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计算机图形学
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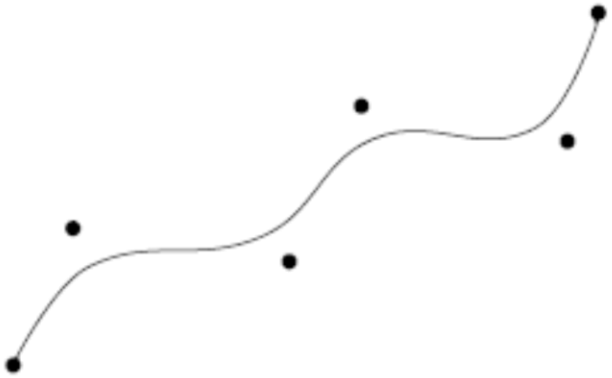
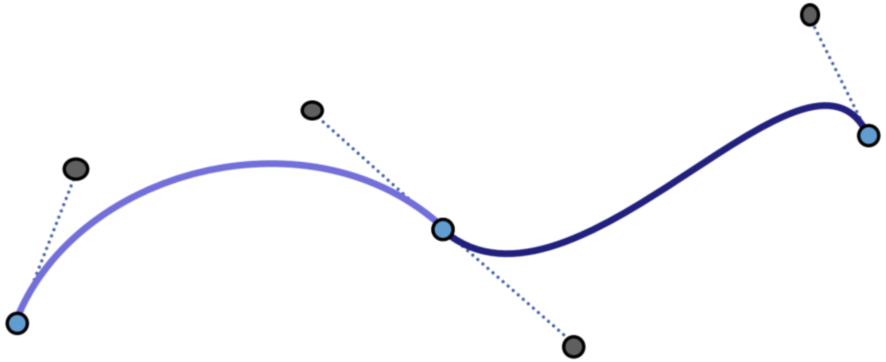
Recap

- **Bézier curves and curve design:**
  - The rough form is specified by the position of the control points
  - Results: smooth curve approximating the control points
  - Computation / Representation
    - de Casteljau algorithm
    - Bernstein form

  - Problems:
    - High polynomial degree
    - Moving a control point can change the whole curve
    - Interpolation of points
    - →**Bézier splines**

Recap





Approximation → Interpolation

Towards Bézier Splines

- **Interpolation problems:**
  - given:

    $\boldsymbol{k}_0, \ldots, \boldsymbol{k}_n \in \mathbb{R}^3$ control points

    $t_0, \ldots, t_n \in \mathbb{R}$ knot sequence

    $t_i < t_{i+1}$, for $i = 0, \ldots, n-1$
  - wanted
    - Interpolating curve $\boldsymbol{x}(i)$, i.e. $\boldsymbol{x}(t_i) = \boldsymbol{k}_i$ for $i = 0, \ldots, n$

  - Approach: "Joining" of n Bézier curves with certain intersection conditions

Towards Bézier Splines

- **The following issues arise when stitching together Bézier curves:**
  - Continuity
  - Parameterization
  - Degree

Bézier Splines

Parametric and Geometric Continuity

Parametric Continuity

Joining curves – continuity

- Given: 2 curves

  $\boldsymbol{x}_1(t)$ over $[t_0, t_1]$

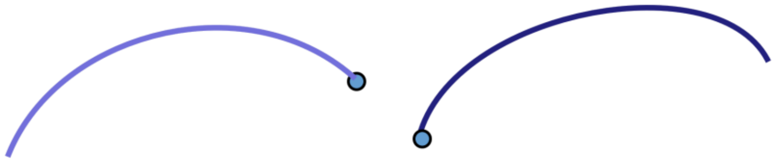
  $\boldsymbol{x}_2(t)$ over $[t_1, t_2]$

- $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are C^r continuous at t_1, if all their 0^{th} to r^{th} derivative vectors coincides at t_1

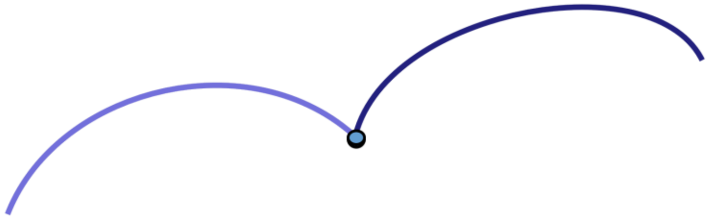
Parametric Continuity

- C^0: **position varies continuously**
- C^1: **First derivative is continuous across junction**
  - In other words: the velocity vector remains the same
- C^2: **Second derivative is continuous across junction**
  - The acceleration vector remains the same

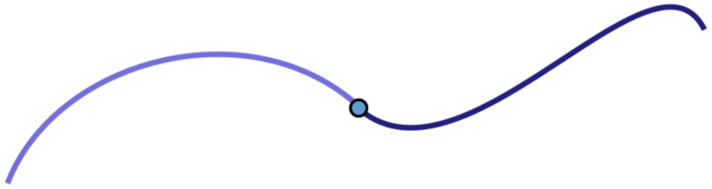
Parametric Continuity



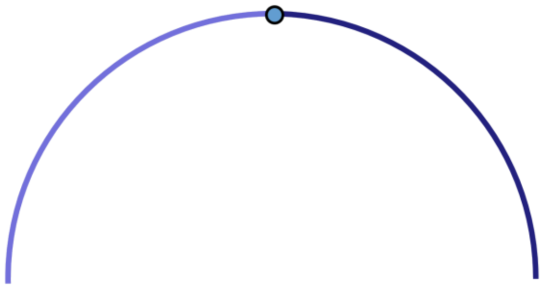
C^{-1} continuity



C^0 continuity



C^1 continuity



C^2 continuity

Continuity

Parametric Continuity C^r:

- C^0, C^1, C^2 ⋯ continuity
- Does a particle moving on this curve have a smooth trajectory (position, velocity, acceleration, ⋯)?
- Depends on parameterization
- Useful for animation (object movement, camera paths)

Geometric Continuity G^r:

- Is the curve itself smooth?
- Independent of parameterization
- More relevant for modeling (curve design)

Bézier Splines

Parameterization

Bézier spline curves

Local and global parameters:

- Given:
  - $b_0,\cdots,b_n$
  - $y(u)$: Bézier curve in interval $[0,1]$
  - $x(t)$: Bézier curve in interval $[t_i, t_{i+1}]$

- Setting $u(t) = \frac{t-t_i}{t_{i+1}-t_i}$

- Results in $x(t) = y\big(u(t)\big)$

The *local* parameter u runs from 0 to 1,
while the *global* parameter t runs from t_i to t_{i+1}

$$u(t) = \frac{t - t_i}{t_{i+1} - t_i}$$

$$x(t) = y\big(u(t)\big)$$

Bézier spline curves

Derivatives:

$$x'(t) = \quad y'\big(u(t)\big) \cdot u'(t) = \frac{y'\big(u(t)\big)}{t_{i+1} - t_i}$$

$$x''(t) = \quad y''\big(u(t)\big) \cdot \big(u'(t)\big)^2 + y'\big(u(t)\big) \cdot u''(t) = \frac{y''\big(u(t)\big)}{(t_{i+1} - t_i)^2}$$

…

$$x^{[n]}(t) = \frac{y^{[n]}\big(u(t)\big)}{(t_{i+1} - t_i)^n}$$

Bézier Curve

$$\boldsymbol{f}(t) = \sum_{i=0}^{n} B_i^n(t)\,\boldsymbol{p}_i$$

- Function value at $\{0,1\}$:

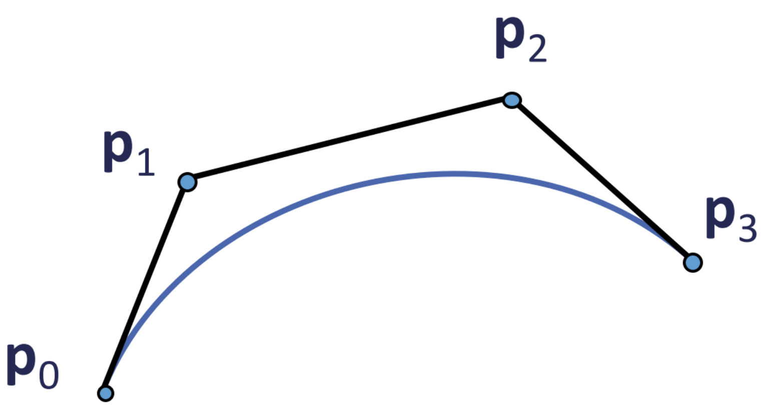
$$\boldsymbol{f}(0) = \boldsymbol{p}_0$$
$$\boldsymbol{f}(1) = \boldsymbol{p}_1$$

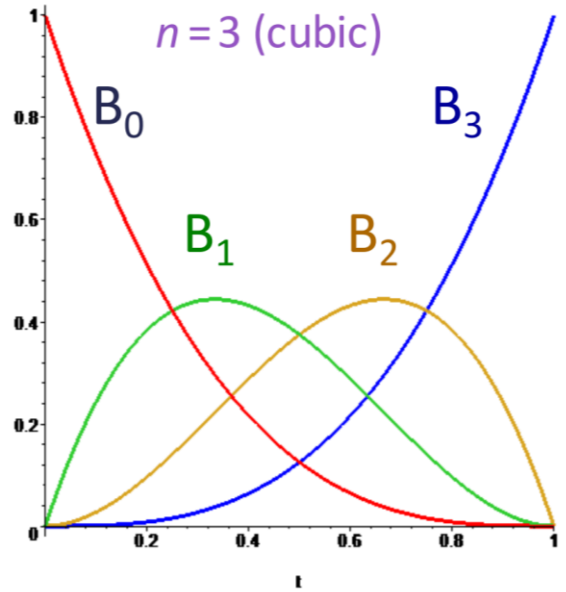
- First derivative vector at $\{0,1\}$

$$\boldsymbol{f}'(0) = n[\boldsymbol{p}_1 - \boldsymbol{p}_0]$$
$$\boldsymbol{f}'(1) = n[\boldsymbol{p}_n - \boldsymbol{p}_{n-1}]$$

- Second derivative vector at $\{0,1\}$

$$\boldsymbol{f}''(0) = n(n-1)[\boldsymbol{p}_2 - \boldsymbol{2p}_1 + \boldsymbol{p}_0]$$
$$\boldsymbol{f}''(1) = n(n-1)[\boldsymbol{p}_n - 2\boldsymbol{p}_{n-1} + \boldsymbol{p}_{n-2}]$$





Bézier spline curves

Special cases:

$$\boldsymbol{x}'(t_i) = \frac{n\cdot(\boldsymbol{p}_1 - \boldsymbol{p}_0)}{t_{i+1} - t_i}$$

$$\boldsymbol{x}'(t_{i+1}) = \frac{n\cdot(\boldsymbol{p}_n - \boldsymbol{p}_{n-1})}{t_{i+1} - t_i}$$

$$\boldsymbol{x}''(t_i) = \frac{n\cdot(n-1)\cdot(\boldsymbol{p}_2 - 2\boldsymbol{p}_1 + \boldsymbol{p}_0)}{(t_{i+1} - t_i)^2}$$

$$\boldsymbol{x}''(t_{i+1}) = \frac{n\cdot(n-1)\cdot(\boldsymbol{p}_n - 2\boldsymbol{p}_{n-1} + \boldsymbol{p}_{n-2})}{(t_{i+1} - t_i)^2}$$

Bézier Splines

General Case

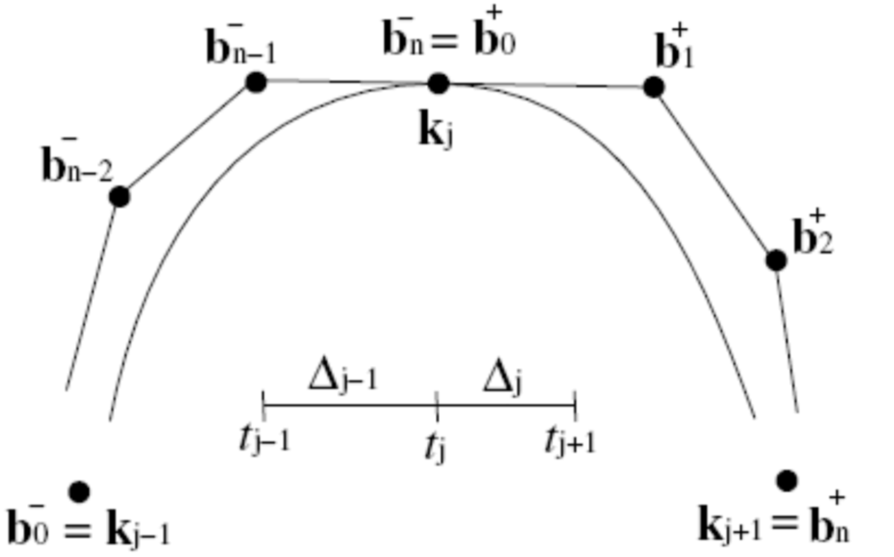
Bézier spline curves

Joining Bézier curves:

- Given: 2 Bézier curves of degree n through

$$\boldsymbol{k}_{j-1} = \boldsymbol{b}_0^-, \boldsymbol{b}_1^-, \dots, \boldsymbol{b}_n^- = \boldsymbol{k}_j$$

$$\boldsymbol{k}_j = \boldsymbol{b}_0^+, \boldsymbol{b}_1^+, \dots, \boldsymbol{b}_n^+ = \boldsymbol{k}_{j+1}$$

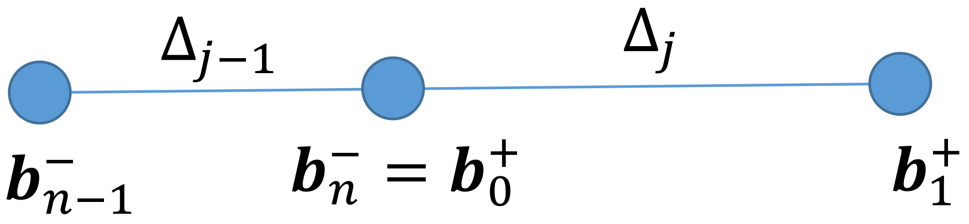


$$\boldsymbol{x}'(t_i) = \frac{n \cdot (\boldsymbol{b}_1 - \boldsymbol{b}_0)}{t_{i+1} - t_i}$$

Bézier spline curves

- Required: C^1-continuity at $\boldsymbol{k}_j$:
- $\boldsymbol{b}_{n-1}^-, \boldsymbol{k}_j, \boldsymbol{b}_1^+$ collinear and

$$\frac{\boldsymbol{b}_n^- - \boldsymbol{b}_{n-1}^-}{t_j - t_{j-1}} = \frac{\boldsymbol{b}_1^+ - \boldsymbol{b}_0^+}{t_{j+1} - t_j}$$



Bézier spline curves

- Required: G^1-continuity at $\boldsymbol{k}_j$:
  - $\boldsymbol{b}_{n-1}^-$, $\boldsymbol{k}_j$, $\boldsymbol{b}_1^+$ collinear
- Less restrictive than C^1-continuity

Bézier Splines

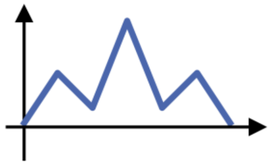
Choosing the degree

Choosing the Degree

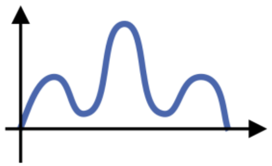
Candidates:

- $d = 0$ (piecewise constant) : not smooth

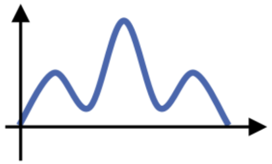
- $d = 1$ (piecewise linear) : not smooth enough



- $d = 2$ (piecewise quadratic) : constant 2nd derivative, still too inflexible



- $d = 3$ (piecewise cubic): degree of choice for computer graphics applications



Cubic Splines

Cubic piecewise polynomials:

- We can attain C^2 continuity without fixing the second derivative throughout the curve

Cubic Splines

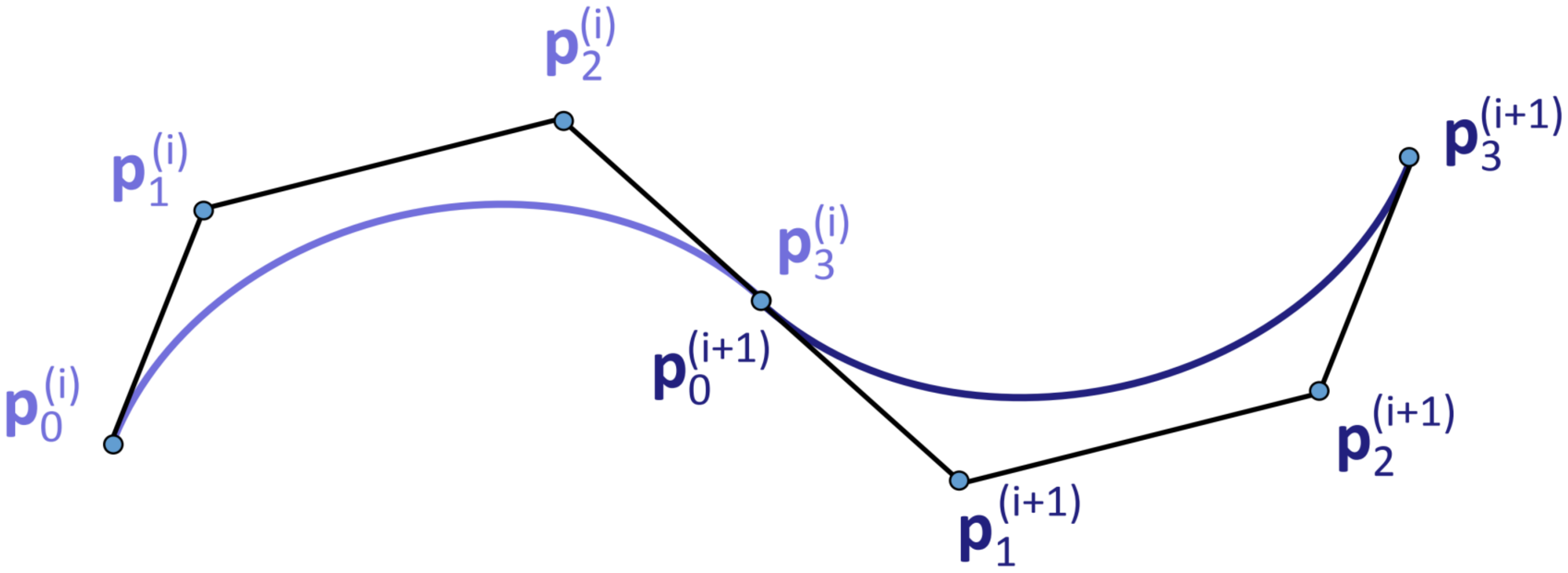
Cubic piecewise polynomials:

- We can attain C^2 continuity without fixing the second derivative throughout the curve
- C^2 continuity is perceptually important
  - Motion: continuous position, velocity & acceleration
    Discontinuous acceleration noticeable (object/camera motion)
  - We can see second order shading discontinuities
    (esp.: reflective objects)

Bézier Splines

Local control: Bézier splines

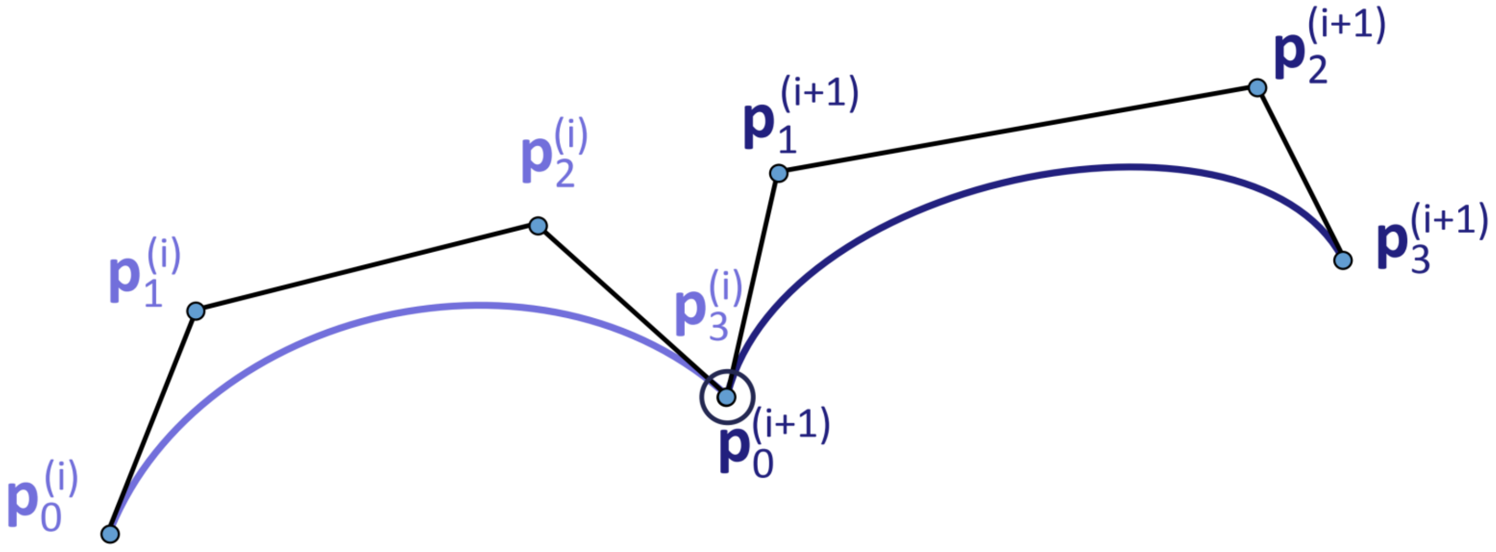
- Concatenate several curve segments
- Question: Which constraints to place upon the control points in order to get C^{-1}, C^0, C^1, C^2 continuity?



Bézier Spline Continuity

Rules for Bézier spline continuity:

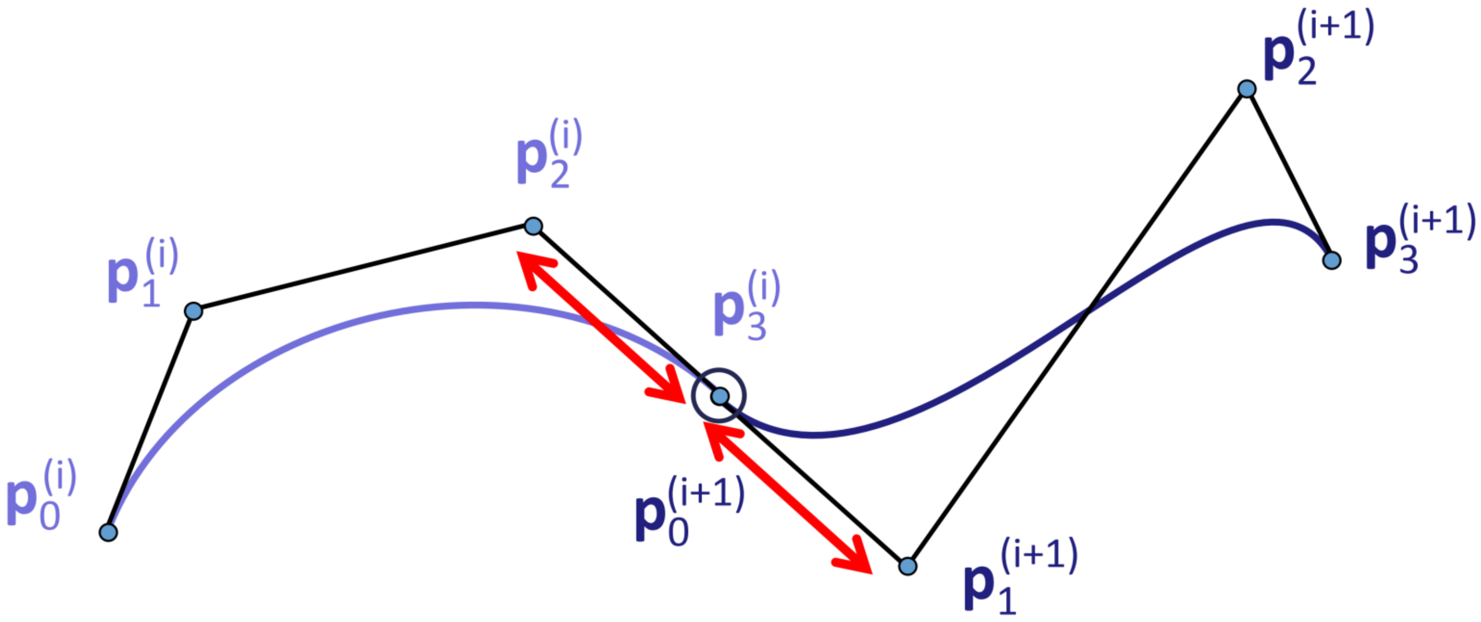
- C^0 continuity:
  - Each spline segment interpolates the first and last control point
  - Therefore: Points of neighboring segments have to coincide for C^0 continuity



Bézier Spline Continuity

Rules for Bézier spline continuity:

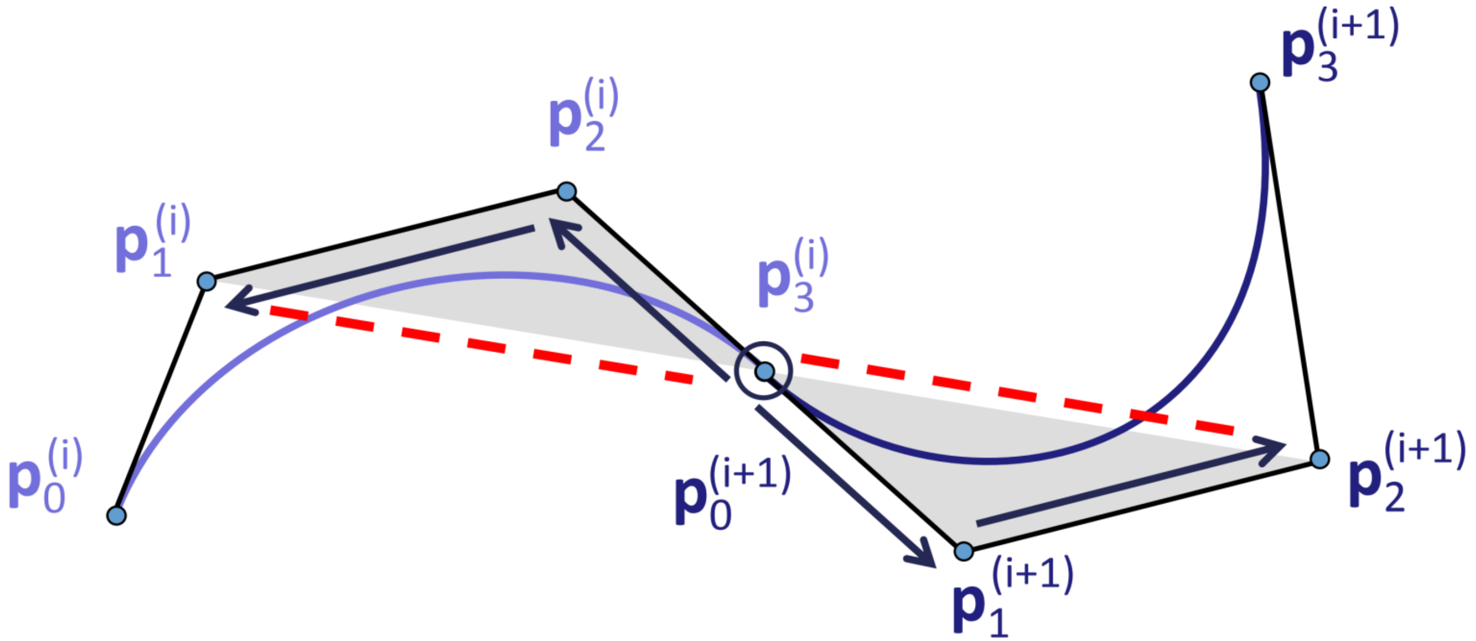
- Additional requirement for C^1 continuity:
  - Tangent vectors are proportional to differences $\boldsymbol{p}_1 - \boldsymbol{p}_0$, $\boldsymbol{p}_n - \boldsymbol{p}_{n-1}$
  - Therefore: These vectors must be identical for C^1 continuity



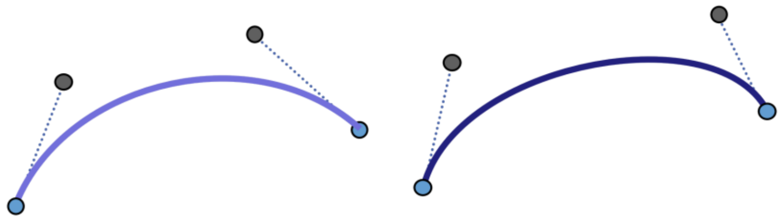
Bézier Spline Continuity

Rules for Bézier spline continuity

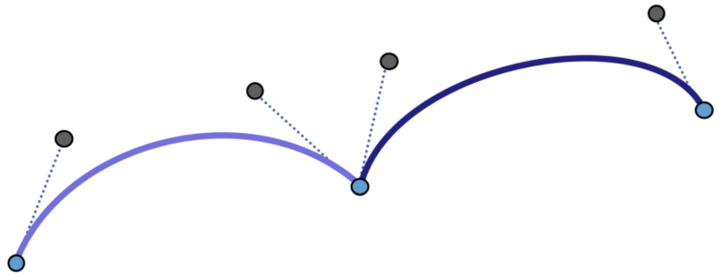
- Additional requirement for C^2 continuity:
  - d^2/dt^2 vectors are prop. to $\boldsymbol{p}_2 - 2\boldsymbol{p}_1 + \boldsymbol{p}_0$, $\boldsymbol{p}_n - 2\boldsymbol{p}_{n-1} + \boldsymbol{p}_{n-2}$
  - Tangents must be the same (C^2 implies C^1)



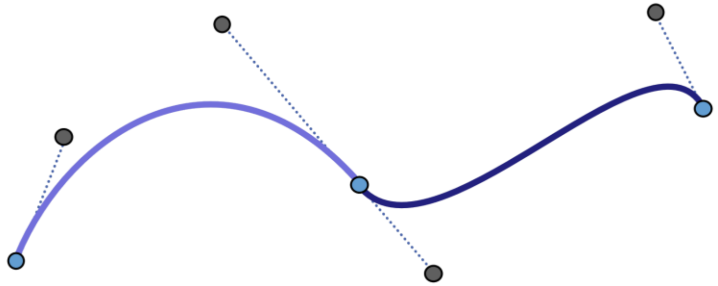
Continuity



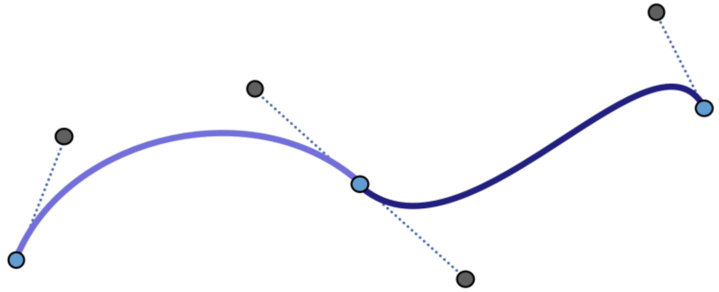
C^{-1} continuity



C^0 continuity



G^1 continuity



C^1 continuity

In Practice

- Everyone is using cubic Bézier curves
- Higher degree are rarely used (some CAD/CAM applications)
- Typically: “points & handles” interface
- Four modes:
  - Discontinuous (two curves)
  - C^0 Continuous (points meet)
  - G^1 continuous: Tangent direction continuous
    - Handles point into the same direction, but different length
  - C^1 continuous
    - Handle points have symmetric vectors
- C^2 is more restrictive: control via k_i

Bézier Splines

C^2 Cubic Bézier Splines

Cubic Bézier Splines

Cubic Bézier spline curves

- Given:

$$\boldsymbol{k}_0, \dots, \boldsymbol{k}_n \in \mathbb{R}^3 \qquad \text{control points}$$

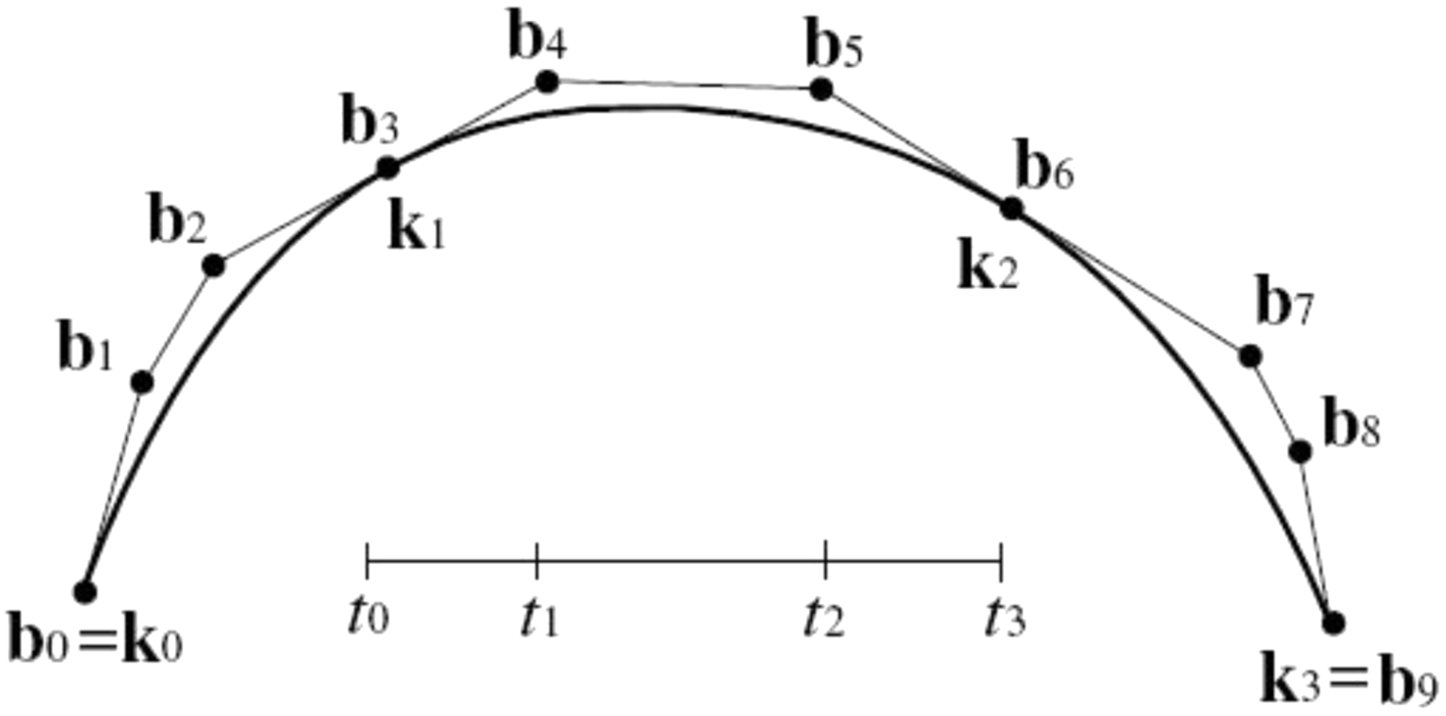
$$t_0, \dots, t_n \in \mathbb{R} \qquad \text{knot sequence}$$

$$t_i < t_{i+1}, \text{ for } i = 0, \dots, n_1$$

- Wanted: Bézier points $\boldsymbol{b}_0, \dots, \boldsymbol{b}_{3n}$ for an interpolating C^2-continuous piecewise cubic Bézier spline curve

Cubic Bézier Splines

Examples: $n = 3$:



Cubic Bézier Splines

- $3n + 1$ unknown points
- $b_{3i} = k_i$ for $i = 0, \dots, n$

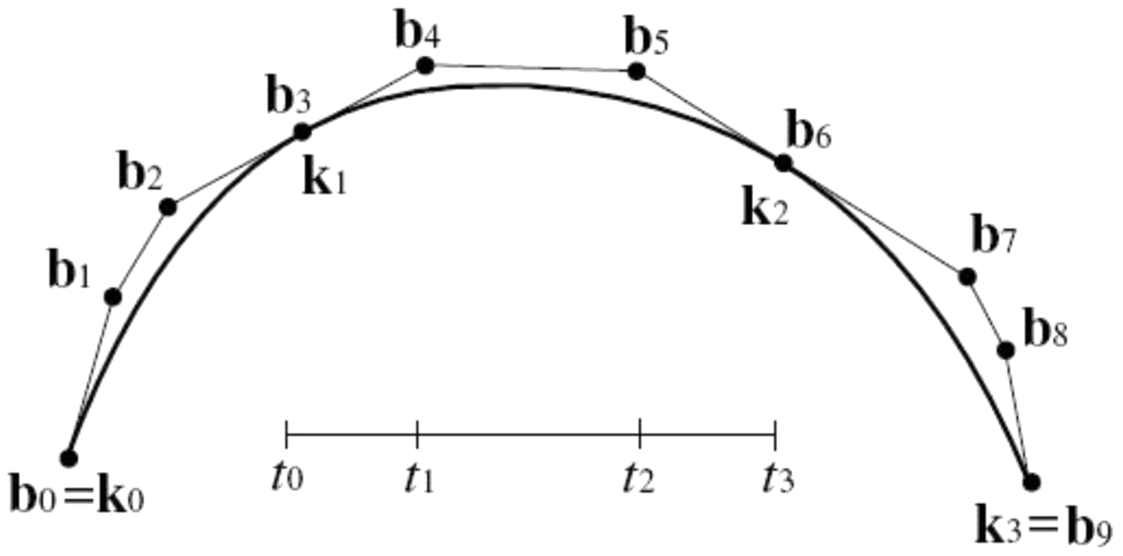
  $n + 1$ equations
- C^1 in points k_i for $i = 1, \dots, n-1$

  $n - 1$ equations
- C^2 in points k_i for $i = 1, \dots, n-1$

  $n - 1$ equations

$3n - 1$ equations

$\Rightarrow$ **2 additional conditions necessary: end conditions**



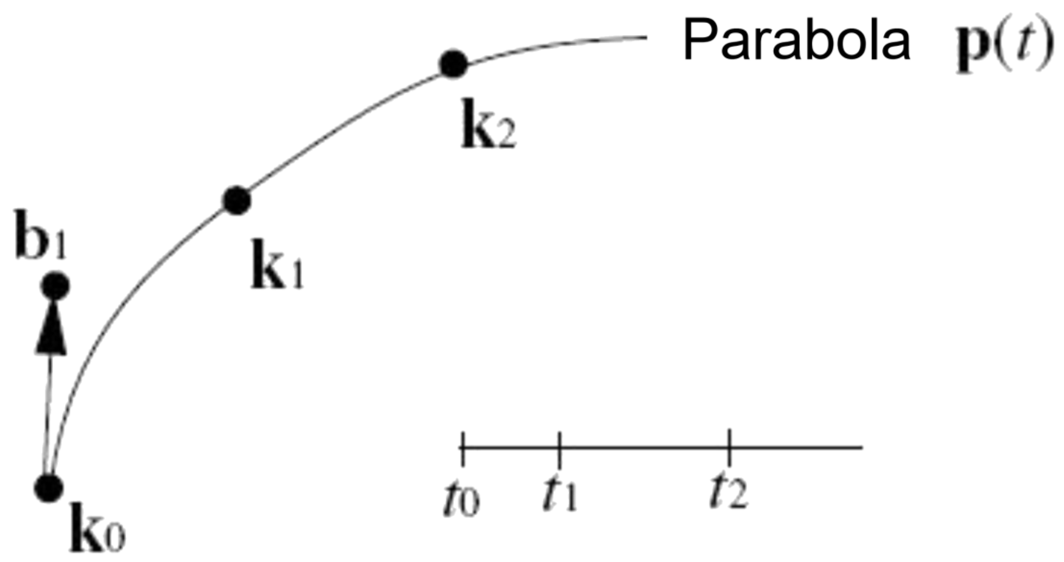
Bézier Splines

C^2 Cubic Bézier Splines: End conditions

Bézier spline curves: End conditions

Bessel's end condition

- The tangential vector in $\boldsymbol{k}_0$ is equivalent to the tangential vector of the parabola interpolating $\{\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2\}$ at $\boldsymbol{k}_0$:



$$\dot{\boldsymbol{x}}(t_i) = \frac{n \cdot (\boldsymbol{b}_1 - \boldsymbol{b}_0)}{t_{i+1} - t_i}$$

Bézier spline curves: End conditions

Parabola Interpolating $\{\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2\}$

$$\boldsymbol{p}(t) = \frac{(t_2 - t)(t_1 - t)}{(t_2 - t_0)(t_1 - t_0)}\boldsymbol{k}_0 + \frac{(t_2 - t)(t - t_0)}{(t_2 - t_1)(t_1 - t_0)}\boldsymbol{k}_1 + \frac{(t_0 - t)(t_1 - t)}{(t_2 - t_1)(t_2 - t_0)}\boldsymbol{k}_2$$

Its derivative

$$\boldsymbol{p}'(t_0) = -\frac{(t_2 - t_0) + (t_1 - t_0)}{(t_2 - t_0)(t_1 - t_0)}\boldsymbol{k}_0 + \frac{(t_2 - t_0)}{(t_2 - t_1)(t_1 - t_0)}\boldsymbol{k}_1 - \frac{(t_1 - t_0)}{(t_2 - t_1)(t_2 - t_0)}\boldsymbol{k}_2$$

Location of $\boldsymbol{b}_1$

$$\boldsymbol{b}_1 = \boldsymbol{b}_0 + \frac{t_1 - t_0}{3}\boldsymbol{p}'(t_0)$$

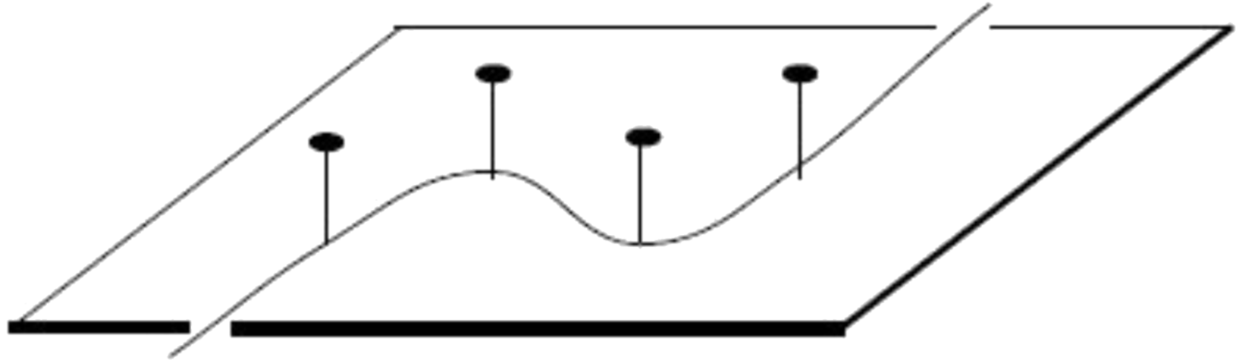
$$\ddot{\boldsymbol{x}}(t_i) = \frac{n \cdot (n-1) \cdot (\boldsymbol{b}_2 - 2\boldsymbol{b}_1 + \boldsymbol{b}_0)}{(t_{i+1} - t_i)^2}$$

Bézier spline curves: End conditions

- Natural end condition:

$$\boldsymbol{x}''(t_0) = 0 \Leftrightarrow \boldsymbol{b}_1 = \frac{\boldsymbol{b}_2 + \boldsymbol{b}_0}{2}$$

$$\boldsymbol{x}''(t_n) = 0 \Leftrightarrow \boldsymbol{b}_{3n-1} = \frac{\boldsymbol{b}_{3n-2} + \boldsymbol{b}_{3n}}{2}$$



End conditions: Examples

***Bessel* end condition**

***Natural* end condition**

Curve: circle of radius 1

Bézier Splines

C^2 Cubic Bézier Splines: parameterization

Bézier spline curves: Parameterization

Approach so far:

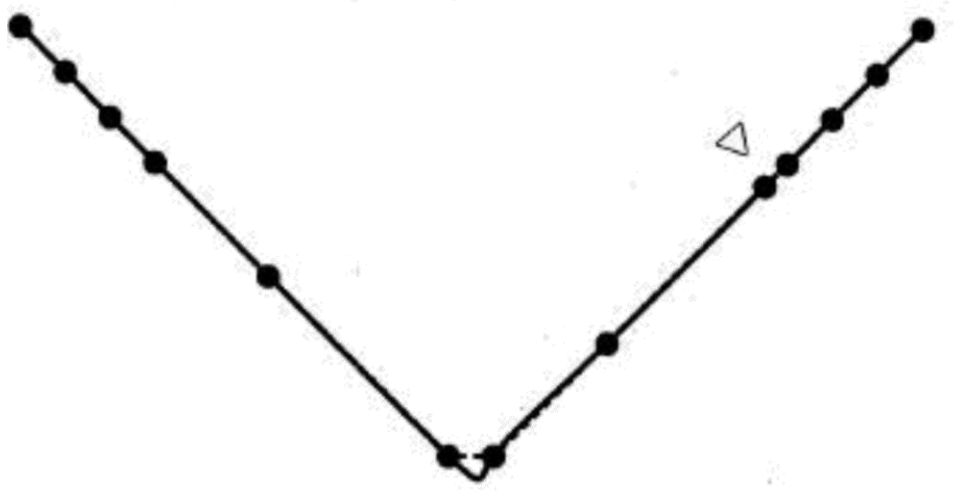
- Given: control points $\boldsymbol{k}_0, \ldots, \boldsymbol{k}_n$ and knot sequence $t_0 < \cdots < t_n$
- Wanted: interpolating curve
- Problem: Normally, the knot sequence is not given, but it influences the curve

Bézier spline curves: Parameterization

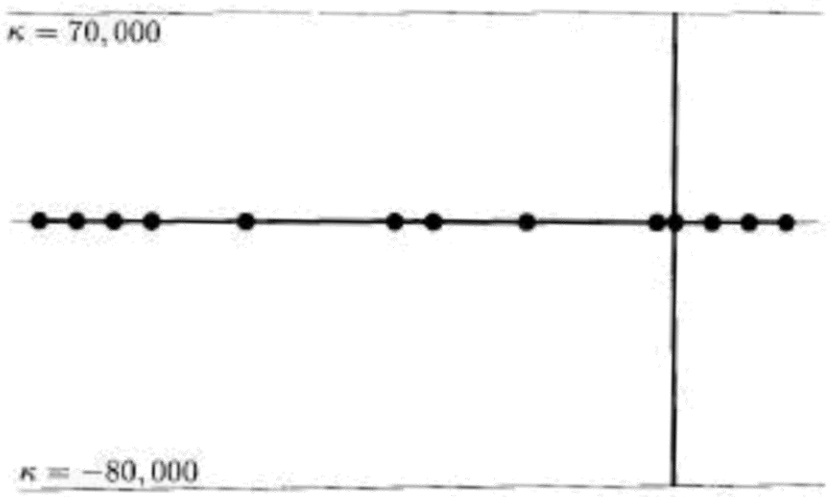
- **Equidistant (uniform) parameterization**
  - $t_{i+1} - t_i = \text{const}$
  - e.g. $t_i = i$
  - Geometry of the data points is not considered
- **Chordal parameterization**
  - $t_{i+1} - t_i = \|\boldsymbol{k}_{i+1} - \boldsymbol{k}_i\|$
  - Parameter intervals proportional to the distances of neighbored control points

Bézier spline curves: Parameterization

- Examples: Uniform parameterization



Curve

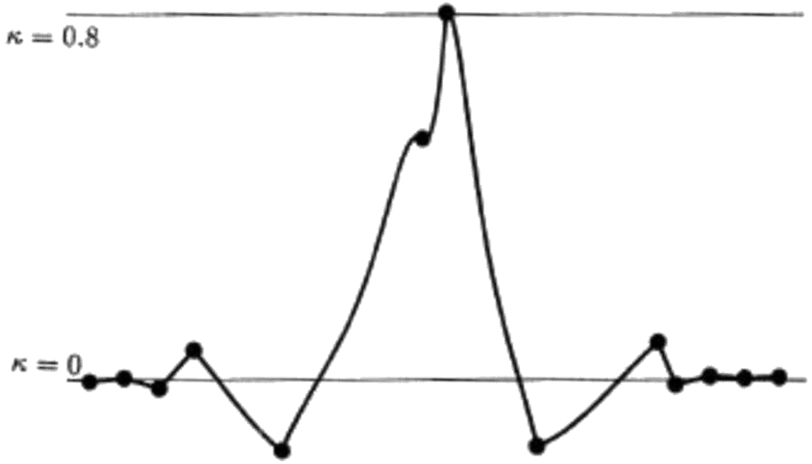


Curvature plot

Bézier spline curves: Parameterization

- Examples: Chordal parameterization

Curve



Curvature plot

Bézier Splines

C^2 Cubic Bézier Splines: closed curves

Closed cubic Bézier spline curves

Closed cubic Bézier spline curves

- Given:

  $\boldsymbol{k}_0, \ldots, \boldsymbol{k}_{n-1}, \boldsymbol{k}_n = \boldsymbol{k}_0$: control points

  $t_0 < \cdots < t_n$: knot sequence

- As an "end condition" for the piecewise cubic curve we place:

$$\boldsymbol{x}'(t_0) = \boldsymbol{x}'(t_n)$$

$$\boldsymbol{x}''(t_0) = \boldsymbol{x}''(t_n)$$

Closed cubic Bézier spline curves

Closed cubic Bézier spline curves

- $\rightarrow C^2$-continuous and closed curve
- Advantage of closed curves: selection of the end condition is not necessary!
- Examples (on the next 3 slides): $n = 3$

Examples