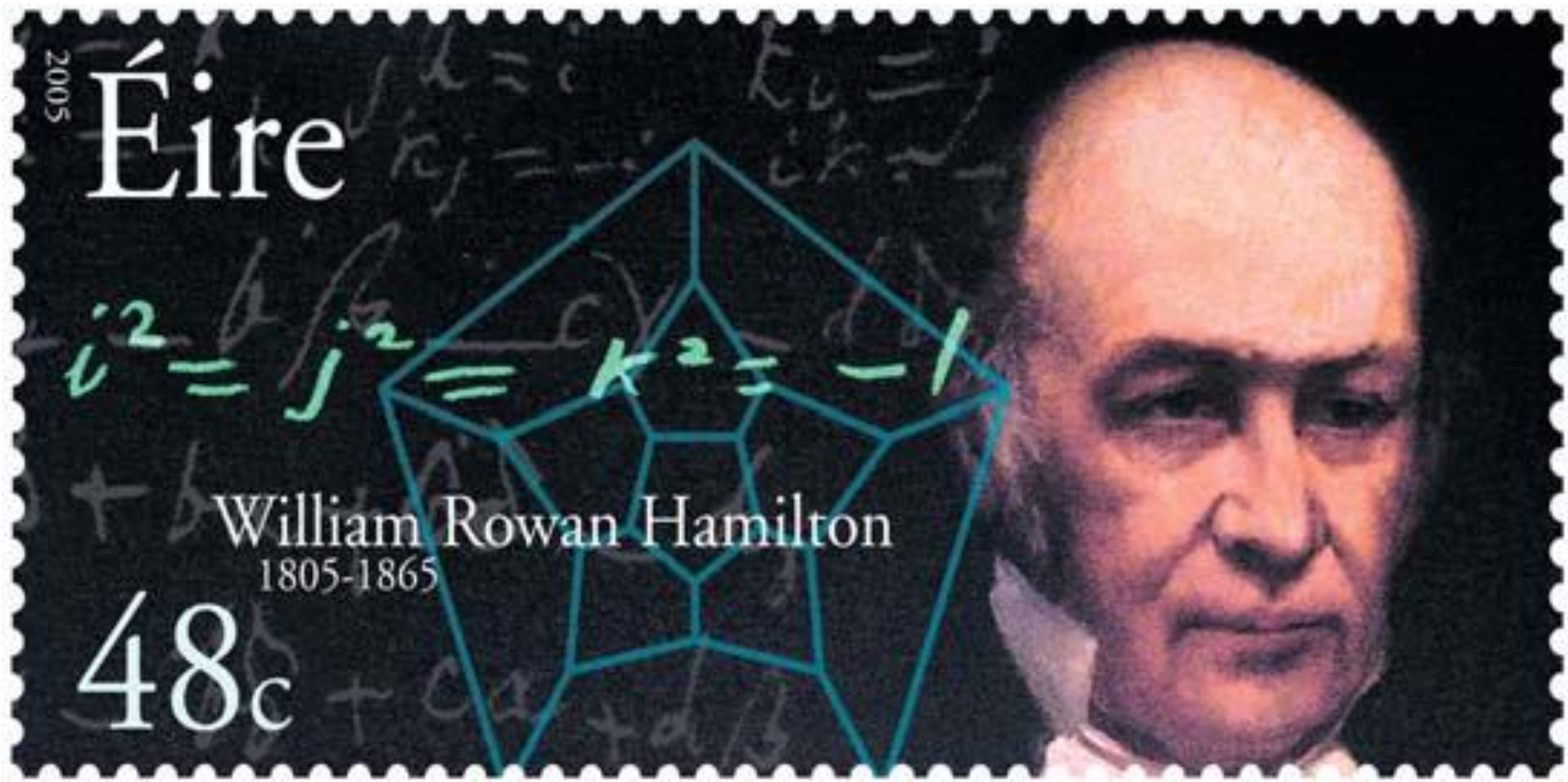
- Generalization of complex numbers
- Three imaginary numbers: $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$

  $\boldsymbol{i}^2 = -1$, $\boldsymbol{j}^2 = -1$, $\boldsymbol{k}^2 = -1$,

  $\boldsymbol{ij} = \boldsymbol{k}$, $\boldsymbol{jk} = \boldsymbol{i}$, $\boldsymbol{ki} = \boldsymbol{j}$, $\boldsymbol{ji} = -\boldsymbol{k}$, $\boldsymbol{kj} = -\boldsymbol{i}$, $\boldsymbol{ik} = -\boldsymbol{j}$

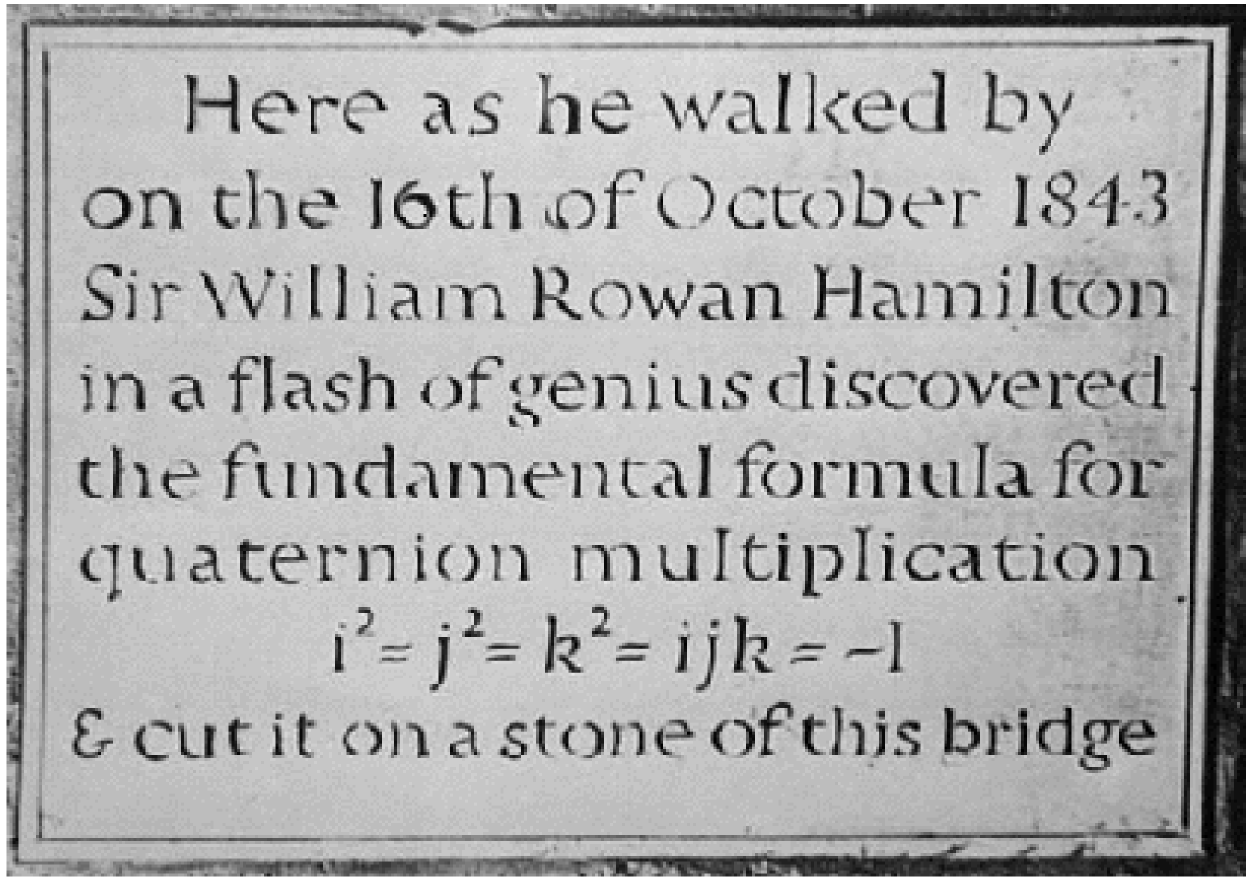
- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$, s,x,y,z are scalars

四元数(Quaternion)



The Broom Bridge in Ireland

The Story


Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
$i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

Quaternions

- Quaternions are **not** commutative!

  $q_1\, q_2 \neq q_2\, q_1$

- However, the following hold:

  $(q_1\, q_2)\, q_3 = q_1\, (q_2\, q_3)$
  $(q_1 + q_2)\, q_3 = q_1\, q_3 + q_2\, q_3$
  $q_1\, (q_2 + q_3) = q_1\, q_2 + q_1\, q_3$
  $\alpha\, (q_1 + q_2) = \alpha\, q_1 + \alpha\, q_2$ (α is scalar)
  $(\alpha q_1)\, q_2 = \alpha\, (q_1 q_2) = q_1\, (\alpha q_2)$ (α is scalar)

- I.e., all usual manipulations are valid, except cannot reverse multiplication order.

Quaternions

- Exercise: multiply two quaternions

  $(2 - \boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})\ (-1 + \boldsymbol{i} + 4\boldsymbol{j} - 2\boldsymbol{k}) = \ldots$

Quaternion Properties

- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$
- Norm: $|q|^2 = s^2 + x^2 + y^2 + z^2$
- Conjugate quaternion: $q = s - x\,\boldsymbol{i} - y\,\boldsymbol{j} - z\,\boldsymbol{k}$
- Inverse quaternion: $q^{-1} = q \,/\, |q|^2$
- Unit quaternion: $|q| = 1$
- Inverse of unit quaternion: $q^{-1} = q$

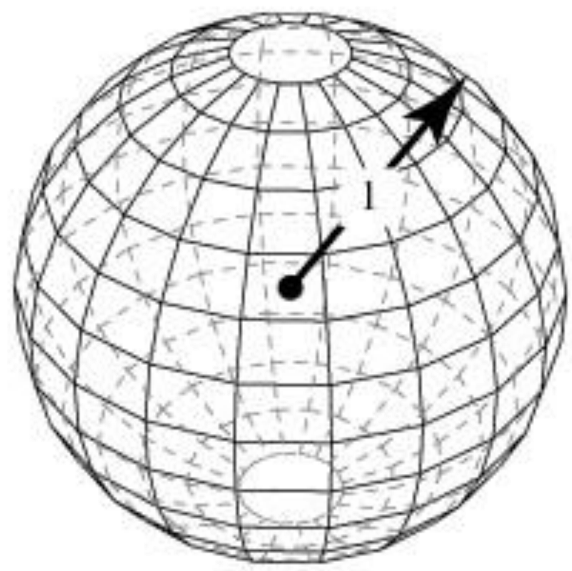
Quaternions and Rotations

- Rotations are represented by ***unit*** quaternions

- $q = s + x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$

  $s^2 + x^2 + y^2 + z^2 = 1$

- Unit quaternion sphere (unit sphere in 4D)



unit sphere in 4D

Rotations to Unit Quaternions

- Let (unit) rotation axis be $[u_x, u_y, u_z]$, and angle θ
- Corresponding quaternion is

$$q = \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right) u_x\boldsymbol{i} + \sin\left(\frac{\theta}{2}\right) u_y\boldsymbol{j} + \sin\left(\frac{\theta}{2}\right) u_z\boldsymbol{k}$$

- Composition of rotations q_1 and q_2 equals

$$q = q_2 q_1$$

- 3D rotations do not commute!

Unit Quaternions to Rotations

- Let v be a (3-dim) vector and let q be a unit quaternion

- Then, the corresponding rotation transforms vector v to $q\boldsymbol{v}q^{-1}$

  ($\boldsymbol{v}$ is a quaternion with scalar part equaling 0, and vector part equaling v)

For $q = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k}$

$$R = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 \end{pmatrix}$$

Quaternions

- Quaternions q and $-q$ give the same rotation!

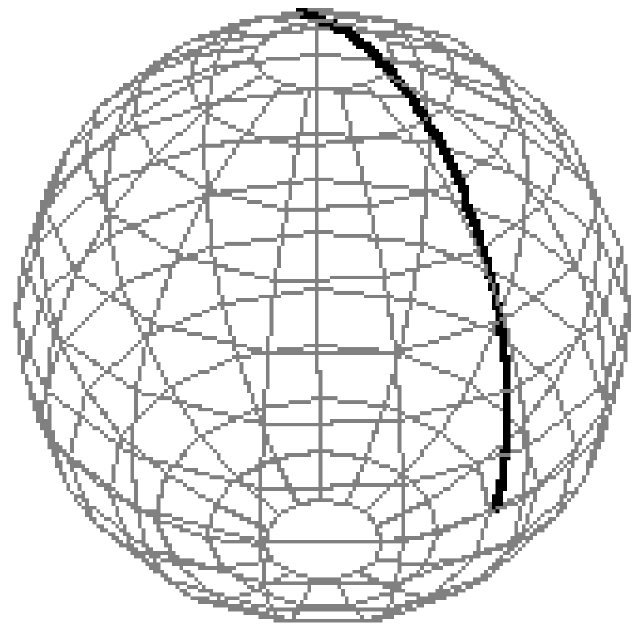
- Other than this, the relationship between rotations and quaternions is unique

Quaternion Interpolation

- Better results than Euler angles

- A quaternion is a point on the 4-D unit sphere

- Interpolating rotations corresponds to curves on the 4-D sphere

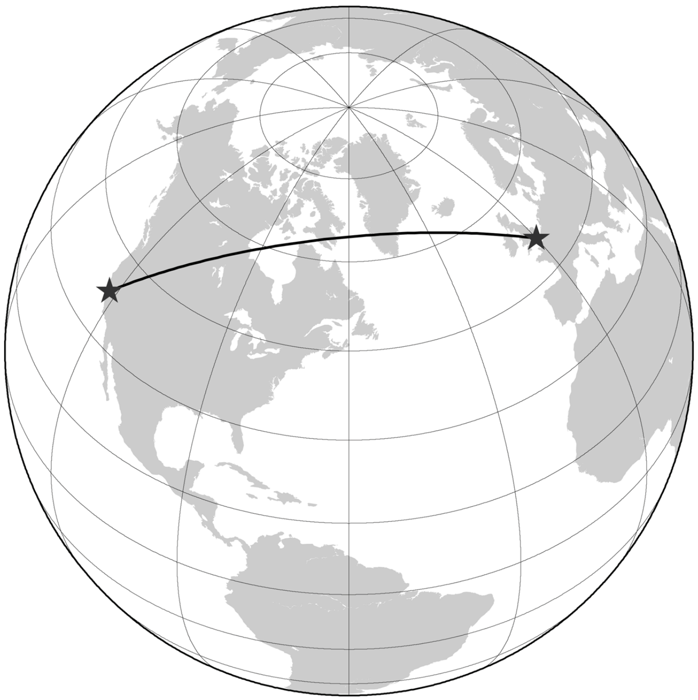


Spherical Linear intERPolation (SLERPing)

- Interpolate along the great circle on the 4-D unit sphere

- Move with constant angular velocity along the great circle between the two points

- Any rotation is given by two quaternions, so there are two SLERP choices; pick the shortest



San Francisco
to London

SLERP

$$\text{Slerp}(q_1, q_2, u) = \frac{\sin((1-u)\theta)}{\sin(\theta)} q_1 + \frac{\sin(u\theta)}{\sin(\theta)} q_2$$

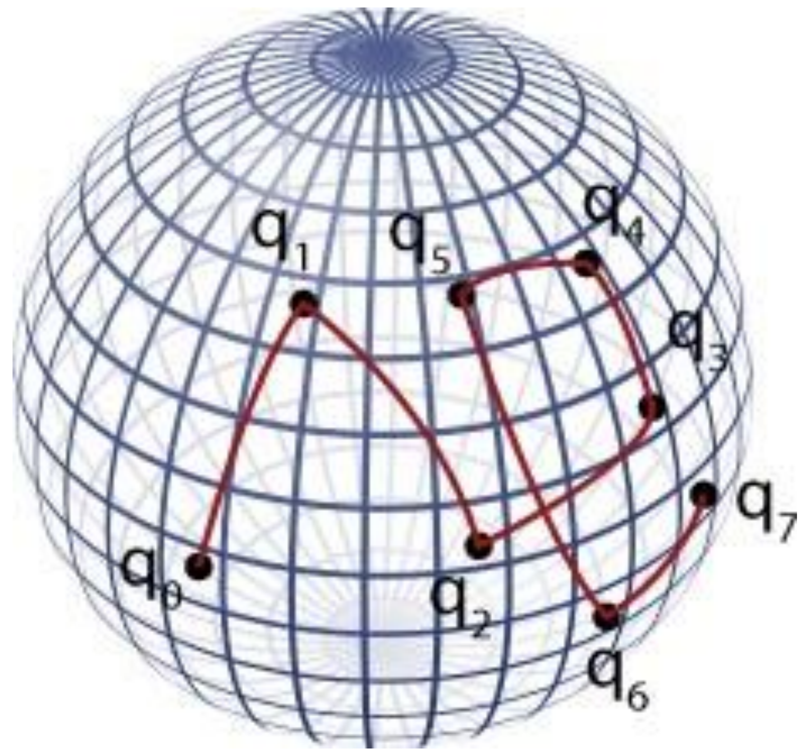
$$\cos(\theta) = q_1 \cdot q_2 =$$

$$= s_1 s_2 + x_1 x_2 + y_1 y_2 + z_1 z_2$$

- u varies from 0 to 1
- $q_m = s_m + x_m \boldsymbol{i} + y_m \boldsymbol{j} + z_m \boldsymbol{k}$,   for $m = 1,2$
- The above formula automatically produces a unit quaternion (not obvious, but true).

Interpolating more than two rotations

- Simplest approach: connect consecutive quaternions using SLERP
- Continuous rotations
- Angular velocity not smooth at the joints

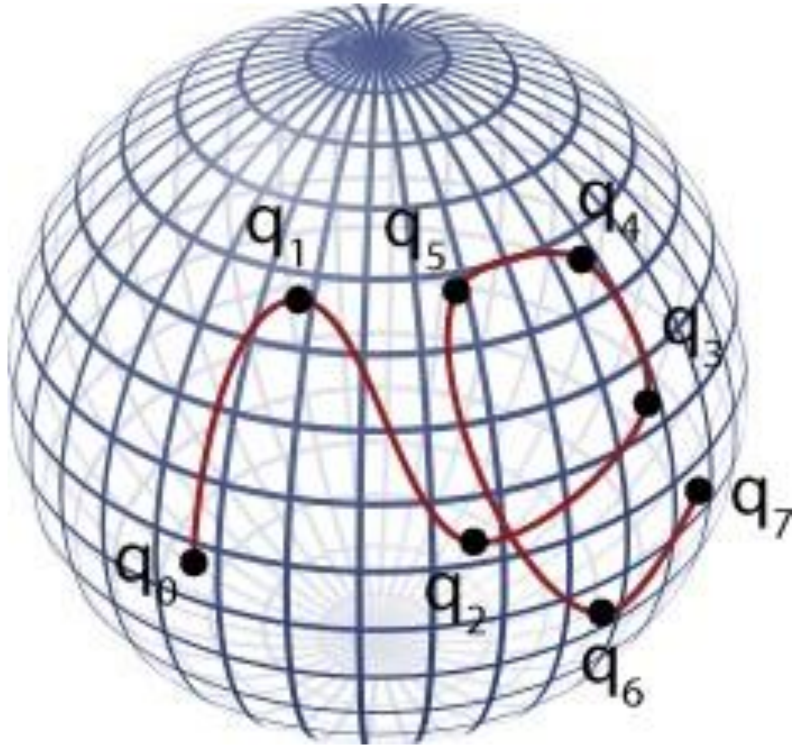


Interpolation with smooth velocities

- Use splines on the unit quaternion sphere

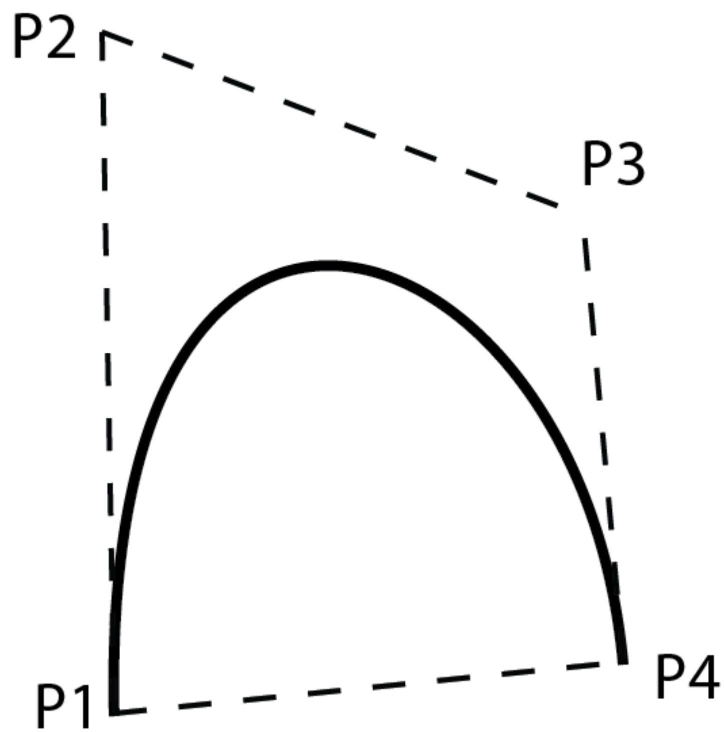
- Reference:

Shoemake, SIGGRAPH '85



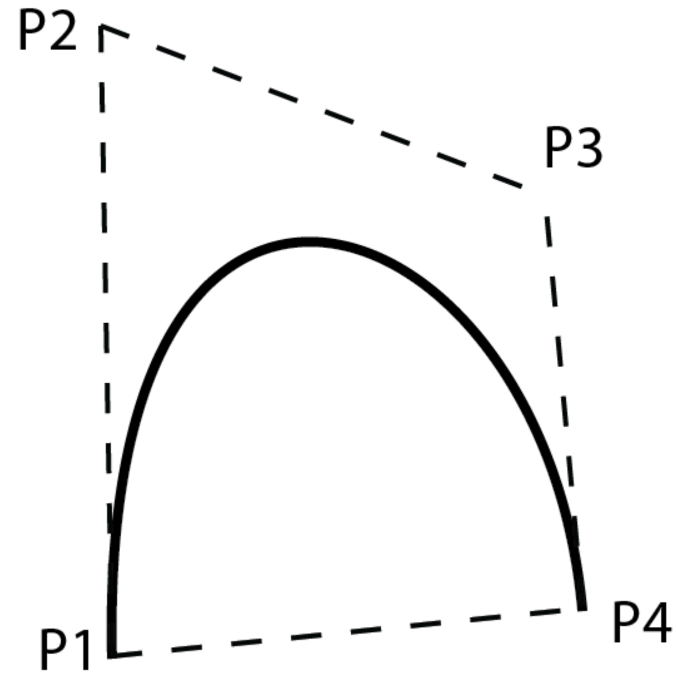
Bezier Spline

- Four control points
  - points P1 and P4 are on the curve
  - points P2 and P3 are off the curve;
    they give curve tangents at beginning and end



Bezier Spline

- p(0) = P1, p(1) = P4,
- p'(0) = 3(P2-P1)
- p'(1) = 3(P4 - P3)
- Convex Hull property:
  curve contained within the
  convex hull of control points
- Scale factor “3” is chosen to
  make “velocity” approximately
  constant



The Bezier Spline Formula

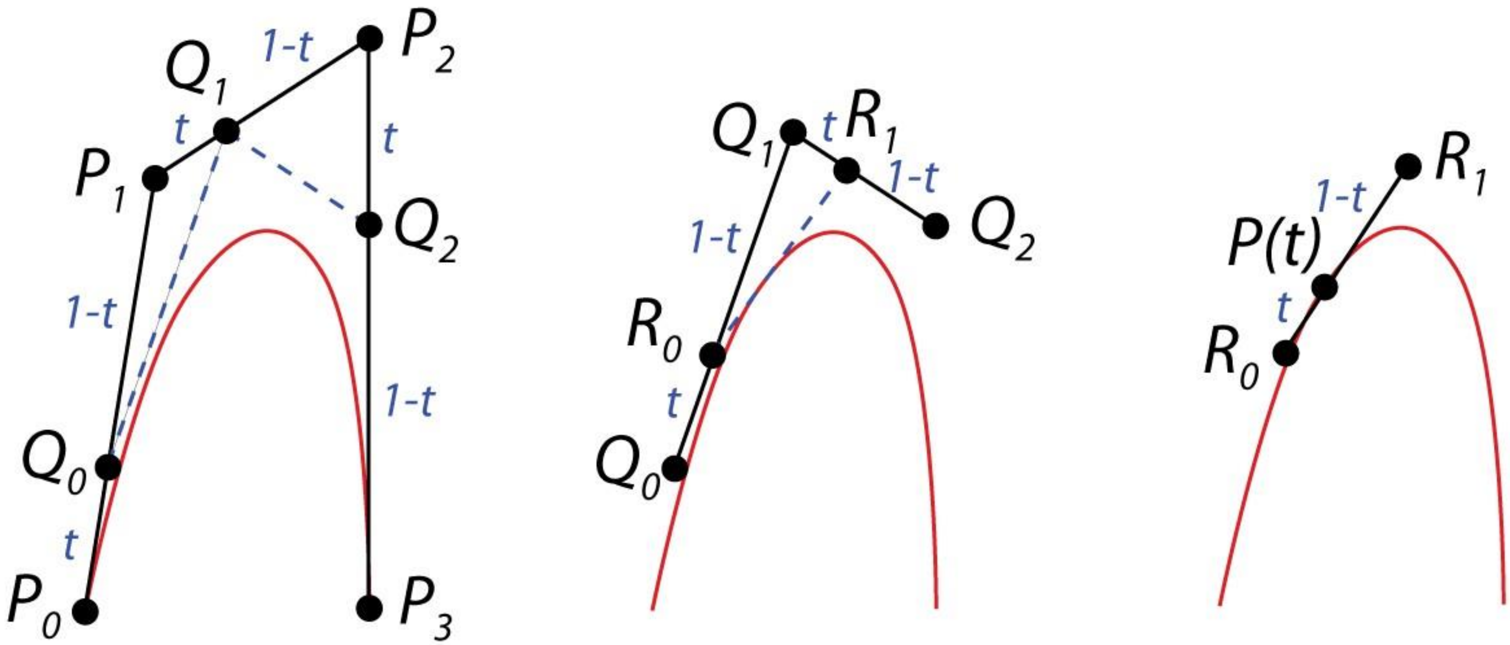
$$[x \;\; y \;\; z] = [u^3 \;\; u^2 \;\; u \;\; 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix}$$

Bezier basis

Bezier control matrix

- [x,y,z] is point on spline corresponding to u
- u varies from 0 to 1
- P1 = $[x_1 \; y_1 \; z_1]$ P2 = $[x_2 \; y_2 \; z_2]$
- P3 = $[x_3 \; y_3 \; z_3]$ P4 = $[x_4 \; y_4 \; z_4]$

DeCasteljau Construction

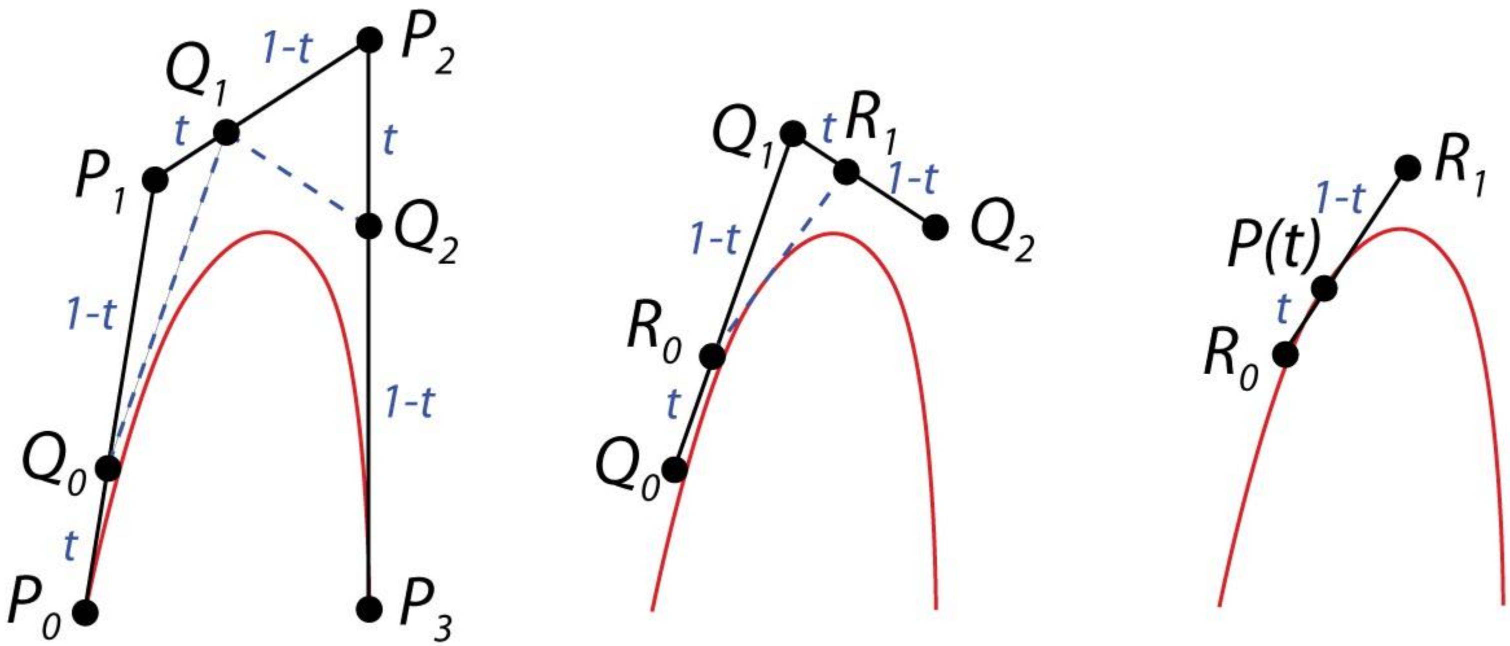


Efficient algorithm to evaluate Bezier splines.

Similar to Horner rule for polynomials.

Can be extended to interpolations of 3D rotations.

DeCasteljau on Quaternion Sphere



Given t, apply DeCasteljau construction:

$Q_0 = \text{Slerp}(P_0,P_1,t)$ $Q_1 = \text{Slerp}(P_1,P_2,t)$

$Q_2 = \text{Slerp}(P_2,P_3,t)$ $R_0 = \text{Slerp}(Q_0,Q_1,t)$

$R_1 = \text{Slerp}(Q_1,Q_2,t)$ $P(t)= \text{Slerp}(R_0,R_1,t)$

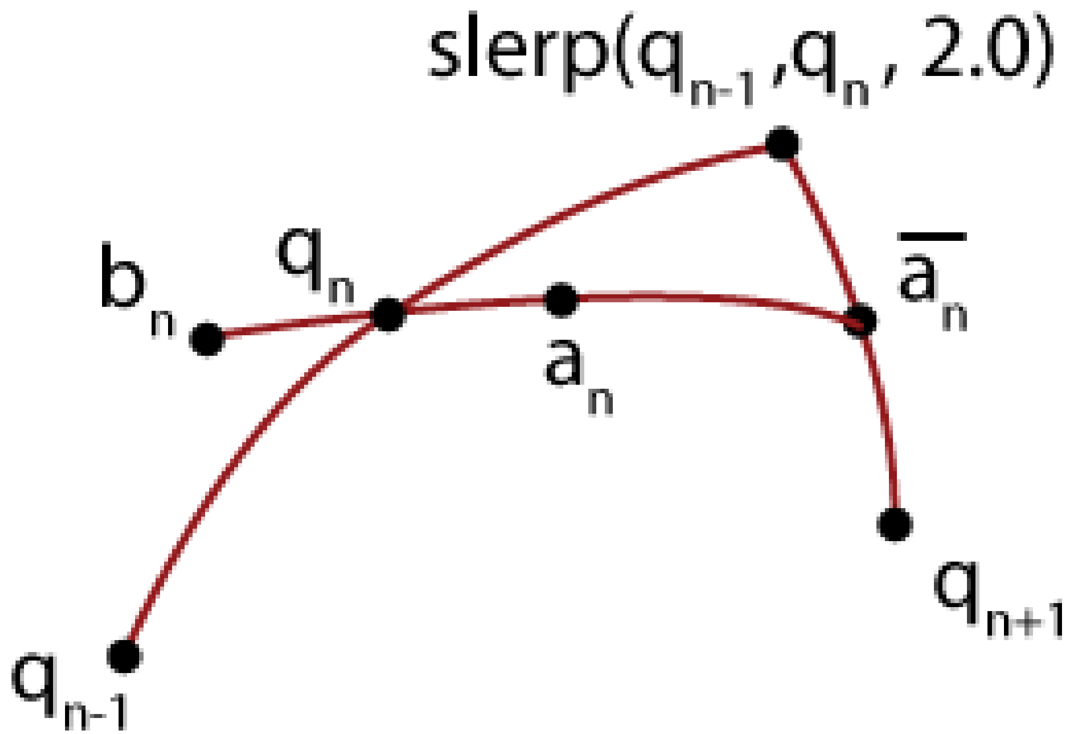
Bezier Control Points for Quaternions

- Given quaternions q_{n-1}, q_n, q_{n+1}, form:

$$\overline{a_n} = Slerp(Slerp(q_{n-1}, q_n, 2), q_{n+1}, 0.5)$$

$$a_n = Slerp(qn\,, \overline{a_n}, 1/3)$$

$$b_n = Slerp(qn\,, \overline{a_n}, -1/3)$$



Interpolating Many Rotations on Quaternion Sphere

- Given quaternions $q_1, ..., q_N$,
  form Bezier spline control points (previous slide)

- Spline 1: q_1, a_1, b_2, q_2
- Spline 2: q_2, a_2, b_3, q_3 etc.

- Need a_1 and b_N; can set
  $a_1 = \text{Slerp}(q_1, \text{Slerp}(q_3, q_2, 2.0), 1.0 / 3)$
  $b_N = \text{Slerp}(q_N, \text{Slerp}(q_{N-2}, q_{N-1}, 2.0), 1.0 / 3)$

- To evaluate a spline at any t, use DeCasteljau construction