中国科学技术大学
University of Science and Technology of China

计算机图形学
Computer Graphics

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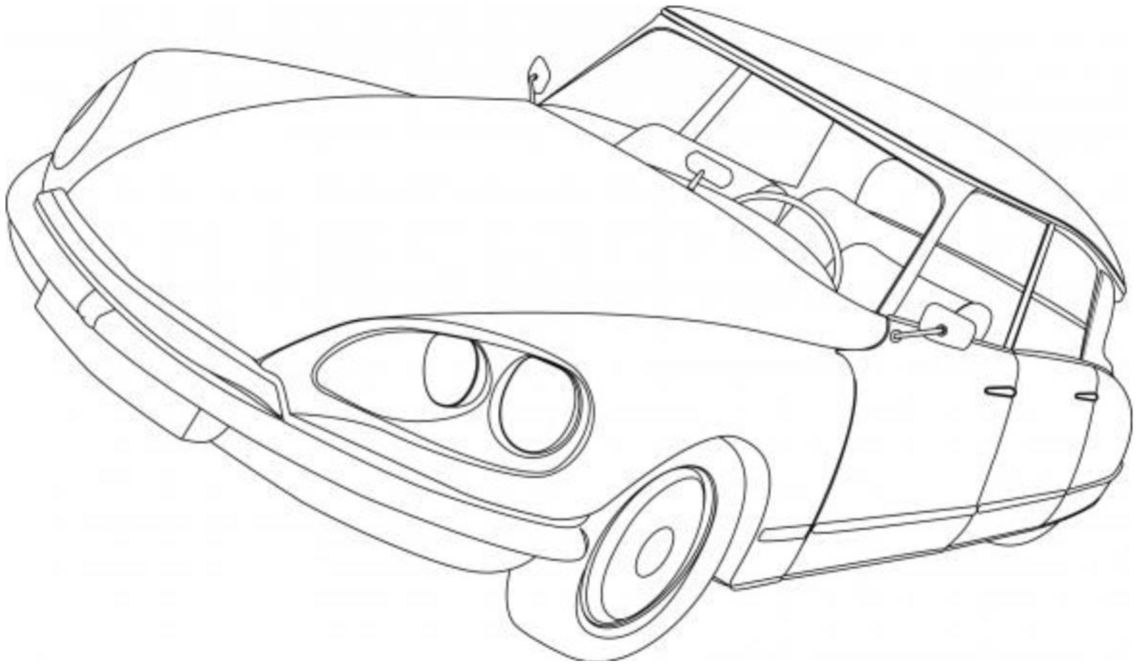
Bézier curves

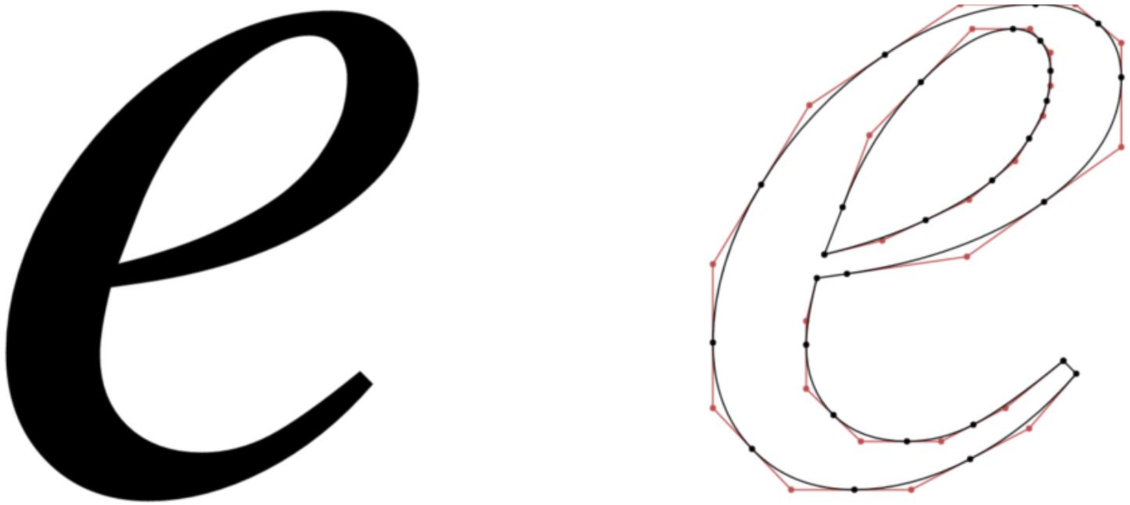
- Bézier curves/splines developed by
  - Paul de Casteljau at Citroen (1959)
  - Pierre Bézier at Renault (1963)

  for free-form parts in automotive design



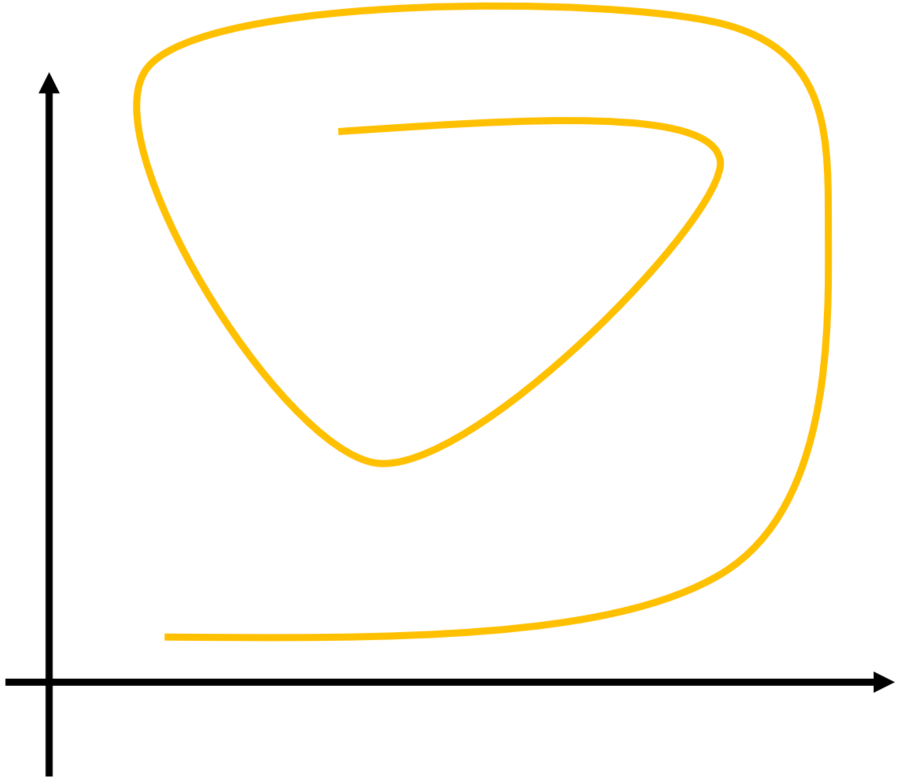
Bézier curves

- Today: Standard tool for 2D curve editing
- Cubic 2D Bézier curves are everywhere:
  - Inkscape, Corel Draw, Adobe Illustrator, Powerpoint, …
  - PDF, Truetype (quadratic curves), Windows GDI, …
- Widely used in 3D curve & surface modeling as well



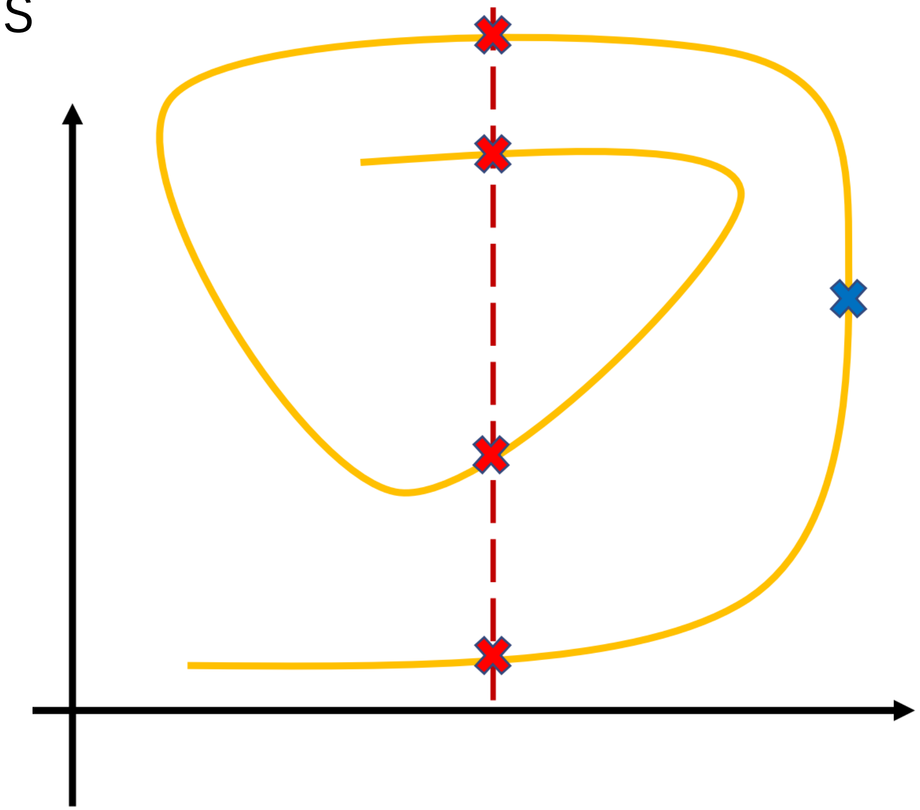
Curve representation

- The implicit curve form $f(x, y) = \mathbf{0}$ suffers from several limitations:



Curve representation

- The implicit curve form $f(x, y) = 0$ suffers from several limitations:
    - Multiple values for the same x-coordinates
    - Undefined derivative $\frac{dy}{dx}$ (see blue cross)
    - Not invariant w.r.t axes transformations

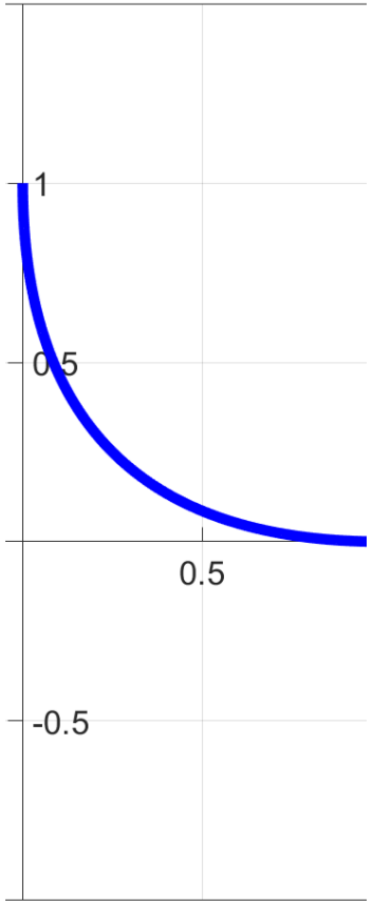


Parametric representation

- Remedy: parametric representation $c(t) = \big(x(t), y(t)\big)$
  - Easy evaluations
  - The parameter t can be interpreted as time
  - The curve can be interpreted as the path traced by a moving particle

Modeling with the power basis, ⋯

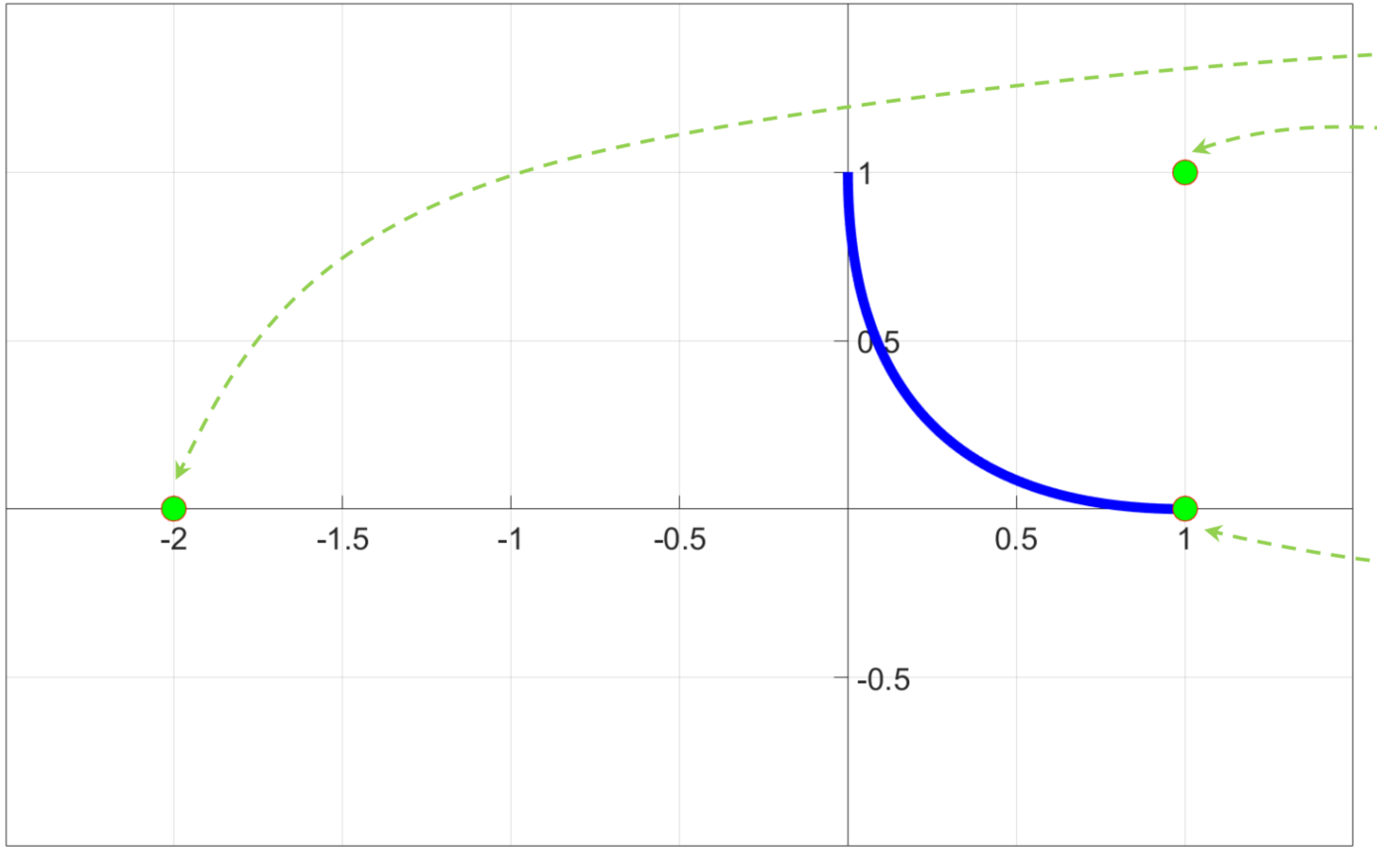
- Example of a parabola: $\boldsymbol{f}(t) = \boldsymbol{a}t^2 + \boldsymbol{b}t + \boldsymbol{c}$



$$\boldsymbol{f}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Modeling with the power basis, ⋯ no thanks!

- Examples of a parabola: $\boldsymbol{f}(t) = \boldsymbol{a}t^2 + \boldsymbol{b}t + \boldsymbol{c}$: the coefficients of the power basis lack intuitive geometric meaning

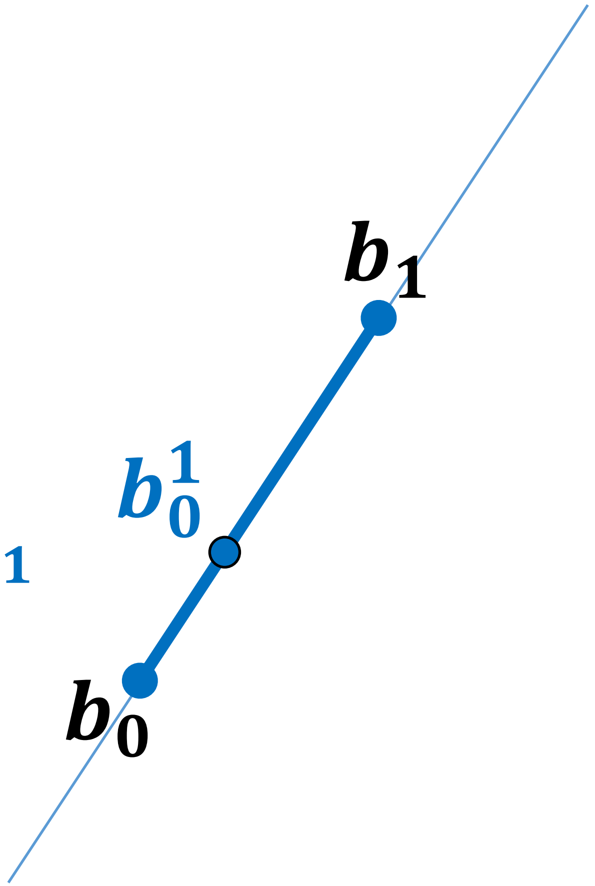


$$\mathbf{f}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Back to the drawing board

- A point on a parametric line

$$\boldsymbol{b}_0^1 = (1-t)\boldsymbol{b}_0 + t\boldsymbol{b}_1$$

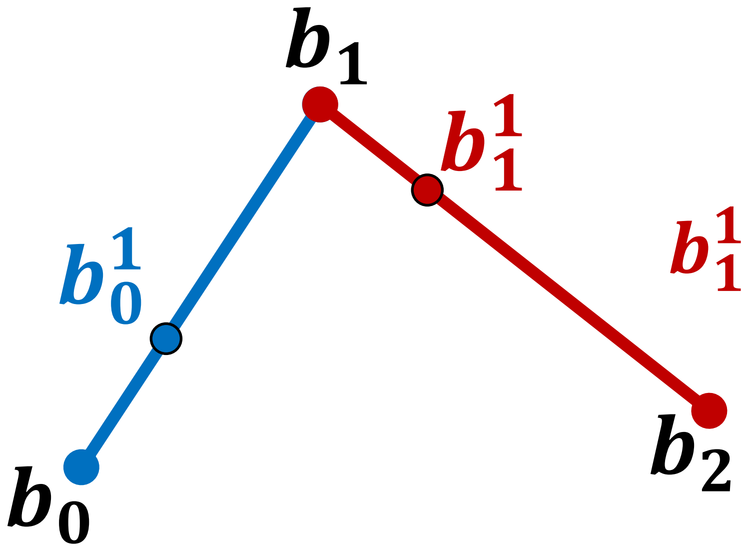


Back to the drawing board

- Another point on a second parametric line

$$\boldsymbol{b}_1^1 = (1-t)\boldsymbol{b}_1 + t\boldsymbol{b}_2$$

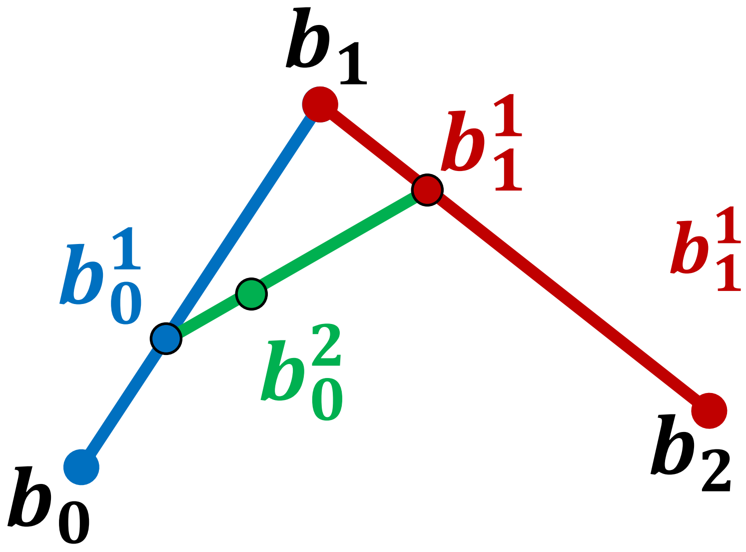
$$\boldsymbol{b}_0^1 = (1-t)\boldsymbol{b}_0 + t\boldsymbol{b}_1$$



Back to the drawing board

- A third point on the line defined by the first two points

$$\boldsymbol{b}_0^1 = (1-t)\boldsymbol{b}_0 + t\boldsymbol{b}_1$$

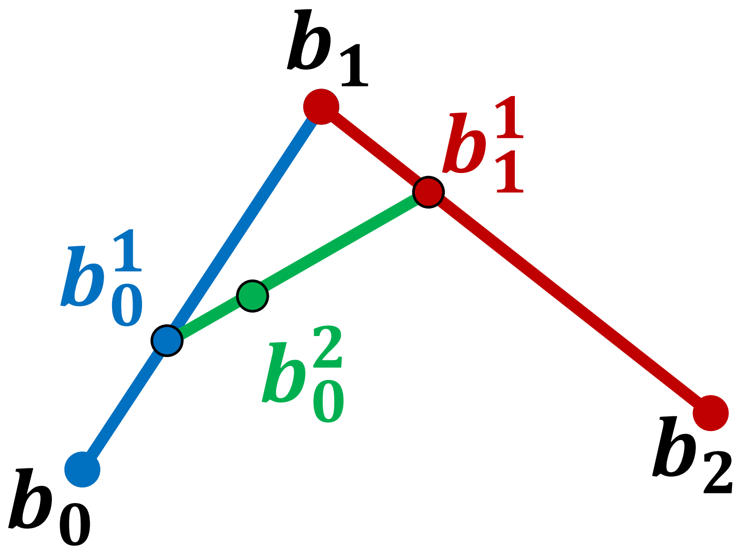


$$\boldsymbol{b}_1^1 = (1-t)\boldsymbol{b}_1 + t\boldsymbol{b}_2$$

$$\boldsymbol{b}_0^2 = (1-t)\boldsymbol{b}_0^1 + t\boldsymbol{b}_1^1$$

Back to the drawing board

- And then simplify⋯



$$\boldsymbol{b}_0^1 = (1-t)\boldsymbol{b}_0 + t\boldsymbol{b}_1$$

$$\boldsymbol{b}_0^2 = (1-t)\boldsymbol{b}_0^1 + t\boldsymbol{b}_1^1$$

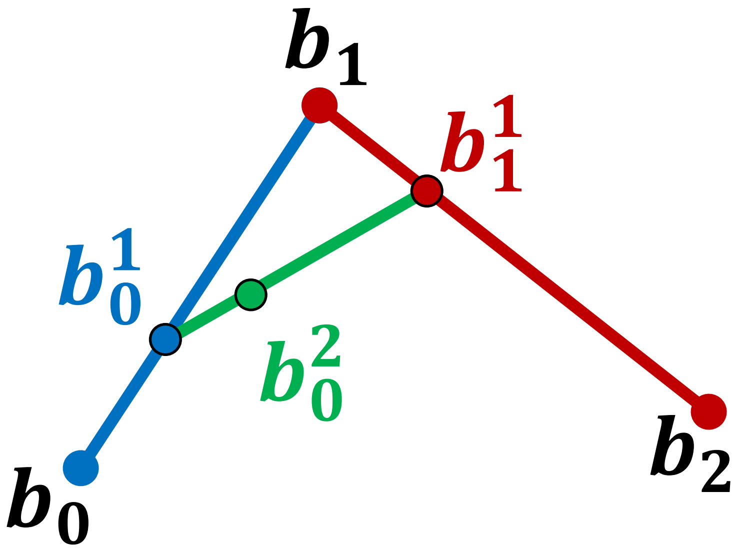
$$\boldsymbol{b}_1^1 = (1-t)\boldsymbol{b}_1 + t\boldsymbol{b}_2$$

$$\boldsymbol{b}_0^2 = (1-t)[(1-t)\boldsymbol{b}_0 + t\boldsymbol{b}_1] + t[(1-t)\boldsymbol{b}_1 + t\boldsymbol{b}_2]$$

$$\boxed{\boldsymbol{b}_0^2 = (1-t)^2\boldsymbol{b}_0 + 2t(1-t)\boldsymbol{b}_1 + t^2\boldsymbol{b}_2}$$

Back to the drawing board

- We obtained another description of parabolic curves
- The coefficients $\boldsymbol{b_0}, \boldsymbol{b_1}, \boldsymbol{b_2}$ have a geometric meaning



$$\boxed{\boldsymbol{b_0^2} = (1-t)^2\boldsymbol{b_0} + 2t(1-t)\boldsymbol{b_1} + t^2\boldsymbol{b_2}}$$

Example re-visited

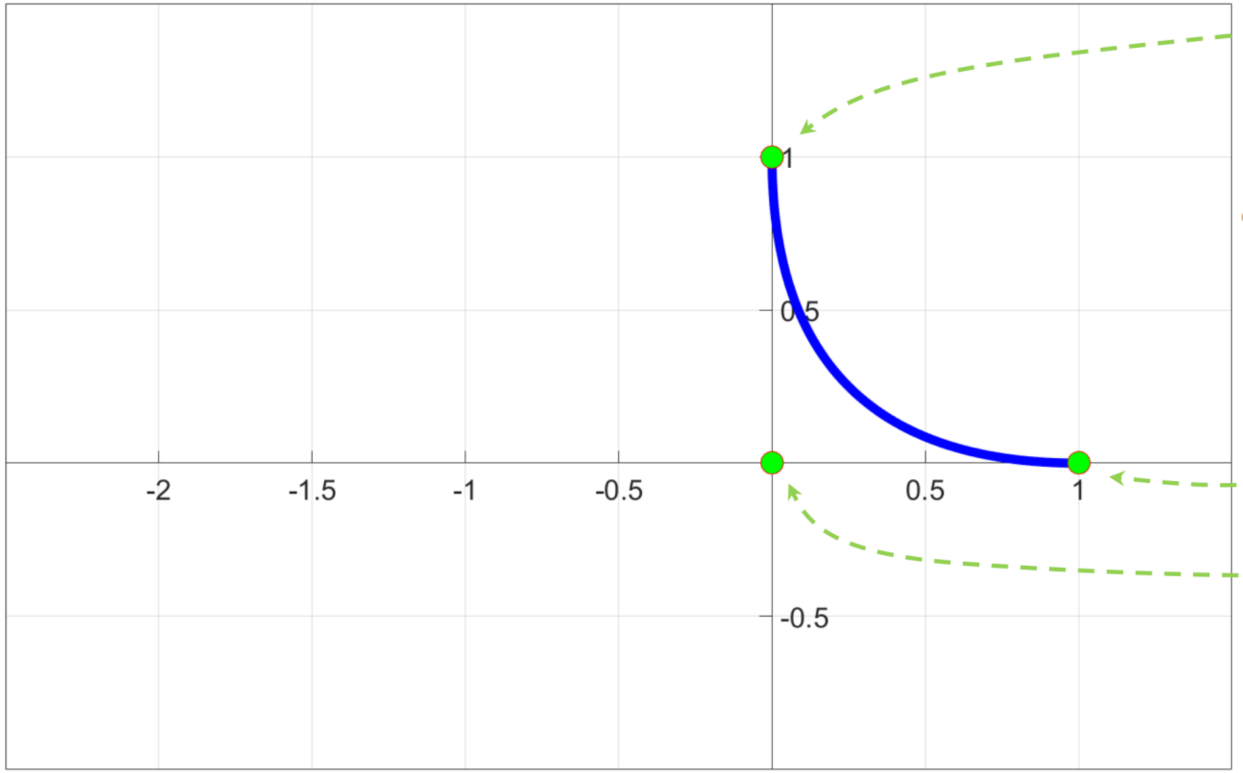
- Let's rewrite our initial parabolic curve example in the new basis

$$\boldsymbol{f}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + \begin{pmatrix} -2 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{f}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1-t)^2 + \begin{pmatrix} 0 \\ 0 \end{pmatrix} 2t(1-t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} t^2$$

Example re-visited

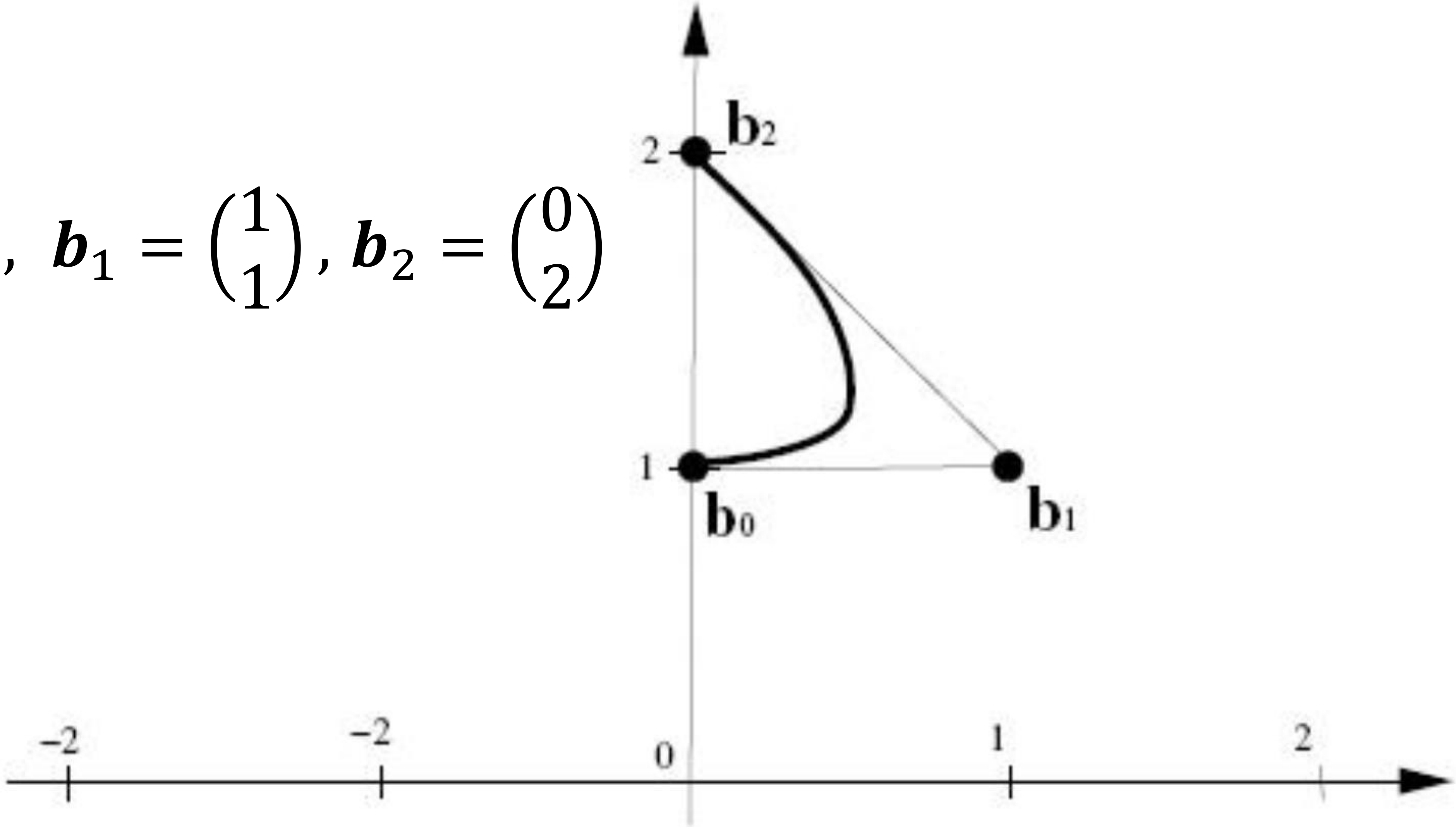
- The coefficient have a geometric meaning
- More intuitive for curve manipulation



$$f(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1-t)^2 + \begin{pmatrix} 0 \\ 0 \end{pmatrix}2t(1-t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}t^2$$

Another example

$$\boldsymbol{b}_0 = \binom{0}{1},\ \boldsymbol{b}_1 = \binom{1}{1}, \boldsymbol{b}_2 = \binom{0}{2}$$



Going further

- Cubic approximation
- Given 4 points: $\boldsymbol{p}_0^0(t) = \boldsymbol{p}_0,\ \ \boldsymbol{p}_1^0(t) = \boldsymbol{p}_1,\ \ \boldsymbol{p}_2^0(t) = \boldsymbol{p}_2,\ \ \boldsymbol{p}_3^0(t) = \boldsymbol{p}_3$
- First iteration

$$\boldsymbol{p}_0^1 = (1-t)\boldsymbol{p}_0 + t\boldsymbol{p}_1$$

$$\boldsymbol{p}_1^1 = (1-t)\boldsymbol{p}_1 + t\boldsymbol{p}_2$$

$$\boldsymbol{p}_2^1 = (1-t)\boldsymbol{p}_2 + t\boldsymbol{p}_3$$

- 2nd iteration

$$\boldsymbol{p}_0^2 = (1-t)^2\boldsymbol{p}_0 + 2t(1-t)\boldsymbol{p}_1 + t^2\boldsymbol{p}_2$$

$$\boldsymbol{p}_1^2 = (1-t)^2\boldsymbol{p}_1 + 2t(1-t)\boldsymbol{p}_2 + t^2\boldsymbol{p}_3$$

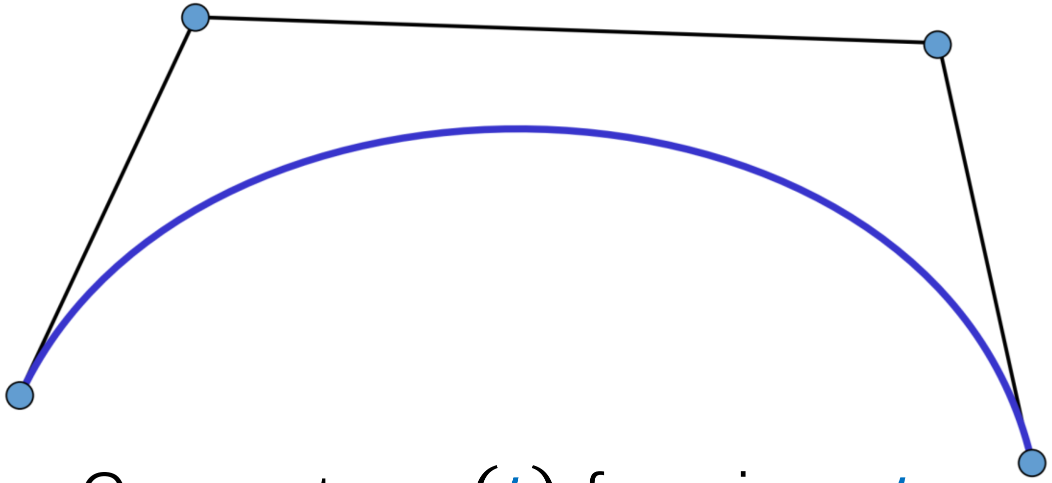
- Curve

$$\boldsymbol{c}(t) = (1-t)^3\boldsymbol{p}_0 + 3t(1-t)^2\boldsymbol{p}_1 + 3t^2(1-t)\boldsymbol{p}_2 + t^3\boldsymbol{p}_3$$

Throughout these examples, we just re-invented a primitive version of the de Casteljau algorithm

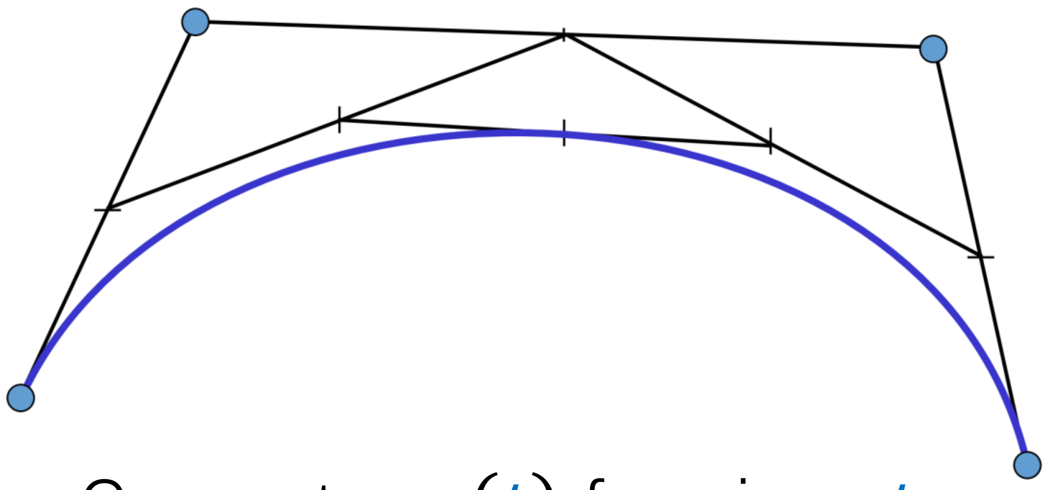
Now let's examine it more closely …

De Casteljau algorithm



- De Casteljau Algorithm: Computes $x(t)$ for given t
  - Bisect control polygon in ratio $t : (1 - t)$
  - Connect the new dots with lines (adjacent segments)
  - Interpolate again with the same ratio
  - Iterate, until only one points is left

De Casteljau algorithm



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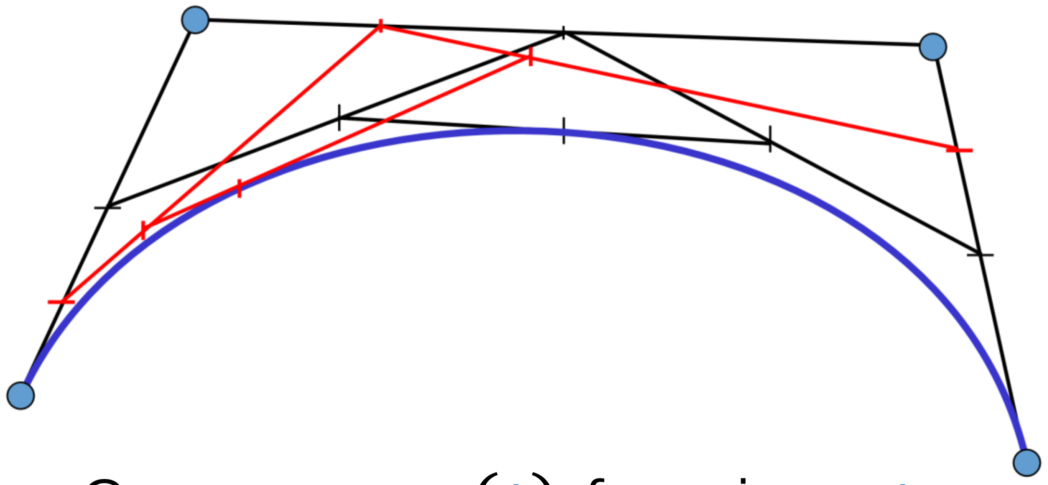

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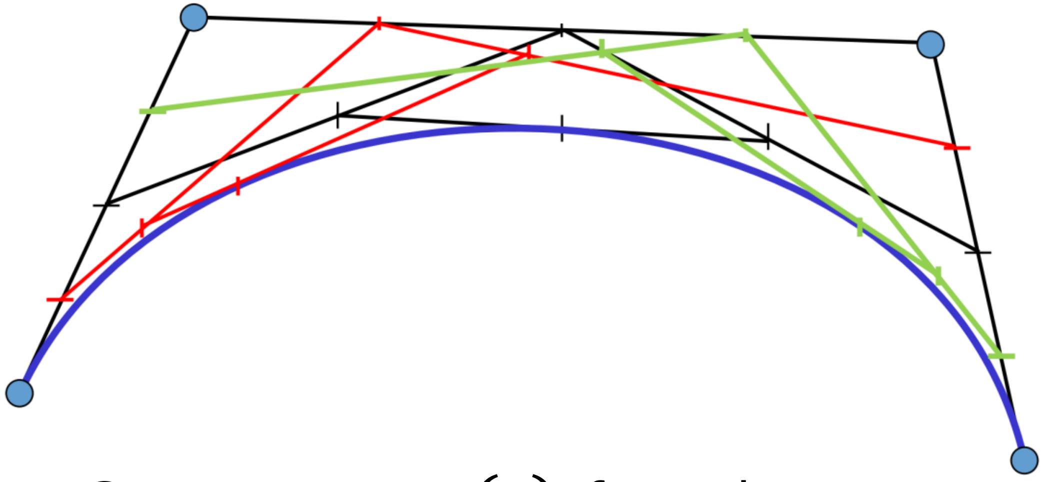

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  - Connect the new dots with lines (adjacent segments)
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De Casteljau algorithm

- Algorithm description
  - Input: points $\boldsymbol{b}_0, \boldsymbol{b}_1, \ldots \boldsymbol{b}_n \in \mathbb{R}^3$
  - Output: curve $\boldsymbol{x}(t), t \in [0,1]$

  - Geometric construction of the points $\boldsymbol{x}(t)$ for given t:

$$\boldsymbol{b}_i^0(t) = \boldsymbol{b}_i, \qquad i = 0, \ldots, n$$

$$\boldsymbol{b}_i^r(t) = (1-t)\boldsymbol{b}_i^{r-1}(t) + t\,\boldsymbol{b}_{i+1}^{r-1}(t)$$

$$r = 1, \ldots, n \qquad i = 0, \ldots, n-r$$

  - Then $\boldsymbol{b}_0^n(t)$ is the searched curve point $\boldsymbol{x}(t)$ at the parameter value t

De Casteljau algorithm

- Repeated convex combination of control points

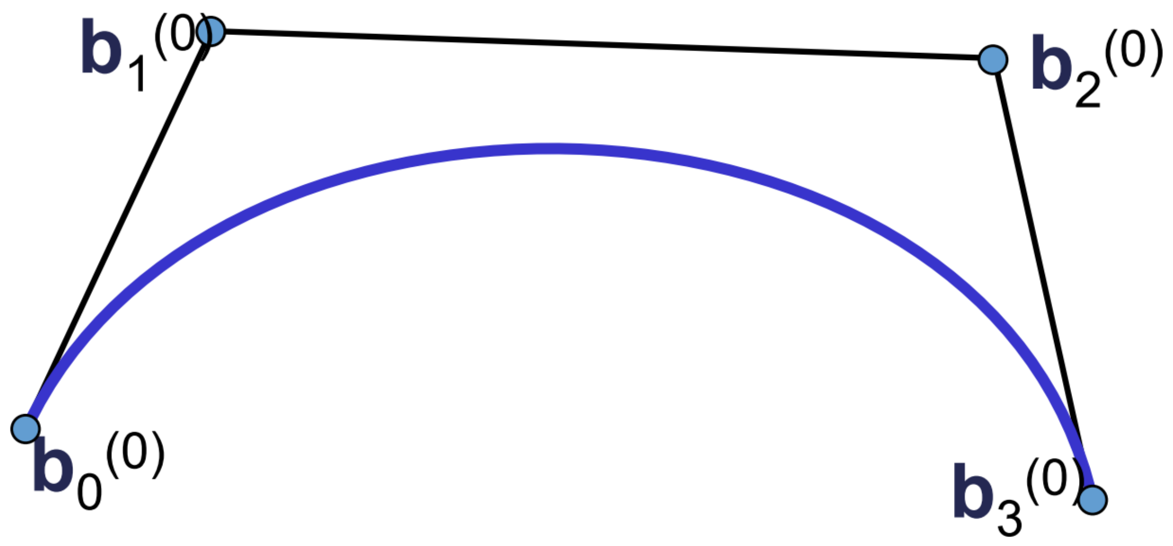
$$\boldsymbol{b}_i^{(r)} = (1-t)\boldsymbol{b}_i^{(r-1)} + t\boldsymbol{b}_{i+1}^{(r-1)}$$

$\boldsymbol{b}_0^{(0)}$

$\boldsymbol{b}_1^{(0)}$

$\boldsymbol{b}_2^{(0)}$

$\boldsymbol{b}_3^{(0)}$

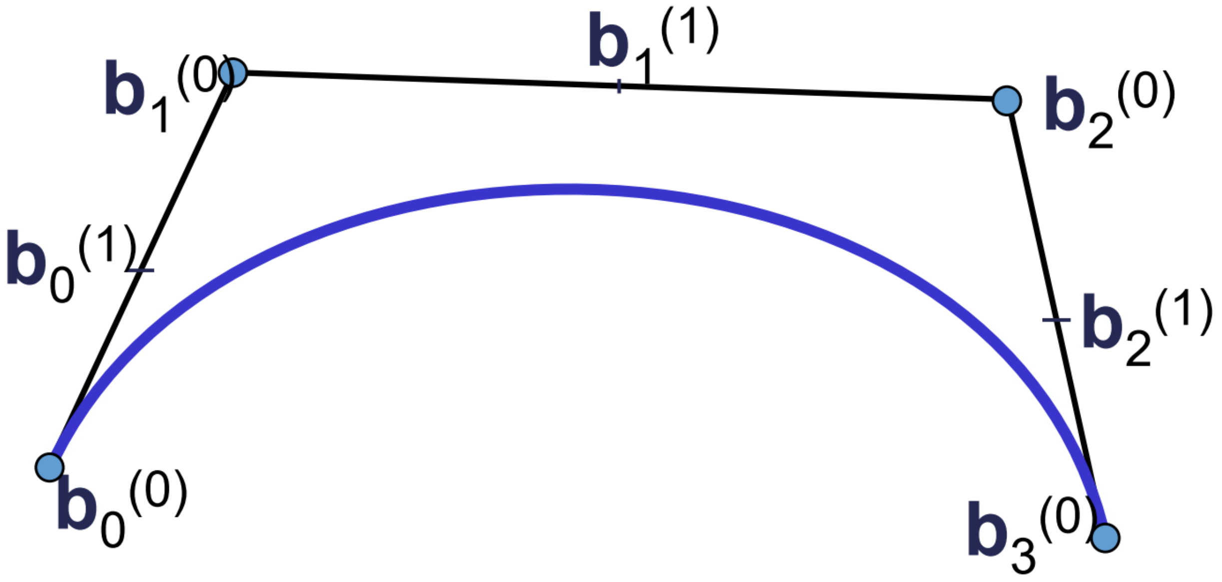


De Casteljau algorithm

- Repeated convex combination of control points

$$\boldsymbol{b}_i^{(r)} = (1 - t)\boldsymbol{b}_i^{(r-1)} + t\boldsymbol{b}_{i+1}^{(r-1)}$$

$$\begin{tikzcd}
\boldsymbol{b}_0^{(0)} \arrow[dr, "1-t"] & \\
\boldsymbol{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(1)} \\
\boldsymbol{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_1^{(1)} \\
\boldsymbol{b}_3^{(0)} \arrow[r, "t"] & \boldsymbol{b}_2^{(1)}
\end{tikzcd}$$



De Casteljau algorithm

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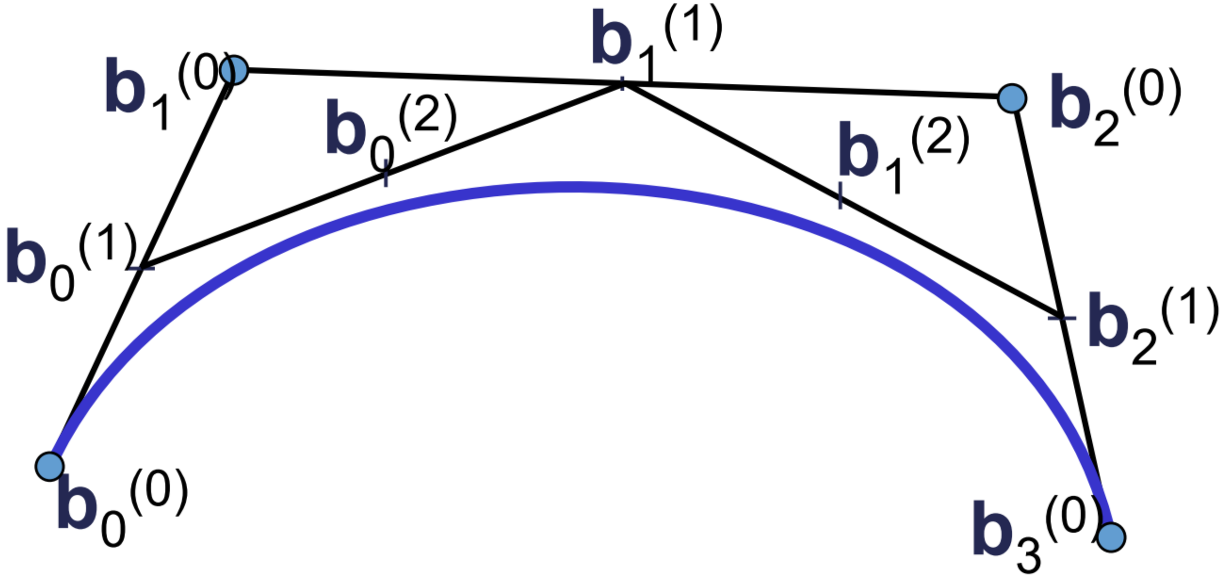
\boldsymbol{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(1)} \arrow[dr, "1-t"] & \\
\boldsymbol{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_1^{(1)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(2)} \\
\boldsymbol{b}_3^{(0)} \arrow[r, "t"] & \boldsymbol{b}_2^{(1)} \arrow[r, "t"] & \boldsymbol{b}_1^{(2)}
\end{tikzcd}$$



De Casteljau algorithm

- Repeated convex combination of control points

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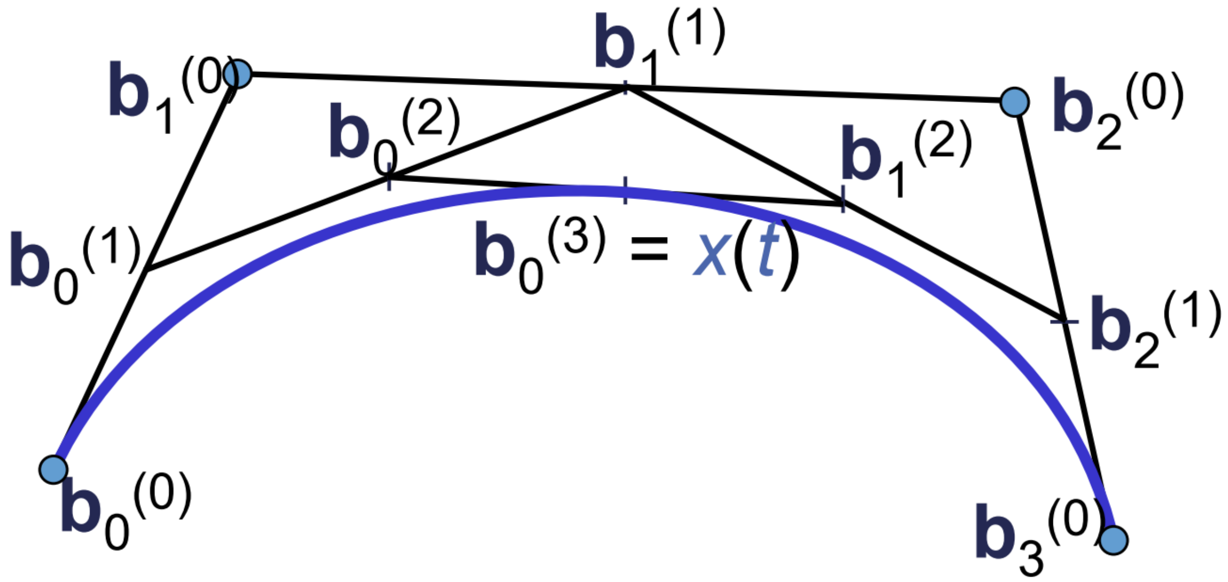
$$\begin{tikzcd}
\boldsymbol{b}_0^{(0)} \arrow[dr, "1-t"] & & & \\
\boldsymbol{b}_1^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(1)} \arrow[dr, "1-t"] & & \\
\boldsymbol{b}_2^{(0)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_1^{(1)} \arrow[r, "t"] \arrow[dr, "1-t"] & \boldsymbol{b}_0^{(2)} \arrow[dr, "1-t"] & \\
\boldsymbol{b}_3^{(0)} \arrow[r, "t"] & \boldsymbol{b}_2^{(1)} \arrow[r, "t"] & \boldsymbol{b}_1^{(2)} \arrow[r, "t"] & \boldsymbol{b}_0^{(3)} = x(t)
\end{tikzcd}$$

De Casteljau scheme

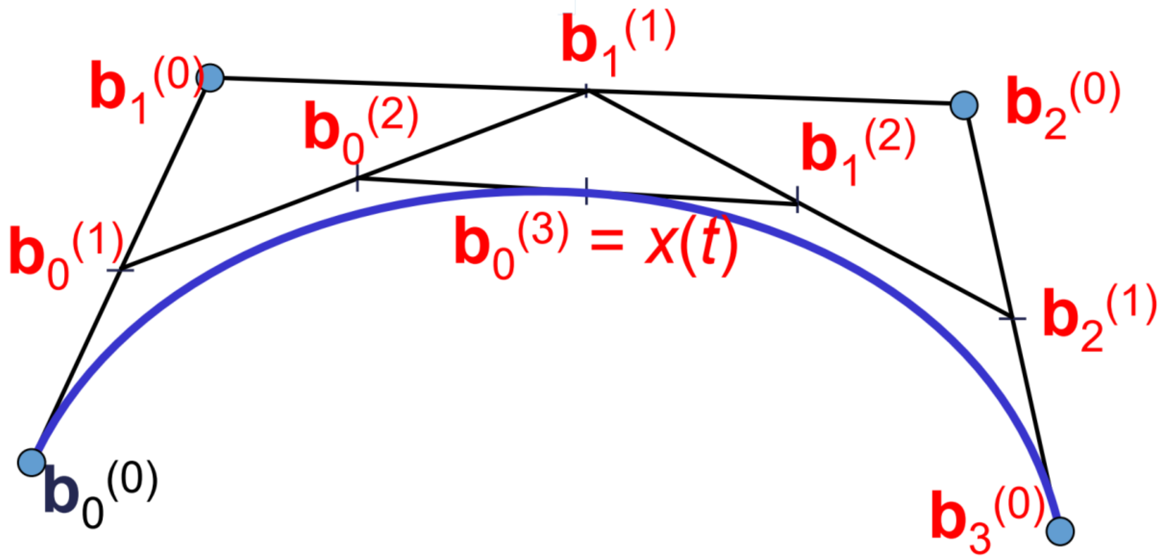


De Casteljau algorithm

- The intermediate coefficients $\boldsymbol{b}_i^r(t)$ can be written in a triangular matrix: the de Casteljau scheme:

$$
\begin{array}{llllll}
\boldsymbol{b}_0 = \boldsymbol{b}_0^0 & & & & & \\
\boldsymbol{b}_1 = \boldsymbol{b}_1^0 & \boldsymbol{b}_0^1 & & & & \\
\boldsymbol{b}_2 = \boldsymbol{b}_2^0 & \boldsymbol{b}_1^1 & \boldsymbol{b}_0^2 & & & \\
\boldsymbol{b}_3 = \boldsymbol{b}_3^0 & \boldsymbol{b}_2^1 & \boldsymbol{b}_1^2 & \boldsymbol{b}_0^3 & & \\
\quad\quad \cdots\cdots\cdots\cdots\cdots & & & & & \\
\boldsymbol{b}_{n-1} = \boldsymbol{b}_{n-1}^0 & \boldsymbol{b}_{n-2}^1 & \ldots & \boldsymbol{b}_0^{n-1} & & \\
\boldsymbol{b}_n \quad = \boldsymbol{b}_n^0 & \boldsymbol{b}_{n-1}^1 & \ldots & \boldsymbol{b}_1^{n-1} & \boldsymbol{b}_0^n = x(t) &
\end{array}
$$

De Casteljau algorithm



Algorithm:

for r=1..n

for i=0..n-r

$$\boldsymbol{b}_i^{(r)} = (1-t)\,\boldsymbol{b}_i^{(r-1)} + t\,\boldsymbol{b}_{i+1}^{(r-1)}$$

end

end

return $\boldsymbol{b}_0^{(n)}$

The whole algorithm consists only of repeated linear interpolations.

De Casteljau algorithm: Properties

- The polygon consisting of the points $\boldsymbol{b_0}, \dots, \boldsymbol{b_n}$ is called Bézier polygon (control polygon)
- The points $\boldsymbol{b_i}$ are called Bézier points (control points)
- The curve defined by the Bézier points $\boldsymbol{b_0}, \dots, \boldsymbol{b_n}$ and the de Casteljau algorithm is called Bézier curve
- The de Casteljau algorithm is numerically stable, since only convex combinations are applied.
- Complexity of the de Casteljau algorithm
  - $O(n^2)$ time
  - $O(n)$ memory
  - with n being the number of Bézier points

De Casteljau algorithm: Properties

- **Properties of Bézier curves:**
  - Given: Bézier points $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_n$
    Bézier curve $\boldsymbol{x}(t)$
  - Bézier curve is polynomial curve of degree n
  - End points interpolation: $\boldsymbol{x}(0) = \boldsymbol{b}_0,\ \boldsymbol{x}(1) = \boldsymbol{b}_n$. The remaining Bézier points are only approximated in general
  - Convex hull property:

  Bézier curve is completely inside the convex hull of its Bézier polygon

De Casteljau algorithm: Properties

- **Variation diminishing**
  - No line intersects the Bézier curve more often than its Bézier polygon
- **Influence of Bézier points**: global but pseudo-local
  - Global: moving a Bézier points changes the whole curve progression
  - Pseudo-local: $\boldsymbol{b}_i$ has its maximal influence on $x(t)$ at $t = \frac{i}{n}$
- **Affine invariance**:
  - Bézier curve and Bézier polygon are invariant under affine transformations
- **Invariance under affine parameter transformations**

De Casteljau algorithm: Properties

- **Symmetry**
  - The following two Bézier curves coincide, they are only traversed in opposite directions:

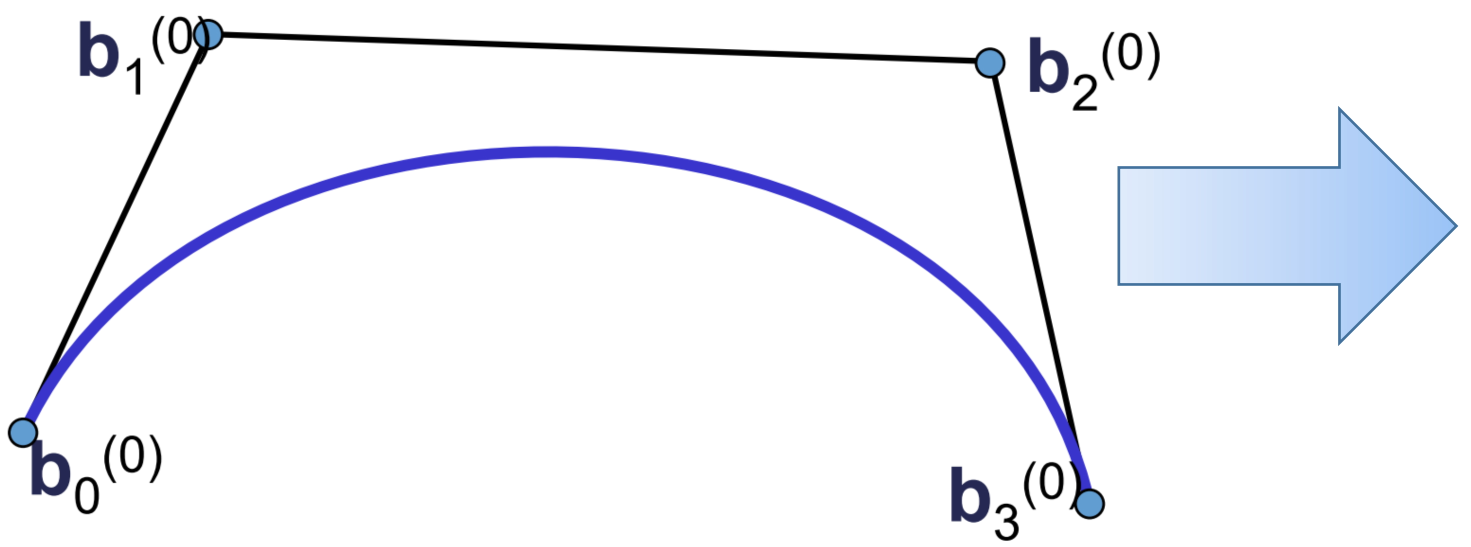
$$\boldsymbol{x}(t) = [\boldsymbol{b}_0, \ldots, \boldsymbol{b}_n] \qquad \boldsymbol{x}'(t) = [\boldsymbol{b}_n, \ldots \boldsymbol{b}_0]$$

- **Linear Precision:**
  - Bézier curve is line segment, if $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_n$ are colinear
- Invariance under barycentric combinations

Bézier Curves

Towards a polynomial description

Bézier Curves
Towards a polynomial description



$$x(t) = \sum_{i=0}^{n} B_i^n(t) \cdot b_i$$

Polynomial description of Bézier curves

- The same problem as before:
  - Given: $(n + 1)$ control points $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_n$
  - Wanted: Bézier curve $\boldsymbol{x}(t)$ with $t \in [0,1]$

- Now with an algebraic approach using basis functions

Desirable Properties

- Useful requirements for a basis:
  - Well behaved curve
    - Smooth basis functions

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- Useful requirements for a basis:
  - Well behaved curve
    - Smooth basis functions
  - Local control (or at least semi-local)
    - Basis functions with compact support
  - Affine invariance:
    - Appling an affine map $\boldsymbol{x} \rightarrow A\boldsymbol{x} + b$ on
      - Control points
      - Curve

    Should have the same effect
    - In particular: rotation, translation
    - Otherwise: interactive curve editing very difficult

Desirable Properties

- Useful requirements for a basis:
  - Convex hull property:
    - The curve lays within the convex hull of its control points
    - Avoids at least too weird oscillations
  - Advantages
    - Computational advantages (recursive intersection tests)
    - More predictable behavior

Summary

- Useful properties
  - Smoothness
  - Local control / support
  - **Affine invariance**
  - **Convex hull property**

Notations

Curve **basis function** **control points**

$$\boldsymbol{f}(t) = \sum_{i=1}^{n} b_i(t)\boldsymbol{p}_i$$

Affine Invariance

- Affine map: $\boldsymbol{x} \rightarrow A\boldsymbol{x} + \boldsymbol{b}$
- **Part I:** Linear invariance – we get this automatically
  - Linear approach: $\boldsymbol{f}(t) = \sum_{i=1}^{n} b_i(t)\boldsymbol{p}_i = \sum_{i=1}^{n} b_i(t) \begin{pmatrix} p_i^{(x)} \\ p_i^{(y)} \\ p_i^{(z)} \end{pmatrix}$
  - Therefore: $A\big(\boldsymbol{f}(t)\big) = A\left(\sum_{i=1}^{n} b_i(t)\boldsymbol{p}_i\right) = \sum_{i=1}^{n} b_i(t)(A\boldsymbol{p}_i)$

Affine Invariance

- Affine Invariance:
  - Affine map: $\boldsymbol{x} \to A\boldsymbol{x} + \boldsymbol{b}$
  - **Part II:** Translational invariance

$$\sum_{i=1}^{n} b_i(t)(\boldsymbol{p}_i + \boldsymbol{b}) = \sum_{i=1}^{n} b_i(t)\boldsymbol{p}_i + \sum_{i=1}^{n} b_i(t)\boldsymbol{b} = \boldsymbol{f}(t) + \left(\sum_{i=1}^{n} b_i(t)\right)\boldsymbol{b}$$

    - For translational invariance, the sum of the basis functions must be one *everywhere* (for all parameter values t that are used).
    - This is called "partition of unity property"
    - The b_i's form an "affine combination" of the control points $\boldsymbol{p}_i$
    - This is very important for modeling

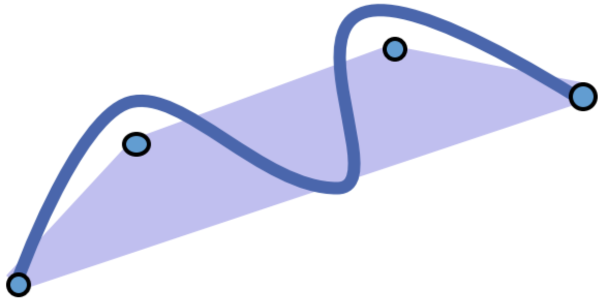
Convex Hull Property

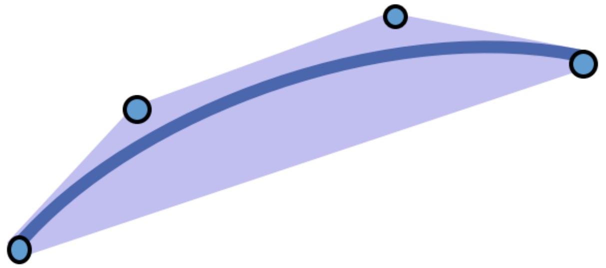
- Convex combinations:
  - A convex combination of a set of points $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ is any point of the form:

$$\sum_{i=1}^{n} \lambda_i \boldsymbol{p}_{\boldsymbol{i}} \text{ with } \sum_{i=1}^{n} \lambda_i = 1 \text{ and } \forall i = 1 \ldots n\colon 0 \leq \lambda_i \leq 1$$

    - (Remark: $\lambda_i \leq 1$ is redundant)
  - The set of all admissible convex combinations forms the convex hull of the point set
    - Easy to see (exercise): The convex hull is the smallest set that contains all points $\{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n\}$ and every complete straight line between two elements of the set

Convex Hull Property





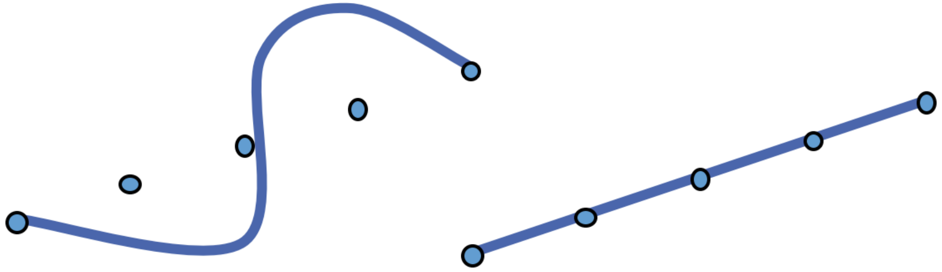
- Accordingly:
  - If we have this property
    $\forall t \in \Omega: \sum_{i=1}^{n} b_i(t) = 1$ and $\forall t \in \Omega, \forall i: b_i(t) \geq 0$
    the constructed curves / surfaces will be:
    - Affine invariant (translations, linear maps)
    - Be restricted to the convex hull of the control points

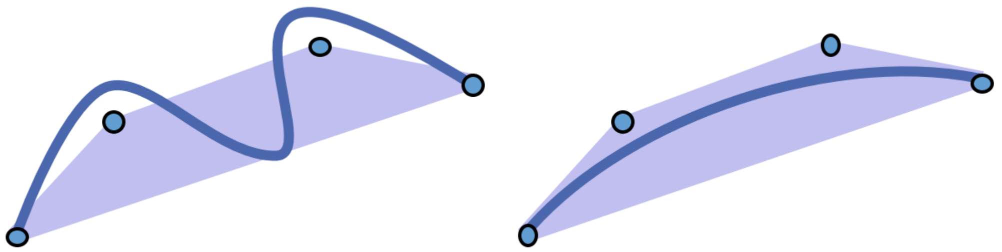
  - Corollary: Curves will have *linear precision*
    - All control points lie on a straight line

    $\Rightarrow$ Curve is a straight line segment
  - Surfaces with planar control points will be flat, too

Convex Hull Property

- Very useful property in practice
  - Avoids at least the worst oscillations
    - no escape from convex hull, unlike polynomial interpolation
  - Linear precision property is intuitive (people expect this)
  - Can be used for fast range checks
    - Test for intersection with convex hull first, then the object
    - Recursive intersection algorithms in conjunctions with subdivision rules (more on this later)





Polynomial description of Bézier curves

- The same problem as before:
  - Given: $(n+1)$ control points $\boldsymbol{b}_0, \dots, \boldsymbol{b}_n$
  - Wanted: Bézier curve $x(t)$ with $t \in [0,1]$
- Now with an algebraic approach using basis functions
- Need to define $n+1$ basis functions
  - Such that this describes a Bézier curve:

$$B_0^n(t), \dots, B_n^n(t) \text{ over } [0,1]$$

$$\boldsymbol{x}(t) = \sum_{i=0}^{n} B_i^n(t) \cdot \boldsymbol{b}_i$$

Bernstein Basis

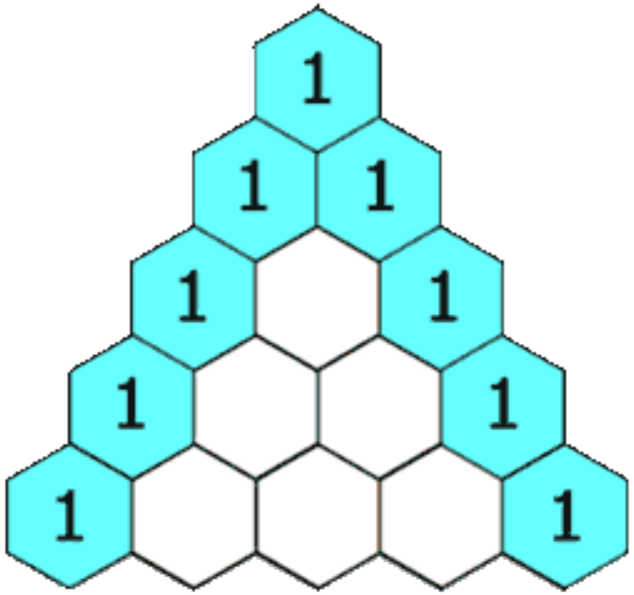
- Let's examine the Bernstein basis: $B = \{B_0^{(n)}, B_1^{(n)}, \dots, B_n^{(n)}\}$
  - Bernstein basis of degree n:

$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i} = B_{i\text{–th basis function}}^{(\text{degree})}$$

where the binomial coefficients are given by:

$$\binom{n}{i} = \begin{cases} \dfrac{n!}{(n-i)!\,i!} & \text{for } 0 \le i \le n \\ 0 & \text{otherwise} \end{cases}$$

Binomial Coefficients and Theorem



$$\binom{n}{i} + \binom{n}{i+1} = \binom{n+1}{i+1}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Examples: The first few

$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

- The first three Bernstein bases:

$$B_0^{(0)} := 1$$

$$B_0^{(1)} := 1-t \qquad B_1^{(1)} := t$$

$$B_0^{(2)} := (1-t)^2 \qquad B_1^{(2)} := 2t(1-t) \qquad B_2^{(2)} := t^2$$

$$B_0^{(3)} := (1-t)^3 \qquad B_1^{(3)} := 3t(1-t)^2 \qquad B_2^{(3)} := 3t^2(1-t) \qquad B_3^{(3)} := t^3$$

Examples: The first few

$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

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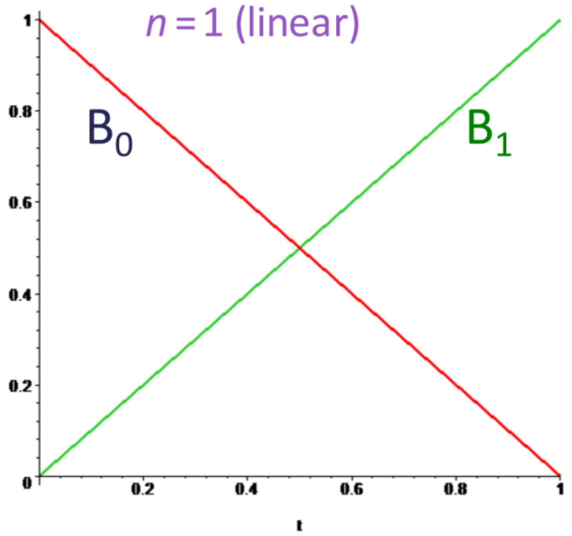
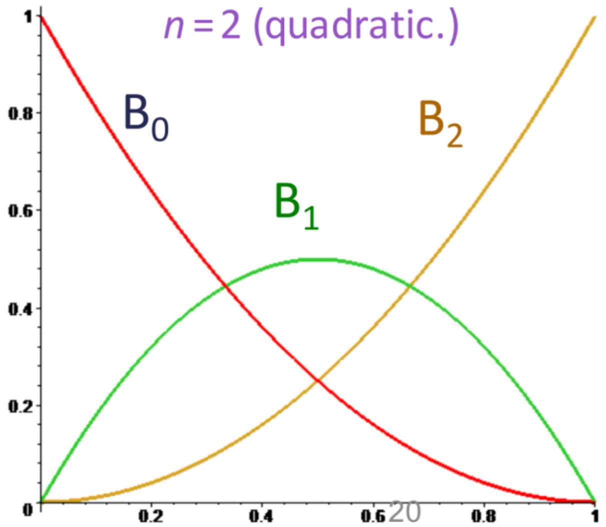
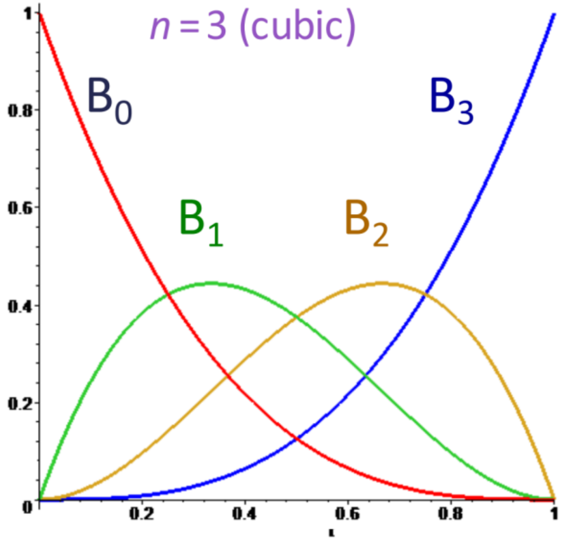


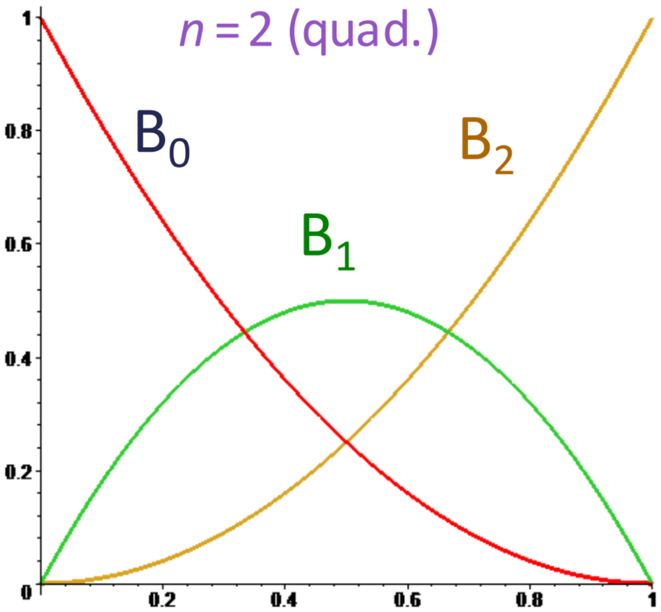


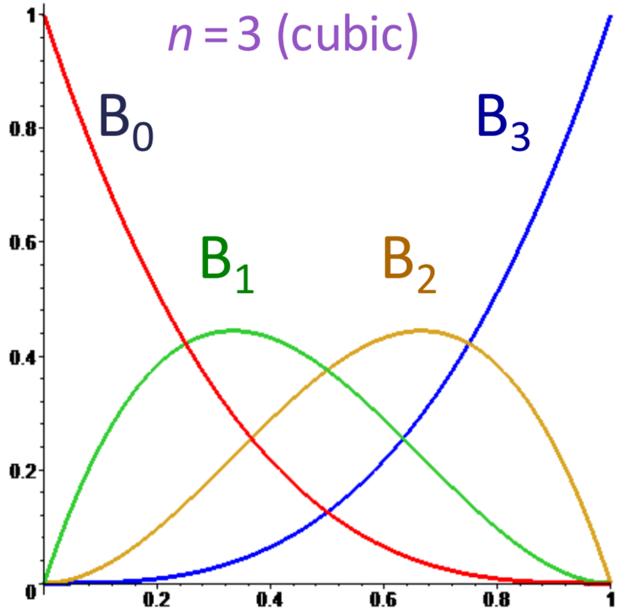


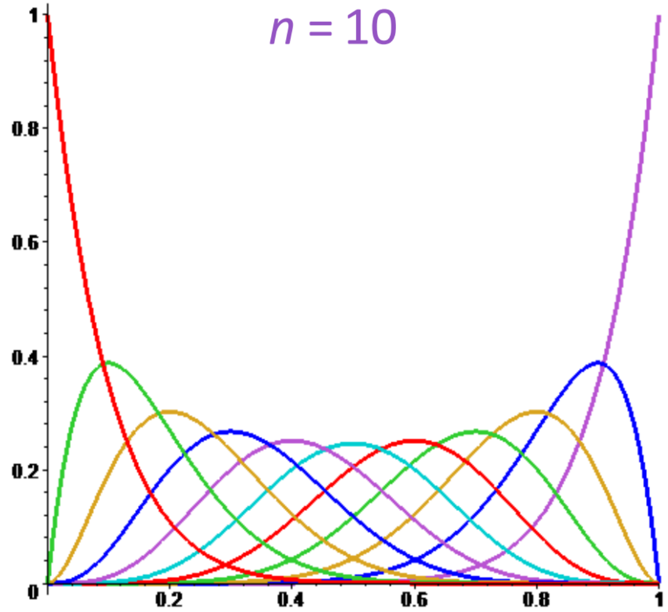
Bernstein Basis

- Bézier curves use the Bernstein basis: $B = \left\{B_0^{(n)}, B_1^{(n)}, \ldots, B_n^{(n)}\right\}$
  - Bernstein basis of degree n:

$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i} = B_{i\text{–th basis function}}^{(\text{degree})}$$







Bernstein Basis

- What about the desired properties?
    - Smoothness
    - Local control / support
    - Affine invariance
    - Convex hull property

Bernstein Basis: Properties

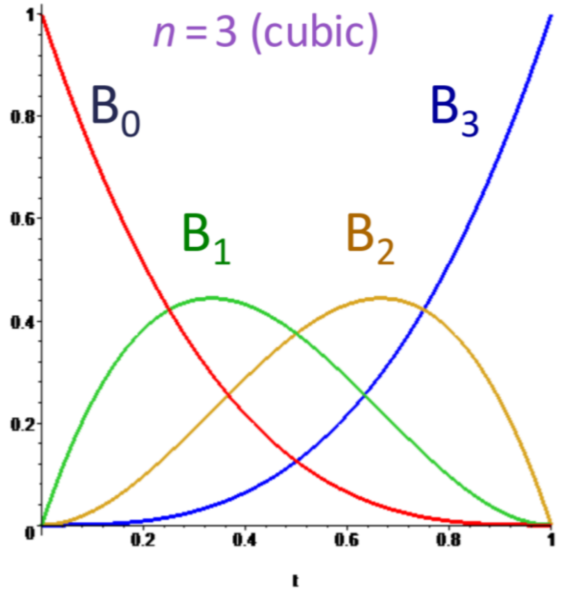
- $B = \left\{B_0^{(n)}, B_1^{(n)}, \dots, B_n^{(n)}\right\}$, $B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

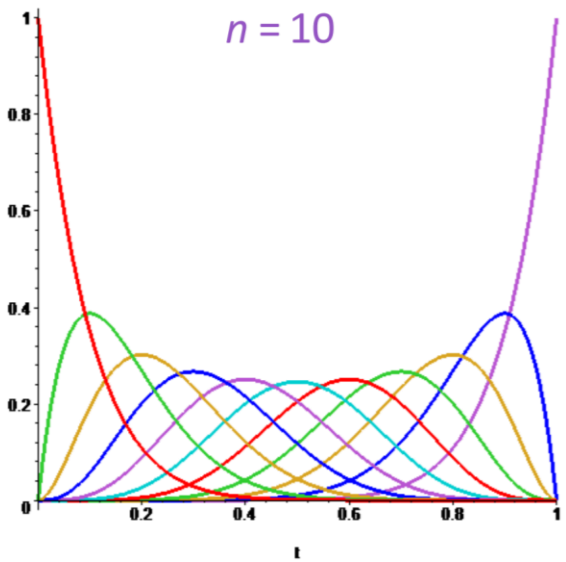
- Basis for polynomials of degree n

Smoothness

- Each basis function $B_i^{(n)}$ has its maximum at $t = \frac{i}{n}$

Local control (semi-local)





Bernstein Basis: Properties

- $B = \left\{B_0^{(n)}, B_1^{(n)}, \dots, B_n^{(n)}\right\}$, $B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

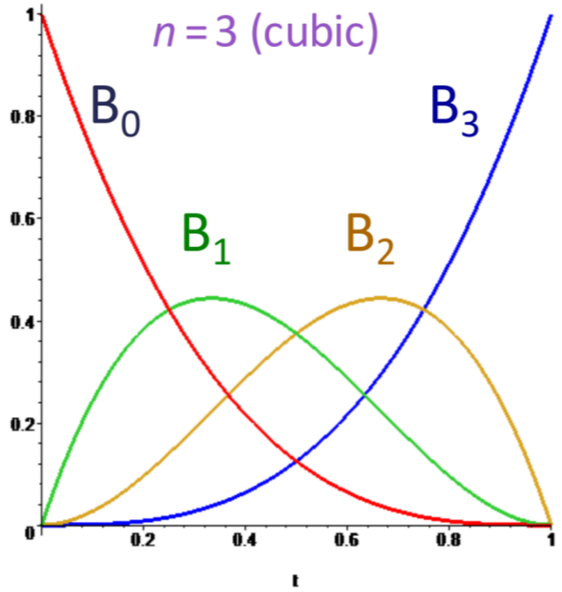
Affine invariance

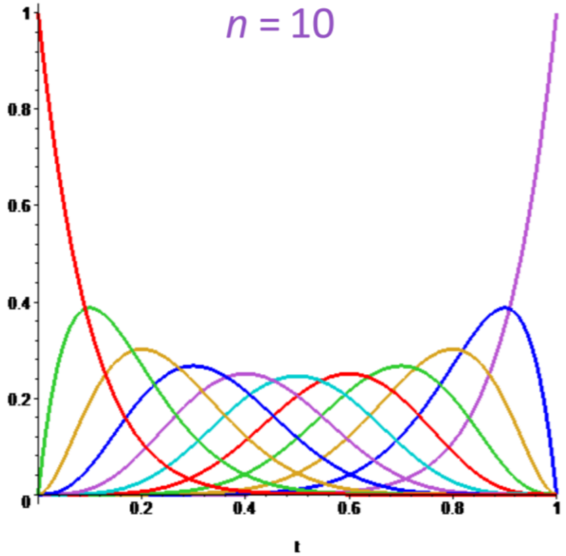
Convex hull property

- Partition of unity (binomial theorem)

$$1 = (1 - t + t)$$

$$\sum_{i=0}^{n} B_i^{(n)}(t) = \left(t + (1-t)\right)^n = 1$$





What about the desired properties?

• Smoothness	Yes
• Local control / support	To some extent
• Affine invariance	Yes
• Convex hull property	Yes

$$\binom{n-1}{i} + \binom{n-1}{i-1} = \binom{n}{i}$$

Bernstein Basis: Properties

- $B = \left\{B_0^{(n)}, B_1^{(n)}, \ldots, B_n^{(n)}\right\}, B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$
- Recursive computation

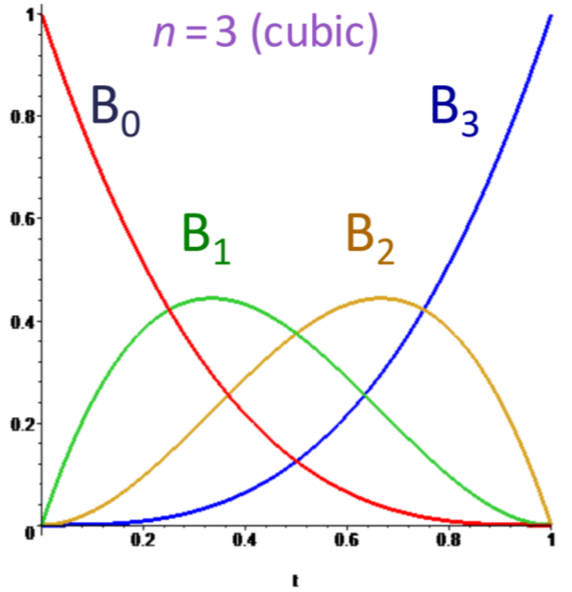
$$B_i^n(t) := (1-t)B_i^{(n-1)}(t) + tB_{i-1}^{(n-1)}(1-t)$$

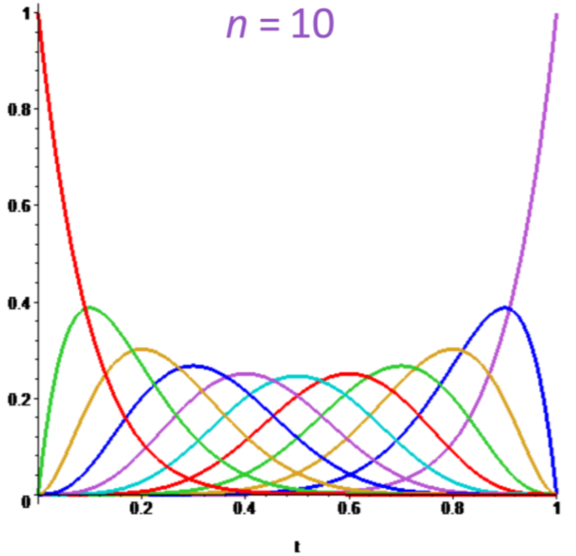
with $B_0^0(t) = 1, B_i^n(t) = 0$ for $i \notin \{0 \ldots n\}$

- Symmetry

$$B_i^n(t) = B_{n-i}^n(1-t)$$

- Non-negativity: $B_i^{(n)}(t) \geq 0$ for $t \in [0..1]$





Bernstein Basis: Properties

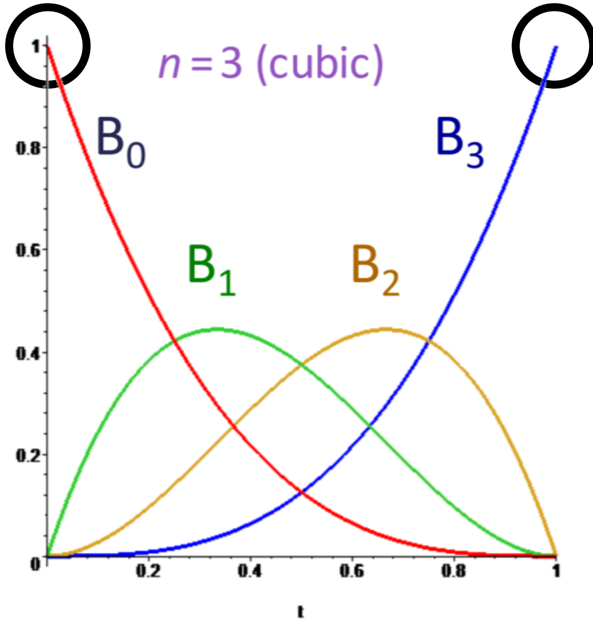
- $B = \left\{B_0^{(n)}, B_1^{(n)}, \ldots, B_n^{(n)}\right\}$, $B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

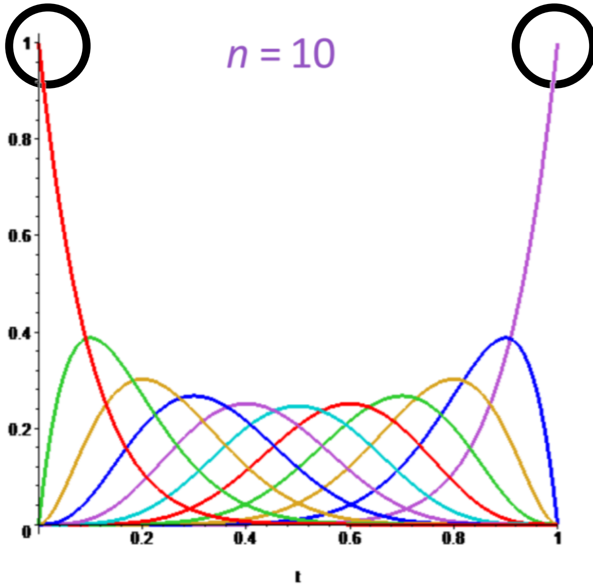
- Non-negativity II

$$B_i^n(t) > 0 \text{ for } 0 < t < 1$$

$$B_0^n(0) = 1, \qquad B_1^n(0) = \cdots = B_n^n(0) = 0$$

$$B_0^n(1) = \cdots = B_{n-1}^n(1) = 0, \qquad B_n^n(1) = 1$$





$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Derivatives

- Bernstein basis properties
  - Derivatives:

$$\frac{d}{dt} B_i^{(n)}(t) =$$

$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

Derivatives

- Bernstein basis properties
  - Derivatives:

$$\begin{aligned}
\frac{d}{dt} B_i^{(n)}(t) &= \binom{n}{i} \left(i t^{\{i-1\}} (1-t)^{n-i} - (n-i) t^i (1-t)^{\{n-i-1\}} \right) \\
&= \frac{n!}{(n-i)!\, i!} i t^{\{i-1\}} (1-t)^{n-i} - \frac{n!}{(n-i)!\, i!} (n-i) t^i (1-t)^{\{n-i-1\}} \\
&= n \left[\binom{n-1}{i-1} t^{\{i-1\}} (1-t)^{n-i} - \binom{n-1}{i} t^i (1-t)^{\{n-i-1\}} \right] \\
&= n \left[B_{i-1}^{(n-1)}(t) - B_i^{(n-1)}(t) \right]
\end{aligned}$$

(Notation: $\{\boldsymbol{k}\} = \boldsymbol{k}$ if $\boldsymbol{k} > \mathbf{0}$, zero otherwise)

Derivatives

$$B_i^{(n)}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

- Bernstein basis properties
  - Derivatives:

$$\frac{d^2}{dt^2} B_i^{(n)}(t) = \frac{d}{dt} n \left[B_{i-1}^{(n-1)}(t) - B_i^{(n-1)}(t) \right]$$

$$= n \left[(n-1) \left(B_{i-2}^{(n-2)}(t) - B_{i-1}^{(n-2)}(t) \right) - (n-1) \left(B_{i-1}^{(n-2)}(t) - B_i^{(n-2)}(t) \right) \right]$$

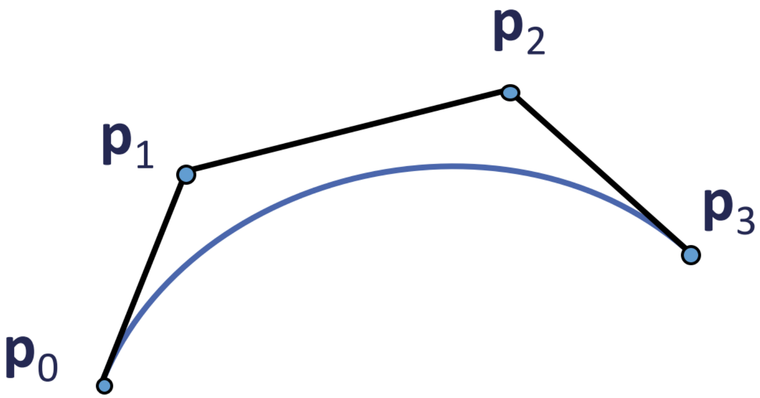
$$= n(n-1) \left[B_{i-2}^{(n-2)}(t) - 2B_{i-1}^{(n-2)}(t) + B_i^{(n-2)}(t) \right]$$

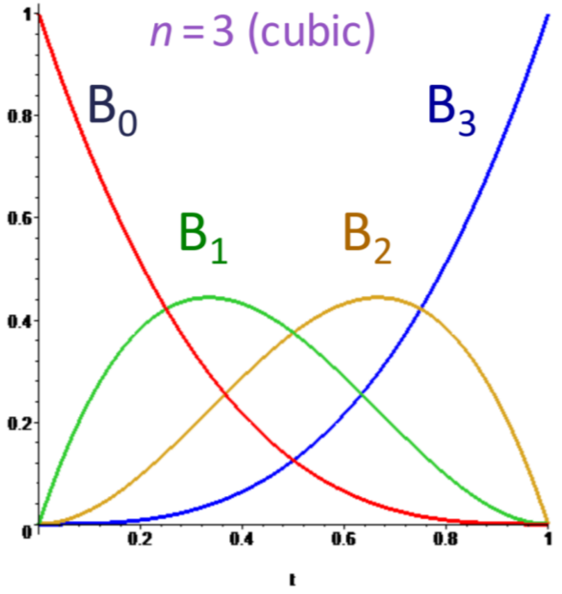
(Notation: $\{\boldsymbol{k}\} = \boldsymbol{k}$ if $\boldsymbol{k} > \mathbf{0}$, zero otherwise)

Bézier Curves in Bernstein form

- **Bézier Curves:**

$$\boldsymbol{f}(t) = \sum_{i=1}^{n} B_i^n \boldsymbol{p}_i \,, t \in [0,1]$$





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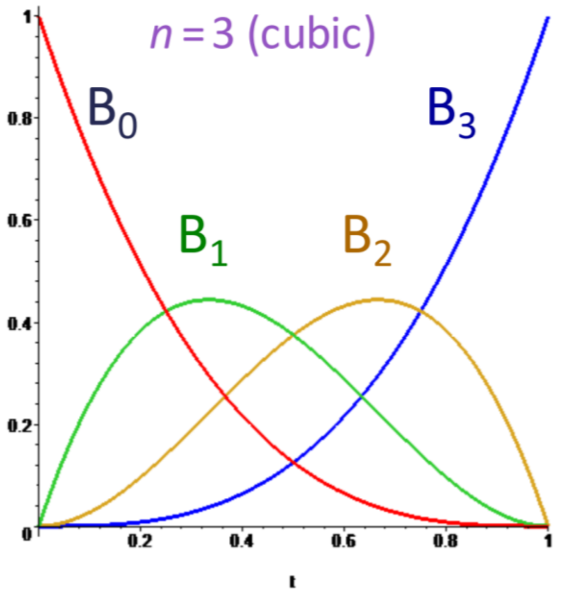

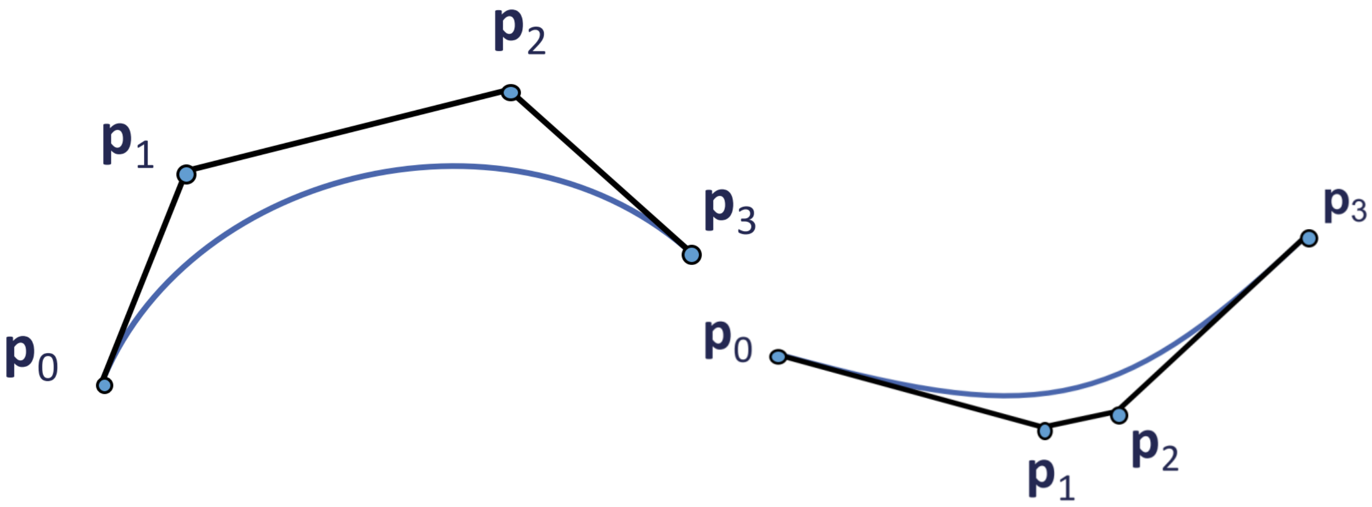

Bézier Curves in Bernstein form

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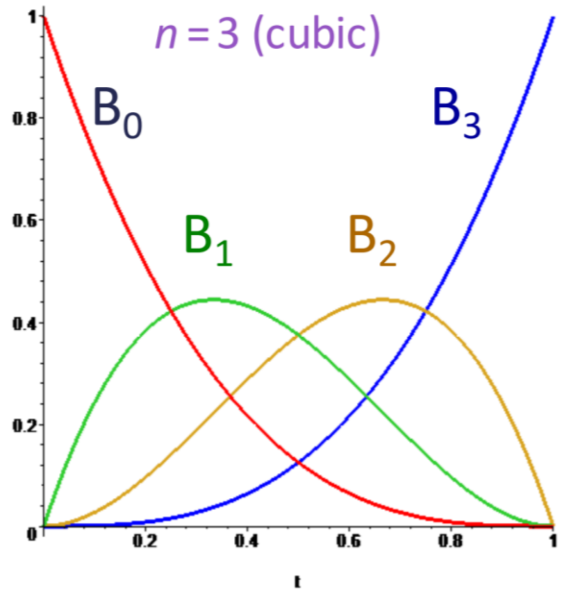
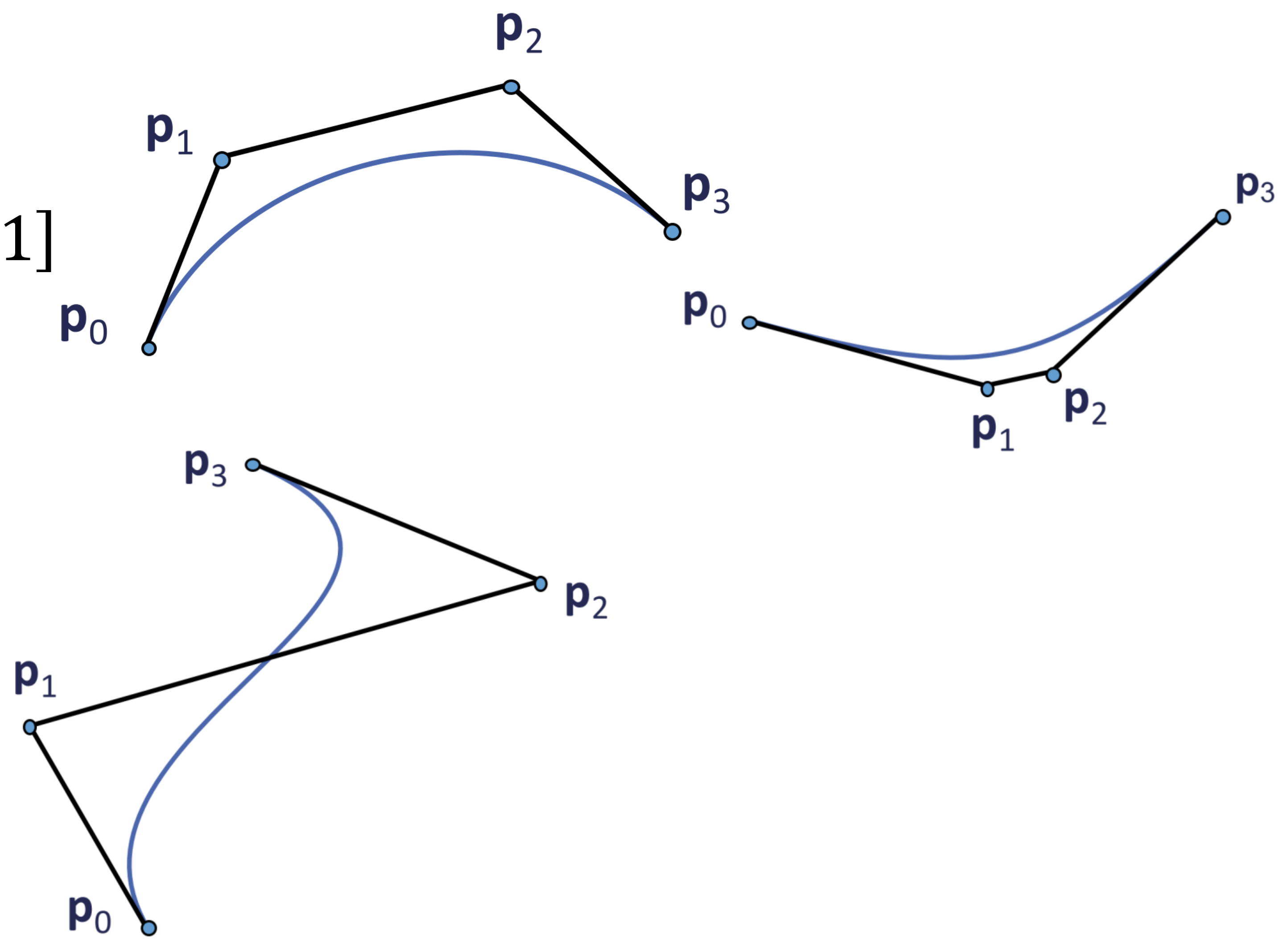
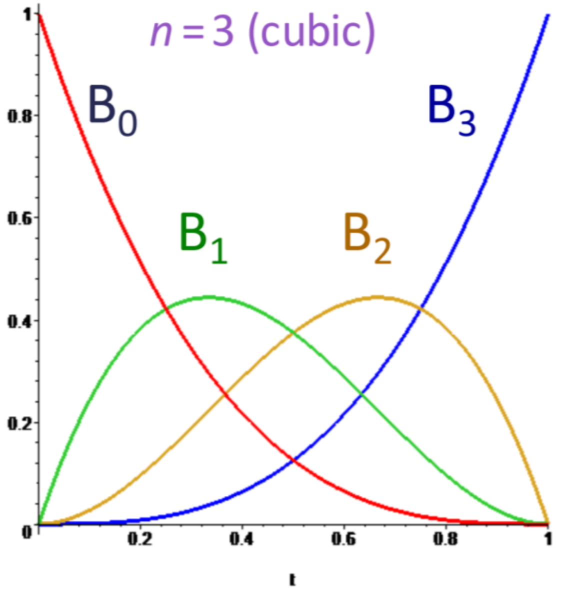


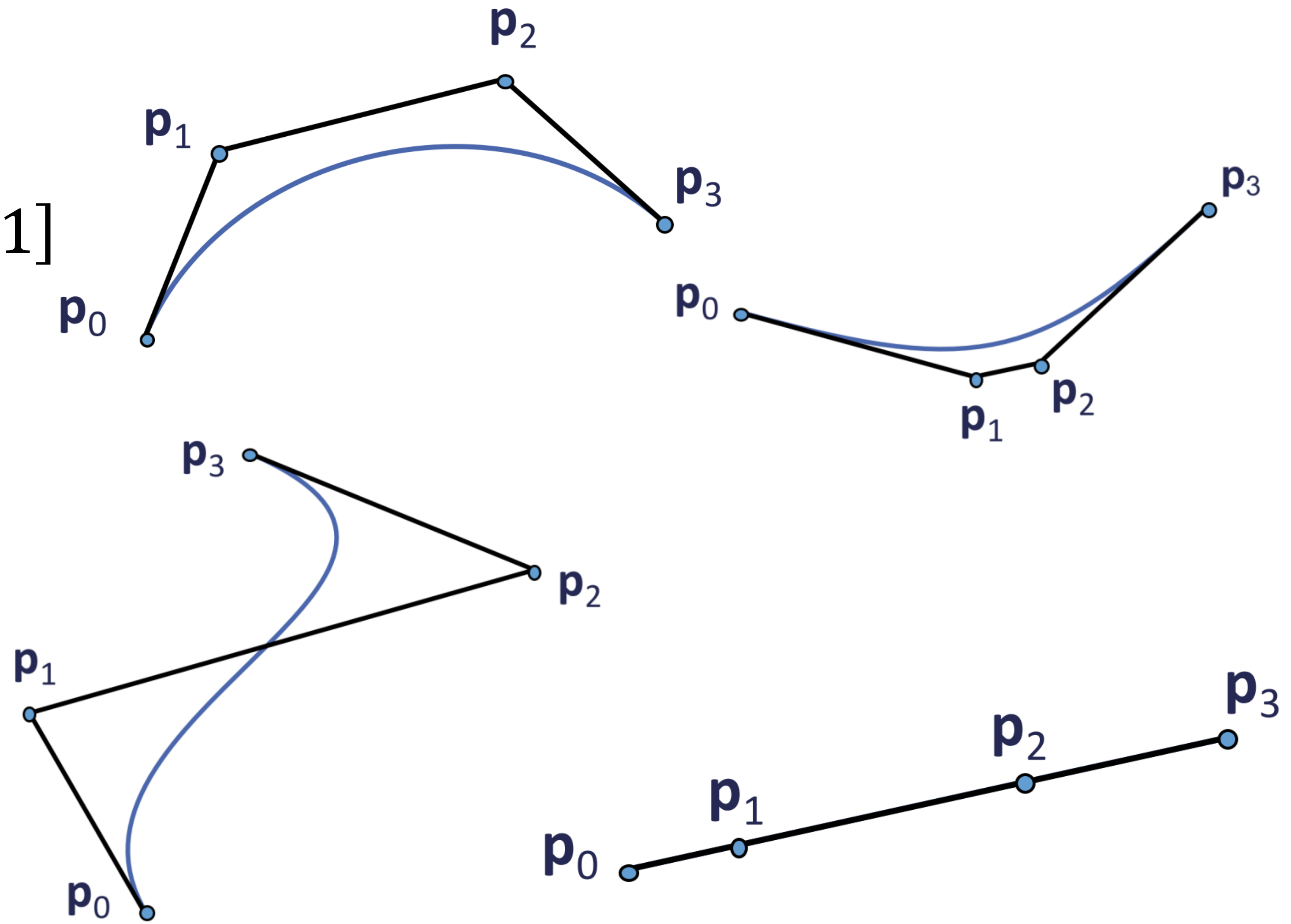


Bézier Curves in Bernstein form

- **Bézier Curves:**

$$\boldsymbol{f}(t) = \sum_{i=1}^{n} B_i^n \boldsymbol{p}_i \,, t \in [0,1]$$

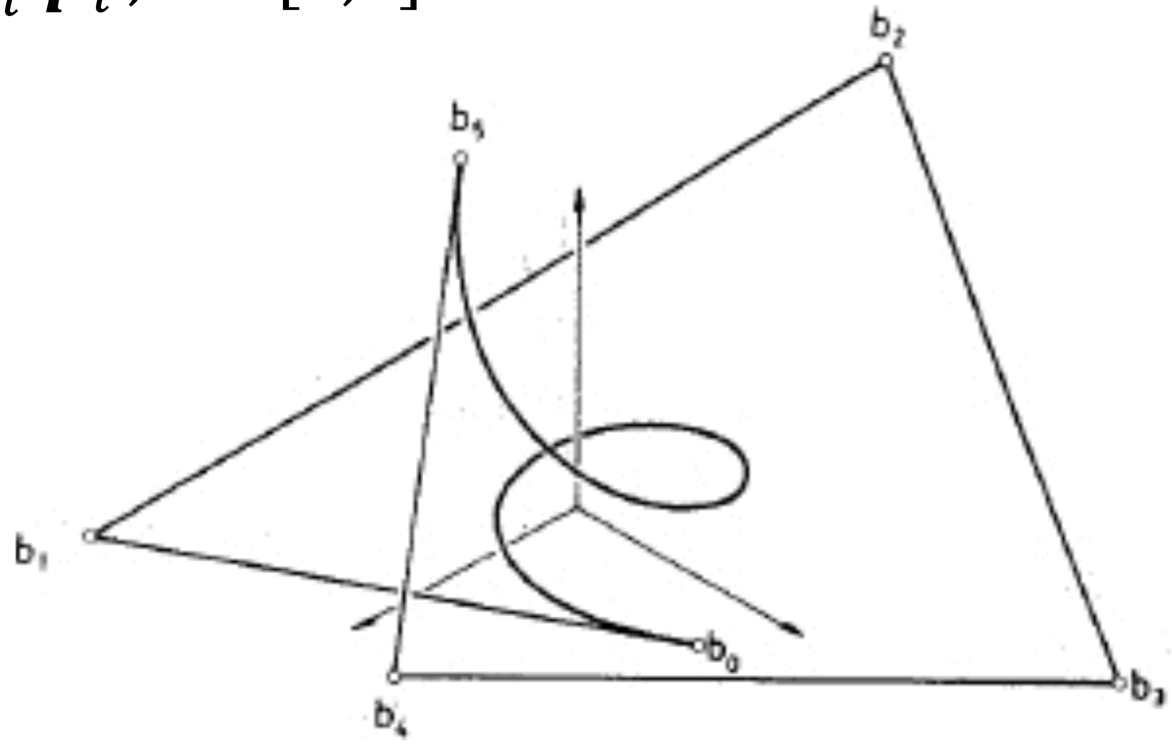




Bézier Curves in Bernstein form

- **Bézier Curves, also in 3D**

$\boldsymbol{f}(t) = \sum_{i=1}^{n} B_i^n \boldsymbol{p}_i \,, t \in [0,1]$



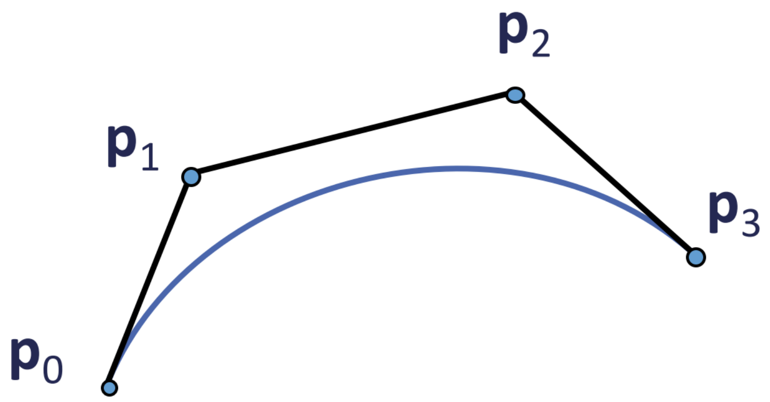
Bézier Curves in Bernstein form

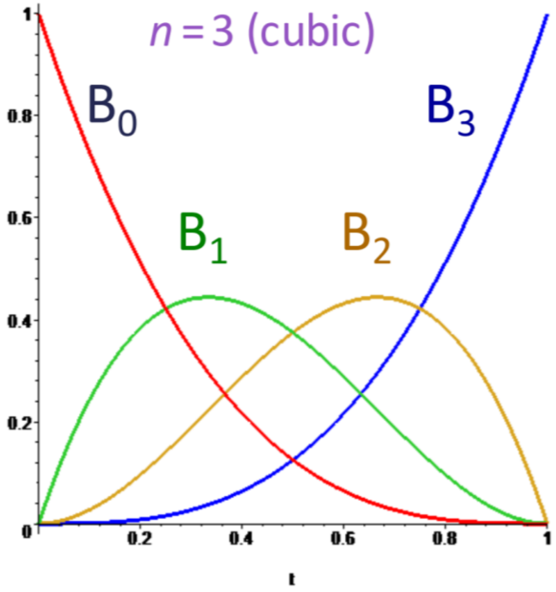
- Bézier curves:
  - Curves: $\boldsymbol{f}(t) = \sum_{i=1}^{n} B_i^n \boldsymbol{p}_i$
  - Considering the interval $t \in [0..1]$
  - Properties as discussed before:
    - Affine invariant
    - Curves contained in the convex hull
    - Influence of control points

    Moving along the curve with index i

    Largest influence at $t = \frac{i}{n}$

    Single curve segments: no full local control





Bézier Curve Properties: another look at derivatives

- Given: $\boldsymbol{p}_0, \dots, \boldsymbol{p}_n$, $\boldsymbol{f}(t) = \sum_{i=0}^{n} B_i^n(t)\, \boldsymbol{p}_i$
- Then: $\boldsymbol{f}'(t) = n \sum_{i=0}^{n-1} B_i^{n-1}(t)\, (\boldsymbol{p}_{i+1} - \boldsymbol{p}_i)$
- Proof: $\boldsymbol{f}'(t) = \sum_{i=0}^{n} \frac{d}{dt} B_i^n(t) \boldsymbol{p}_i = n \sum_{i=0}^{n} \left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right) \boldsymbol{p}_i$

$$= n \sum_{i=0}^{n} B_{i-1}^{n-1}(t) \boldsymbol{p}_i - n \sum_{i=0}^{n} B_i^{n-1}(t) \boldsymbol{p}_i$$

Index change

$$= n \sum_{i=-1}^{n-1} B_i^{n-1}(t) \boldsymbol{p}_{i+1} - n \sum_{i=0}^{n} B_i^{n-1}(t) \boldsymbol{p}_i = n \sum_{i=0}^{n-1} B_i^{n-1}(t) \boldsymbol{p}_{i+1} - n \sum_{i=0}^{n-1} B_i^{n-1}(t) \boldsymbol{p}_i$$

$$= n \sum_{i=0}^{n-1} B_i^{n-1}(t) (\boldsymbol{p}_{i+1} - \boldsymbol{p}_i)$$

Bézier Curve Properties

- Higher order derivatives:

$$f^{[r]}(t) = \frac{n!}{(n-r)!} \cdot \sum_{i=0}^{n-r} B_i^{n-r}(t) \cdot \Delta^r \boldsymbol{p}_i$$

Bézier Curve Properties

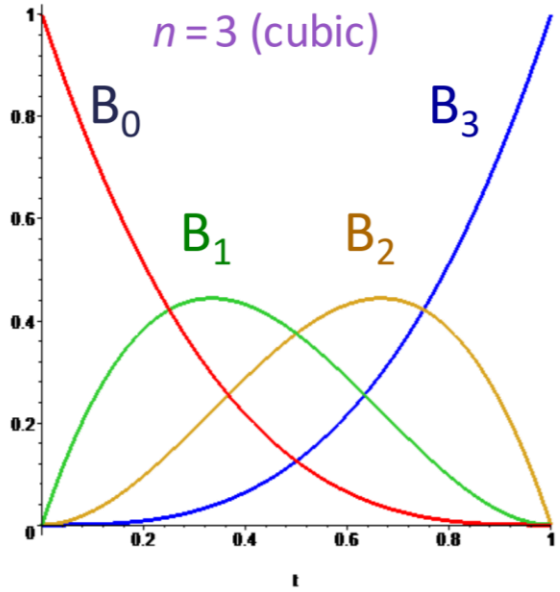
- Imporant for continuous concatenation:
  - Function value at $\{0,1\}$:

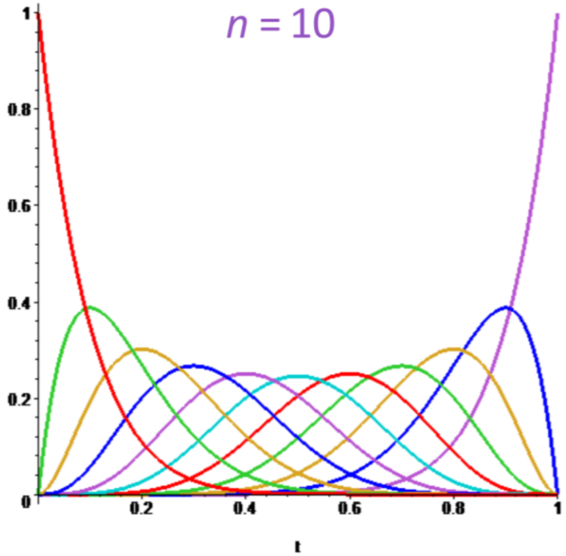
$$\boldsymbol{f}(t) = \sum_{i=0}^{n-1} \binom{n}{i} t^i (1-t)^{n-i} \boldsymbol{p}_i$$

$$\Rightarrow \boldsymbol{f}(0) = \boldsymbol{p}_0$$

$$\boldsymbol{f}(1) = \boldsymbol{p}_1$$

  - First derivative vector at $\{0,1\}$
  - Second derivative vector at $\{0,1\}$

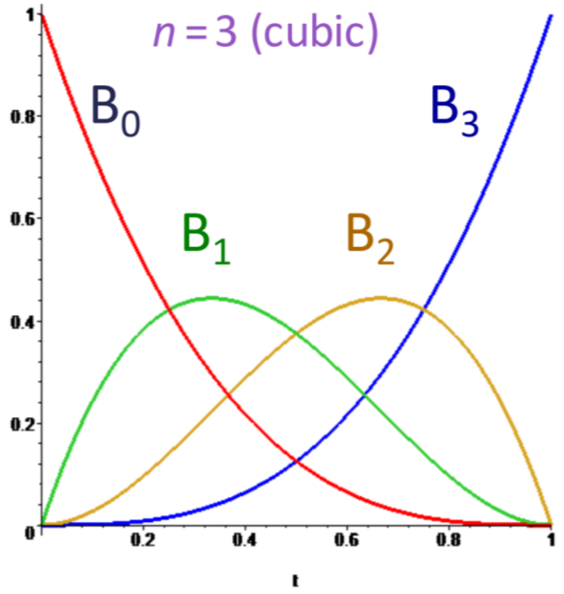


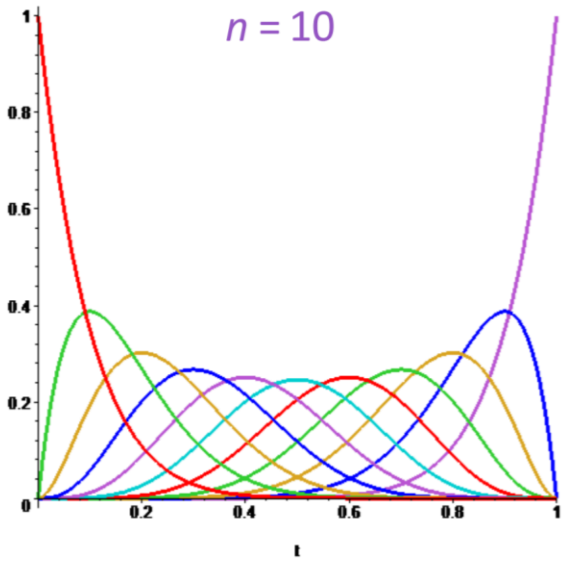


Bézier Curve Properties

First derivative vector at $\{0,1\}$

$$\frac{d}{dt}\boldsymbol{f}(t) =$$

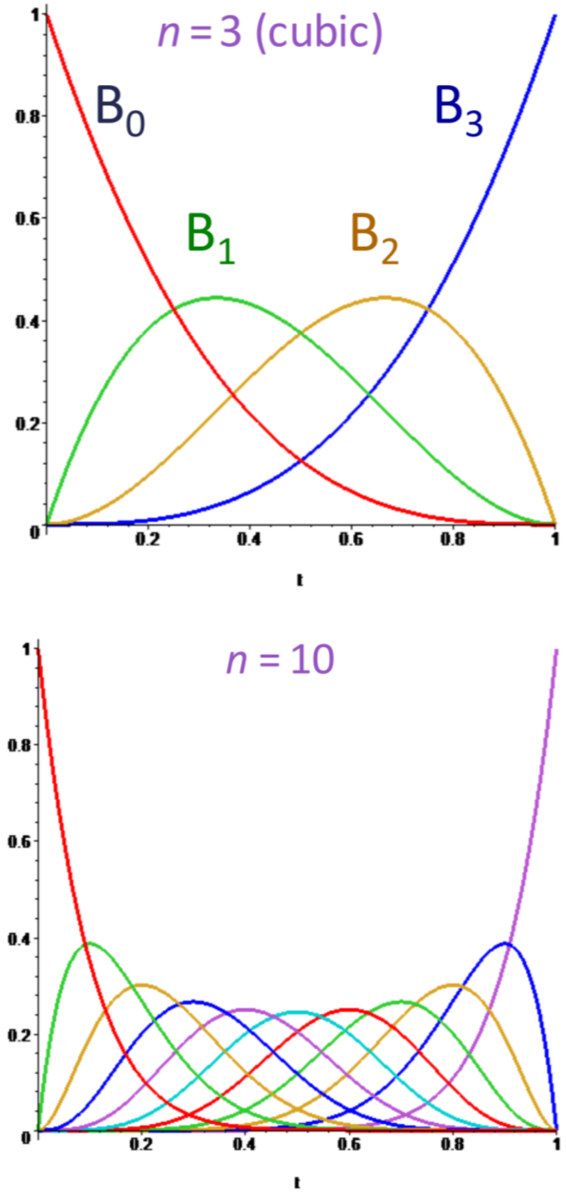




Bézier Curve Properties

First derivative vector at $\{0,1\}$

$$\frac{d}{dt}\boldsymbol{f}(t) = n\sum_{i=0}^{n-1}\left[B_{i-1}^{(n-1)}(t) - B_i^{(n-1)}(t)\right]\boldsymbol{p}_i$$



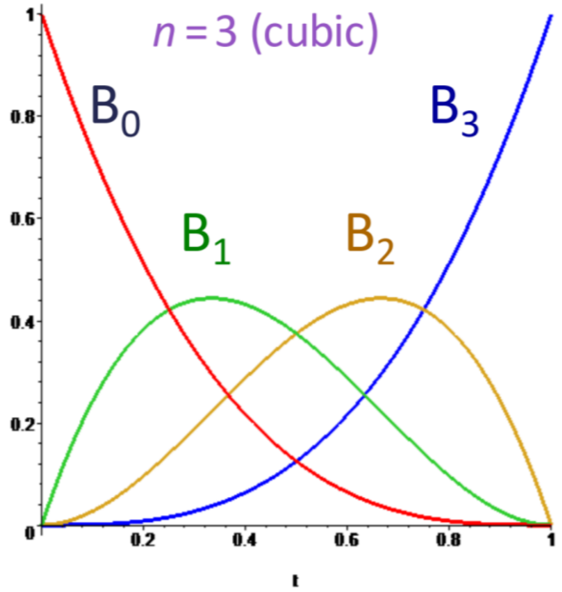
Bézier Curve Properties

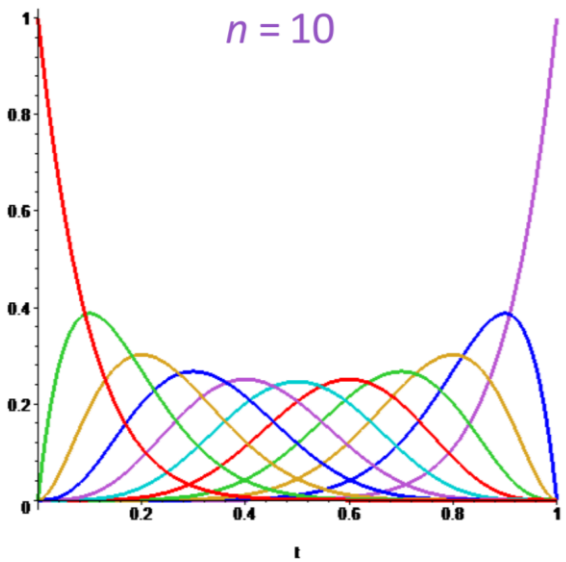
First derivative vector at $\{0,1\}$

$$\frac{d}{dt}\boldsymbol{f}(t) = n\sum_{i=0}^{n-1}\left[B_{i-1}^{(n-1)}(t) - B_i^{(n-1)}(t)\right]\boldsymbol{p}_i$$

$$= n\left(\left[-B_0^{(n-1)}(t)\right]\boldsymbol{p}_0 + \left[B_0^{(n-1)}(t) - B_1^{(n-1)}(t)\right]\boldsymbol{p}_1 + \cdots\right.$$
$$\left. + \left[B_{n-2}^{(n-1)}(t) - B_{n-1}^{(n-1)}(t)\right]\boldsymbol{p}_{n-1} + \left[B_{n-1}^{(n-1)}(t)\right]\boldsymbol{p}_n\right)$$

$$\boxed{\frac{d}{dt}\boldsymbol{f}(0) = n(\boldsymbol{p}_1 - \boldsymbol{p}_0) \qquad \frac{d}{dt}\boldsymbol{f}(1) = n(\boldsymbol{p}_n - \boldsymbol{p}_{n-1})}$$





Bézier Curve Properties

- Imporant for continuous concatenation:
  - Function value at $\{0,1\}$:
  $$\boldsymbol{f}(0) = \boldsymbol{p}_0$$
  $$\boldsymbol{f}(1) = \boldsymbol{p}_1$$
  - First derivative vector at $\{0,1\}$
  $$\boldsymbol{f}'(0) = n[\boldsymbol{p}_1 - \boldsymbol{p}_0]$$
  $$\boldsymbol{f}'(1) = n[\boldsymbol{p}_n - \boldsymbol{p}_{n-1}]$$
  - Second derivative vector at $\{0,1\}$
  $$\boldsymbol{f}''(0) = n(n-1)[\boldsymbol{p}_2 - \boldsymbol{2p}_1 + \boldsymbol{p}_0]$$
  $$\boldsymbol{f}''(1) = n(n-1)[\boldsymbol{p}_n - 2\boldsymbol{p}_{n-1} + \boldsymbol{p}_{n-2}]$$