中国科学技术大学
University of Science and Technology of China

计算机图形学
Computer Graphics

陈仁杰
renjiec@ustc.edu.cn
http://staff.ustc.edu.cn/~renjiec

2D Transformations

2D Linear Transformations

- Each 2D linear map can be represented by a unique 2×2 matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

- Concatenation of mappings corresponds to multiplication of matrices

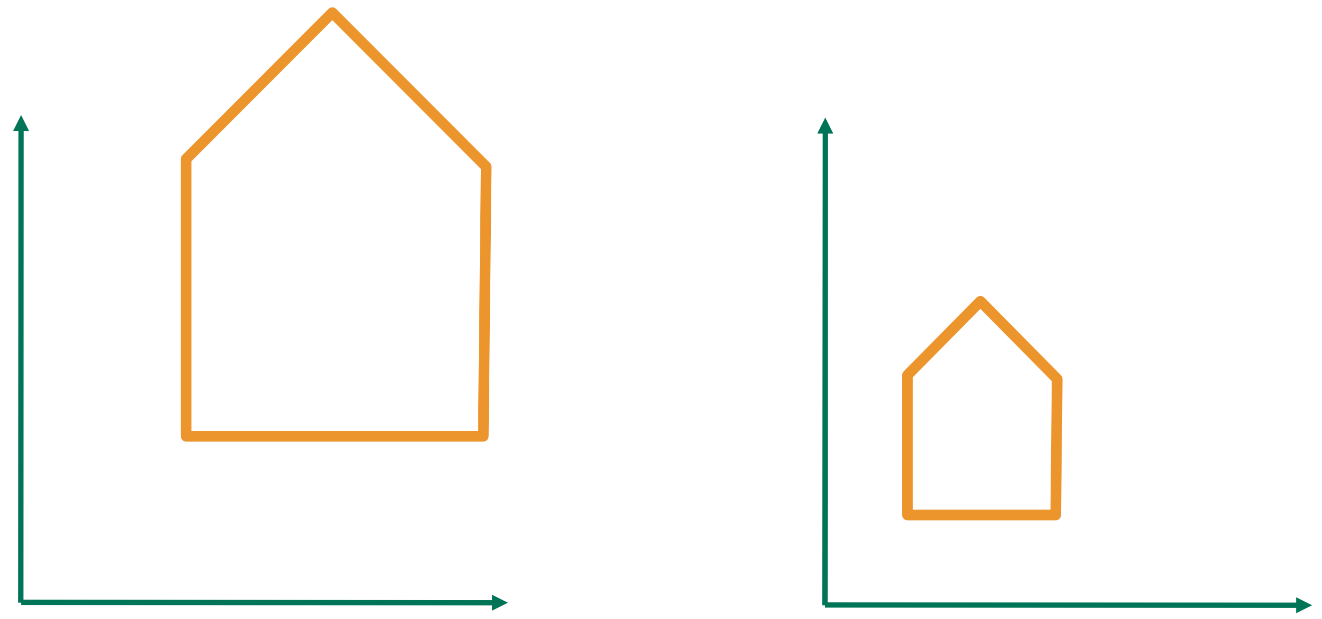
$$L_2(L_1(\mathbf{x})) = \mathbf{L}_2 \, \mathbf{L}_1 \, \mathbf{x}$$

```
L2 * L1 * x;
```

- Linear transformations are very common in computer graphics!

2D Scaling

- Scaling $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}}_{\mathbf{S}(s_x, s_y)} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

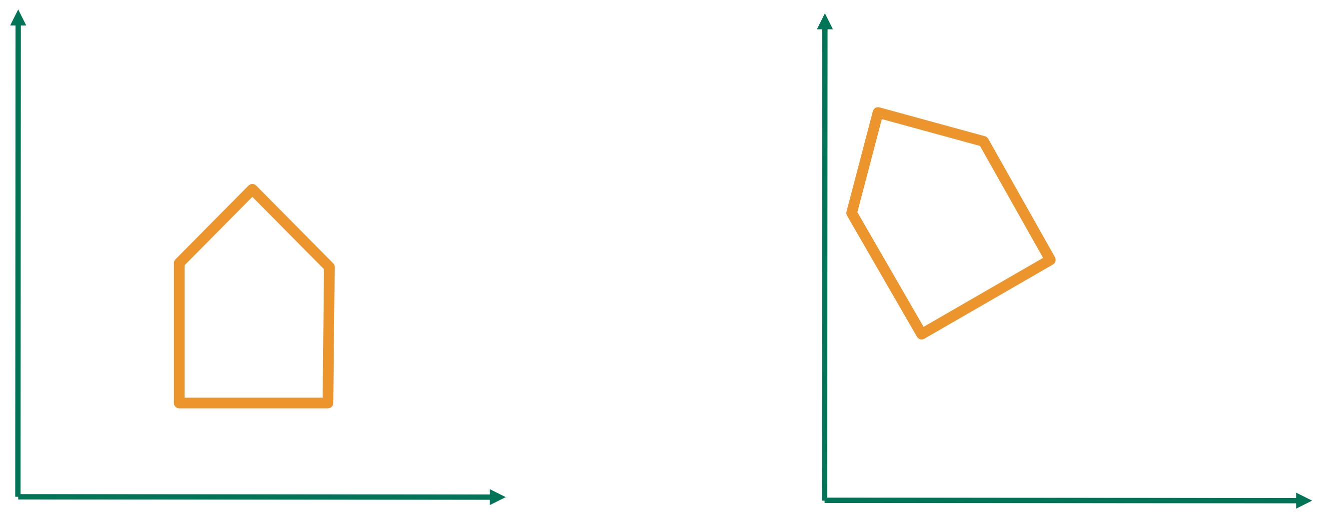


$\mathbf{S}(0.5, 0.5)$

Image Copyright: Mark Pauy

2D Rotation

- Rotation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}}_{\mathbf{R}(\alpha)} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$



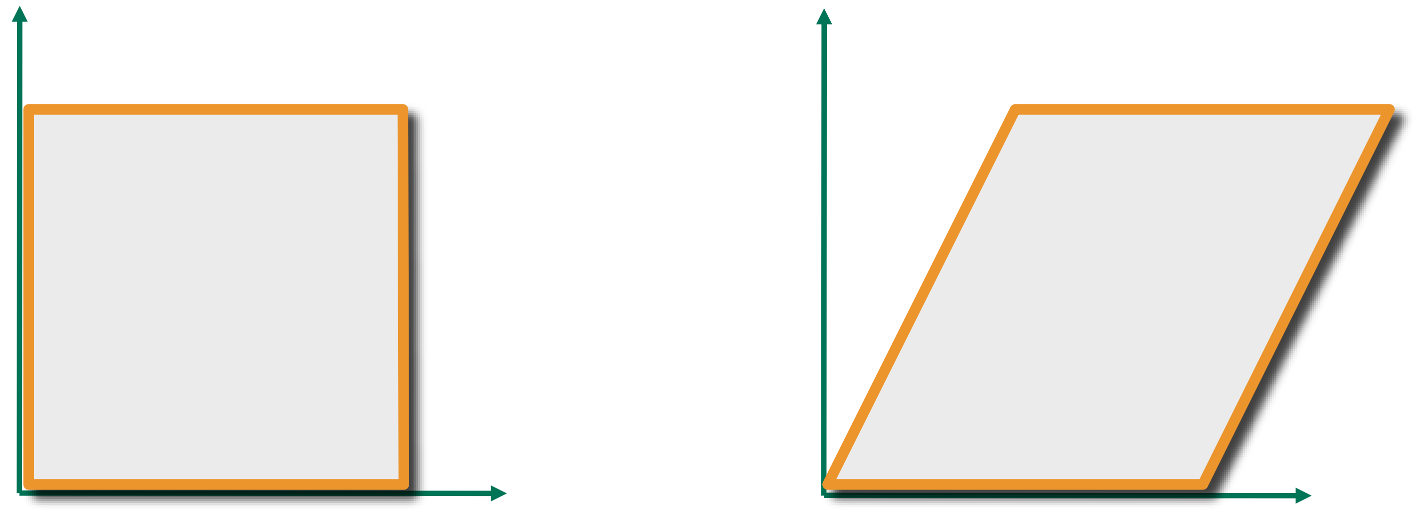
$\mathbf{R}(20^\circ)$

Special case: $\mathbf{R}(90) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

2D Shearing

- Shear along x-axis

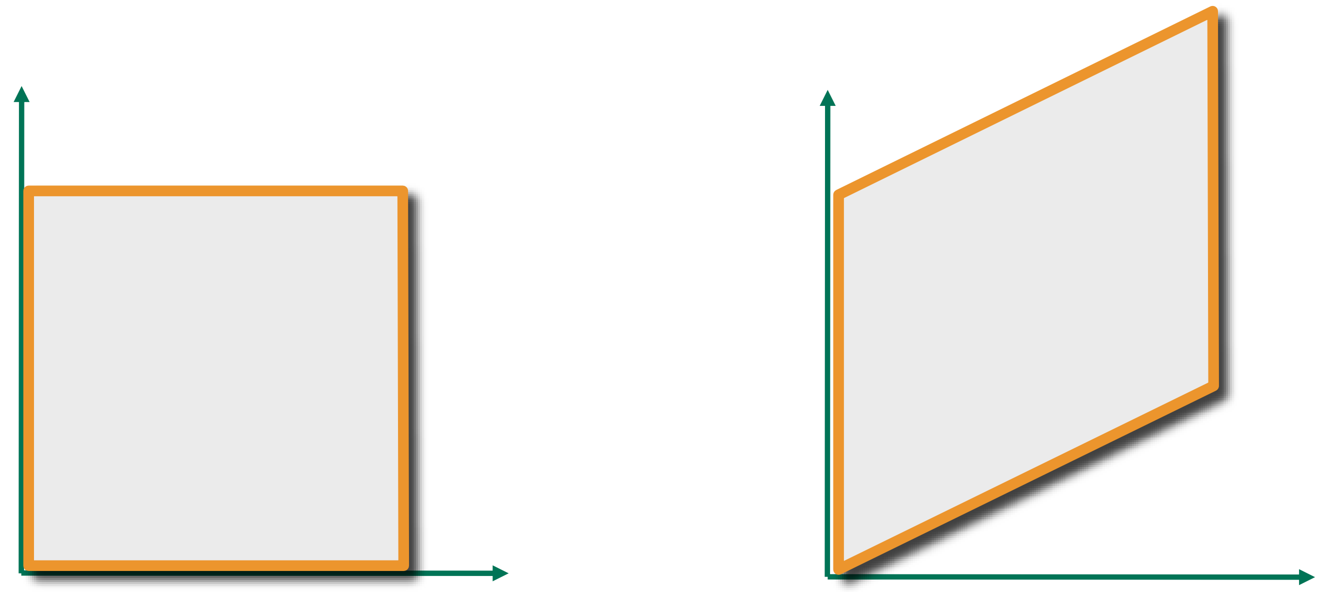
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}}_{\mathbf{H}_x(a)} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$



$\mathbf{H}_x(0.5)$

- Shear along y-axis

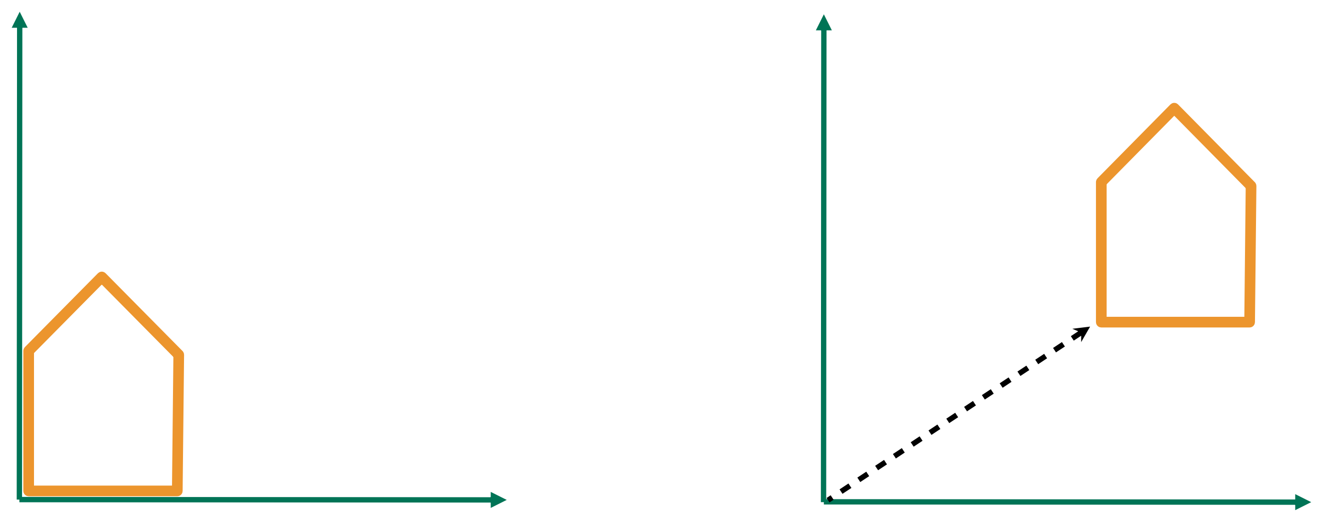
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}}_{\mathbf{H}_y(b)} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$



$\mathbf{H}_y(0.5)$

2D Translation

- Translation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$



- Matrix representation? $\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{T}(t_x, t_y) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

Affine Transformations

- Translation is not linear, but it is **affine**
  - Origin is no longer a fixed point
- Affine map = linear map + translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} = \mathbf{L}\mathbf{x} + \mathbf{t}$$

- Is there a matrix representation for all affine transformations?
  - A unified framework -> simpler to code and optimize

Homogenous Coordinates

- Add a third coordinate (w-coordinate)
  - 2D point = $(x, y, 1)^T$
  - 2D vector = $(x, y, 0)^T$

$$\begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \\ 1 \end{pmatrix}$$

- Matrix representation of translations

Affine Transformations

- Affine map = linear map + translation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

- Using homogenous coordinates:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

2D Transformations

- Scale

$$\mathbf{S}(s_x, s_y) \;=\; \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation

$$\mathbf{R}(\alpha) \;=\; \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Translation

$$\mathbf{T}(t_x, t_y) \;=\; \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

Concatenation of Transformations

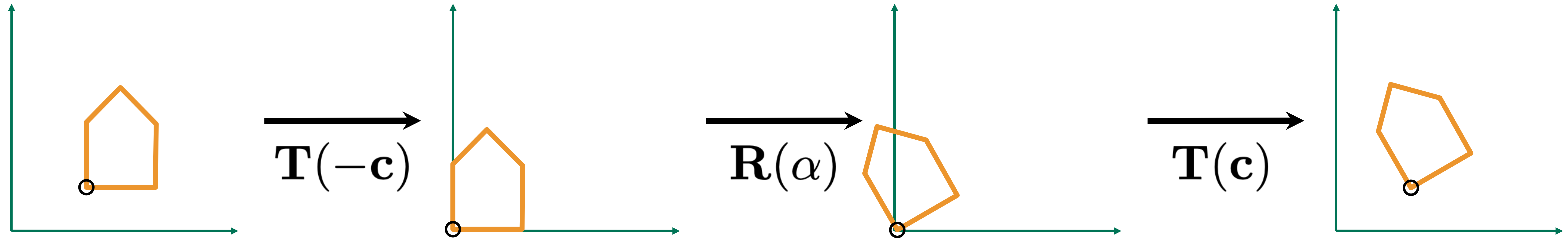
- Sequence of affine maps A_1, A_2, A_3, ...
  - Concatenation by matrix multiplication

$$A_n(\dots A_2(A_1(\mathbf{x}))) \;=\; \mathbf{A}_n \cdots \mathbf{A}_2 \cdot \mathbf{A}_1 \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

  - Very important for performance!
  - Matrix multiplication not commutative, ordering is important!

2D Rotation

- How to rotate around a given point c?
  1. Translate c to origin
  2. Rotate
  3. Translate back

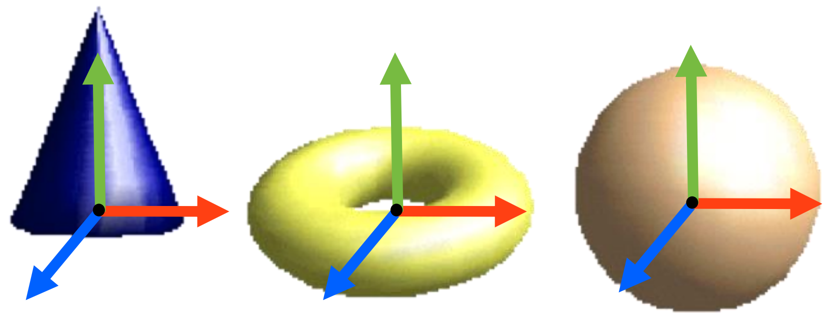


- Matrix representation?

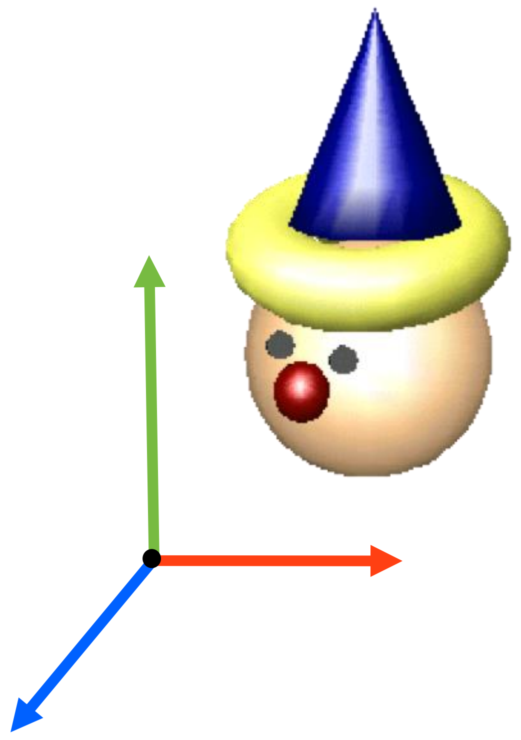
$$\mathbf{T}(\mathbf{c}) \cdot \mathbf{R}(\alpha) \cdot \mathbf{T}(-\mathbf{c})$$

View Transformations

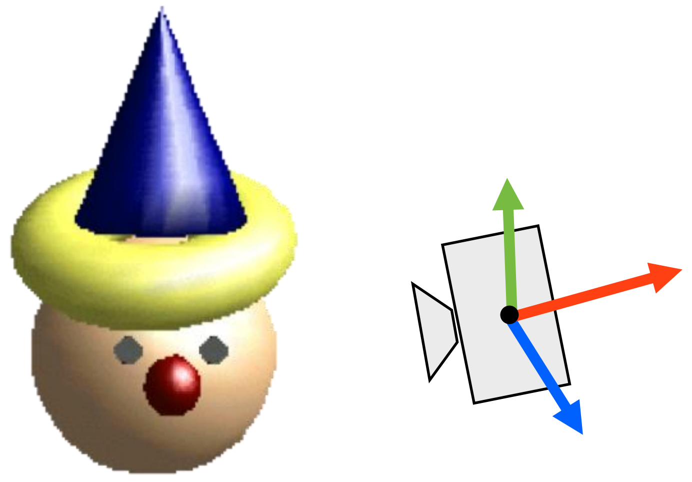
Coordinate Systems



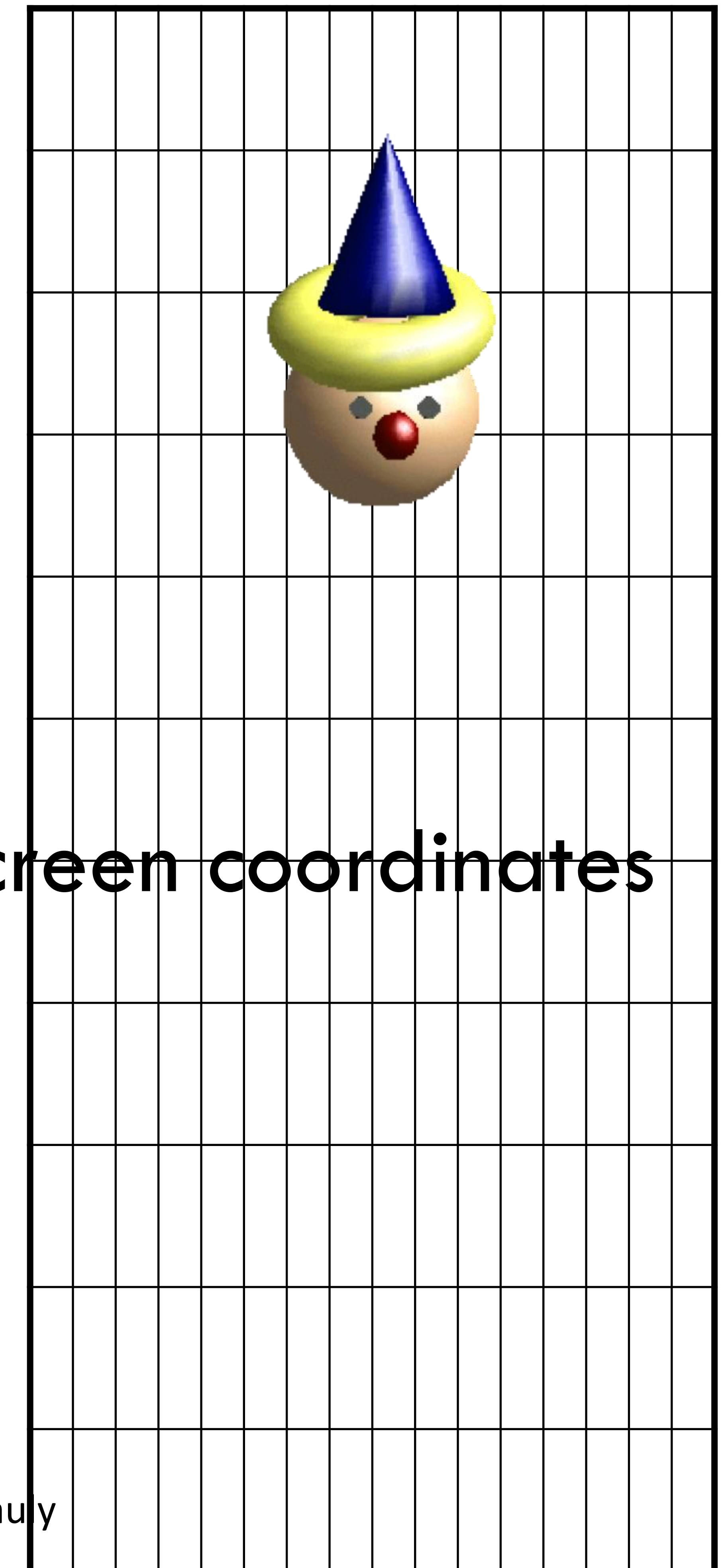
object coordinates



world coordinates

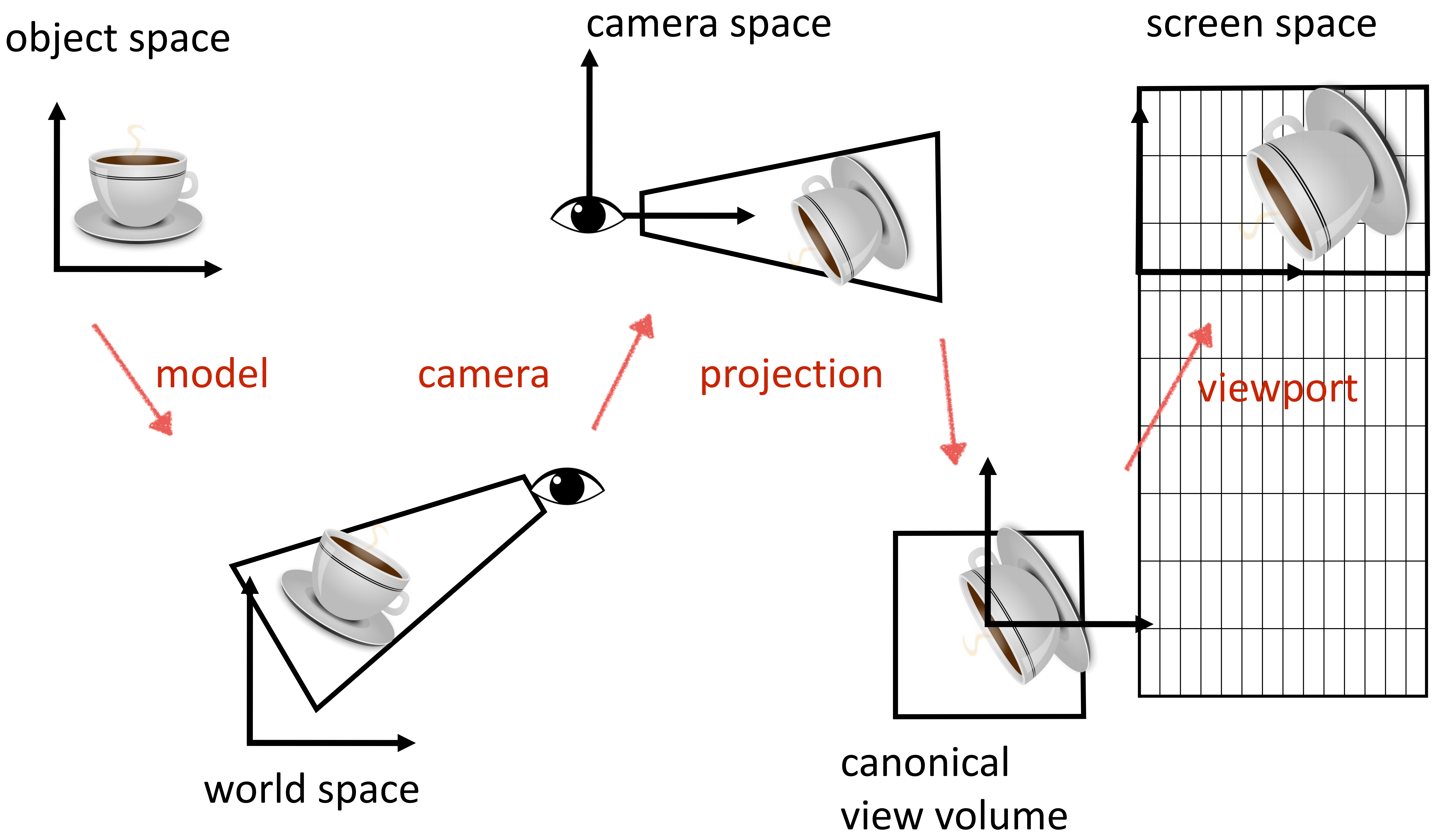


camera coordinates

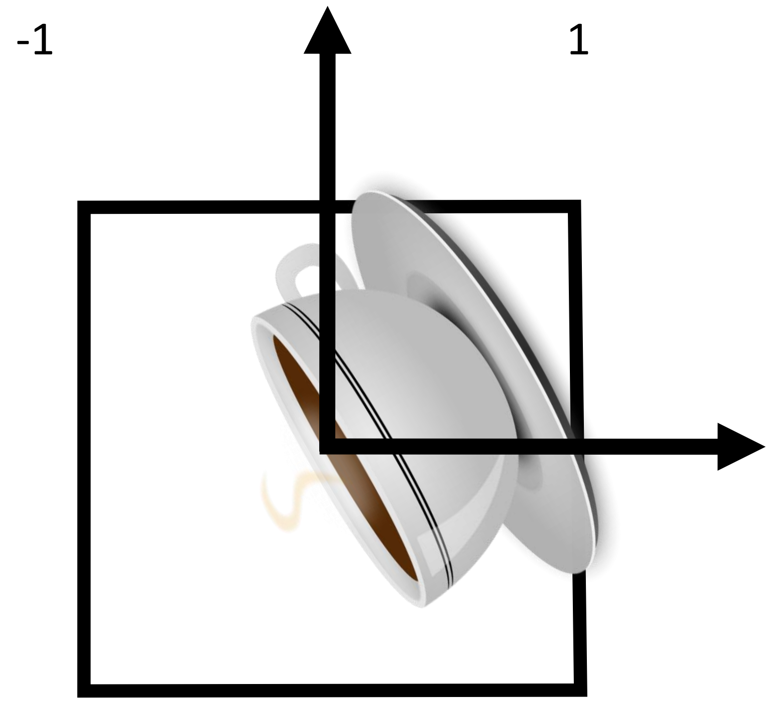


screen coordinates

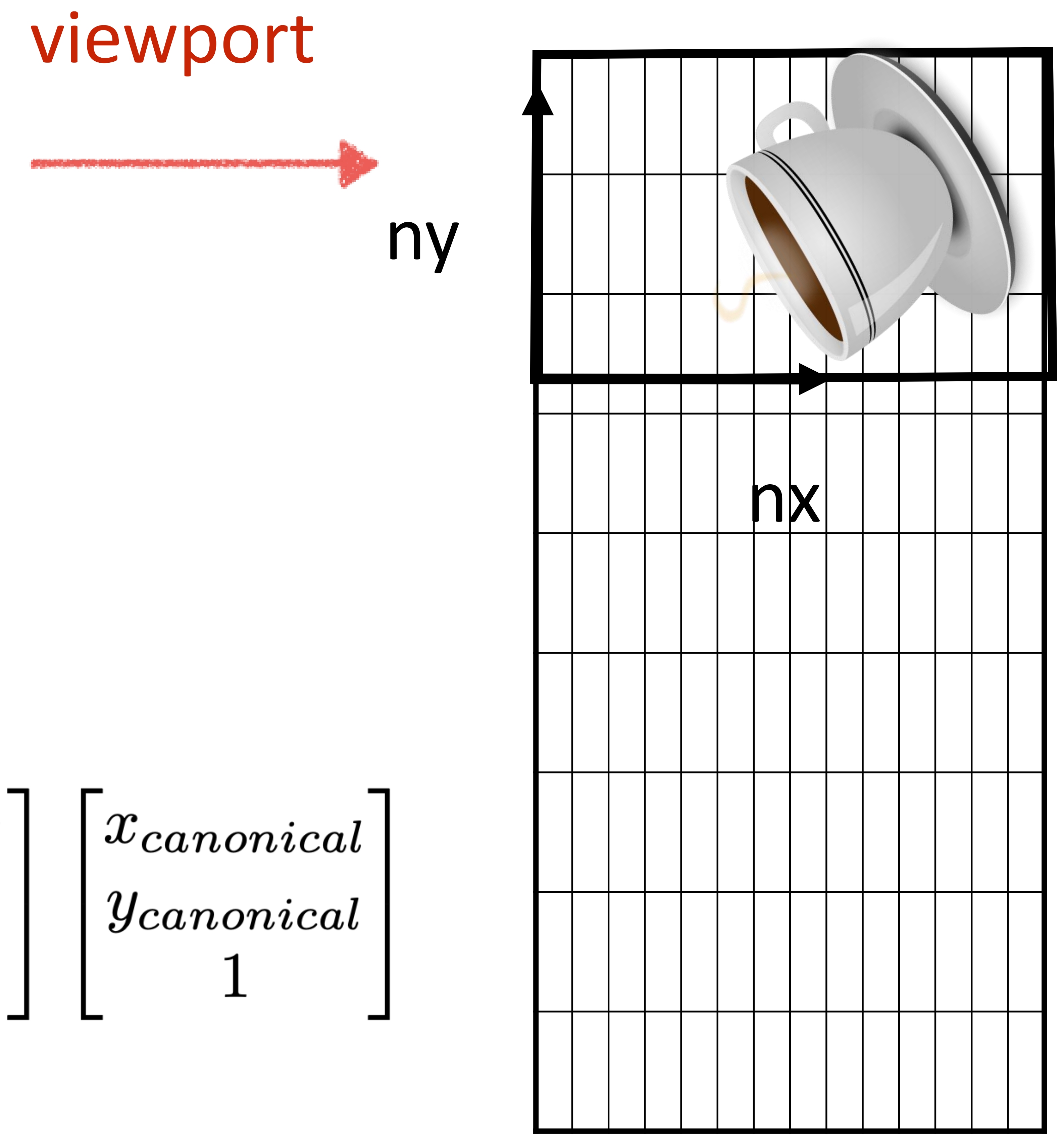
View Transformation



Viewport transformation



canonical
view volume

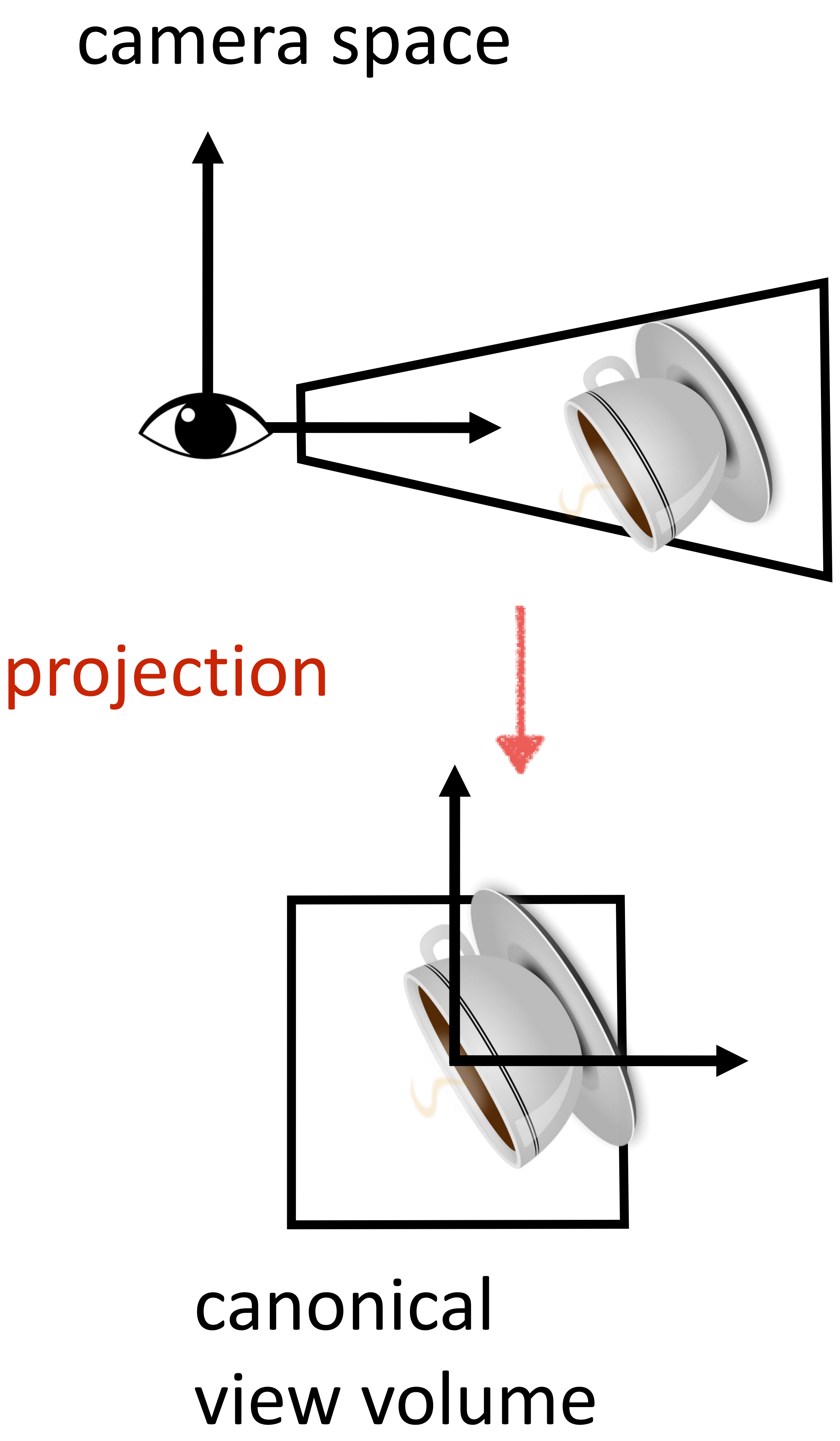


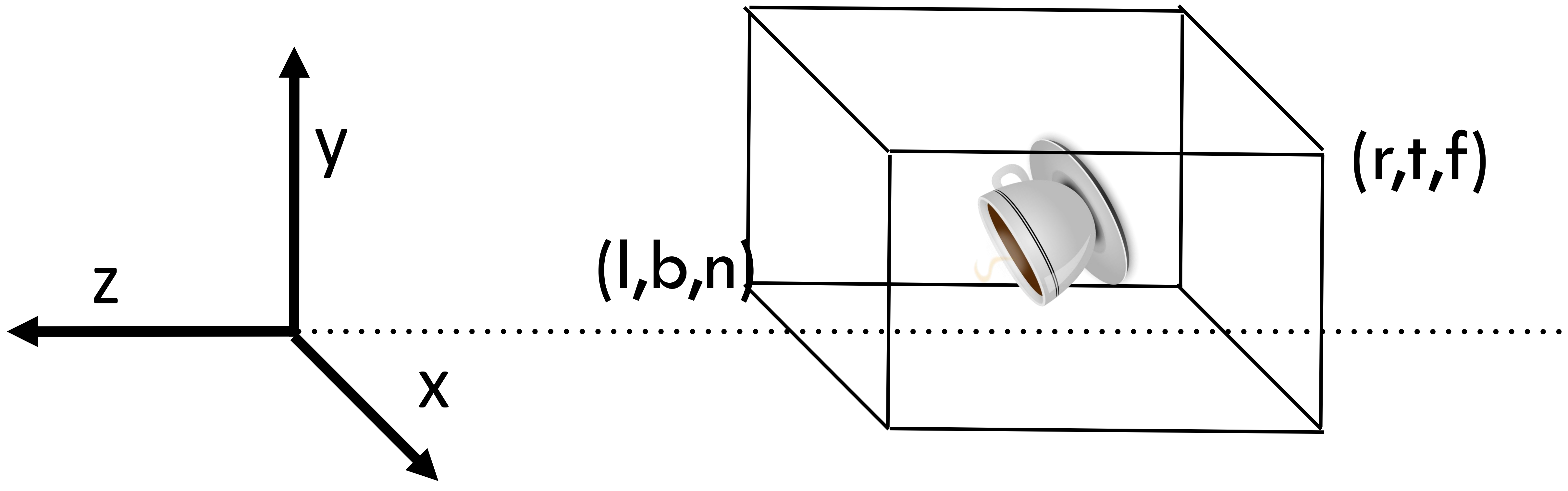
screen space

$$\begin{bmatrix} x_{screen} \\ y_{screen} \\ 1 \end{bmatrix} = \begin{bmatrix} nx/2 & 0 & \frac{n_x-1}{2} \\ 0 & ny/2 & \frac{n_y-1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{canonical} \\ y_{canonical} \\ 1 \end{bmatrix}$$

How does it look in 3D?

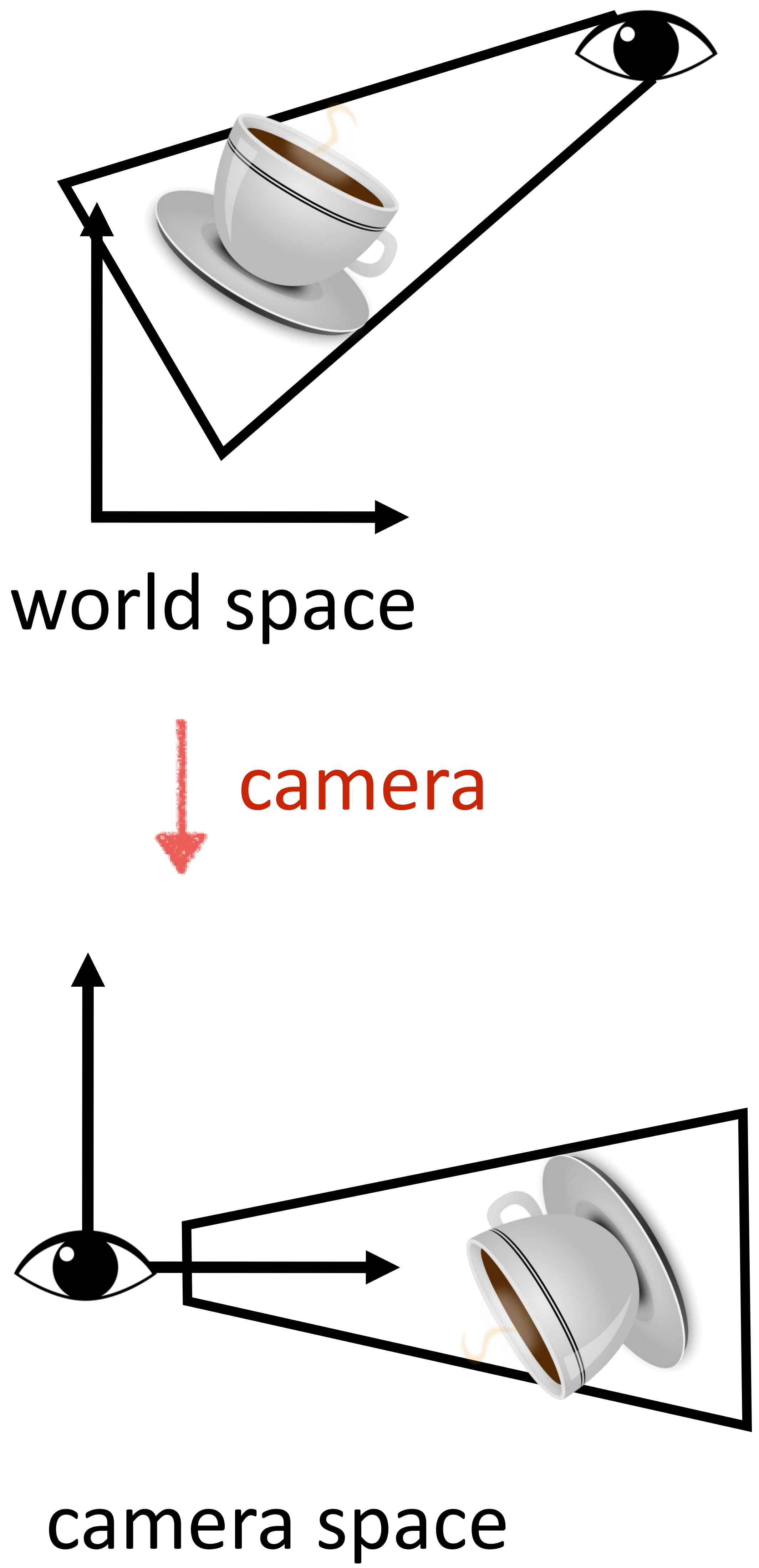
Orthographic Projection



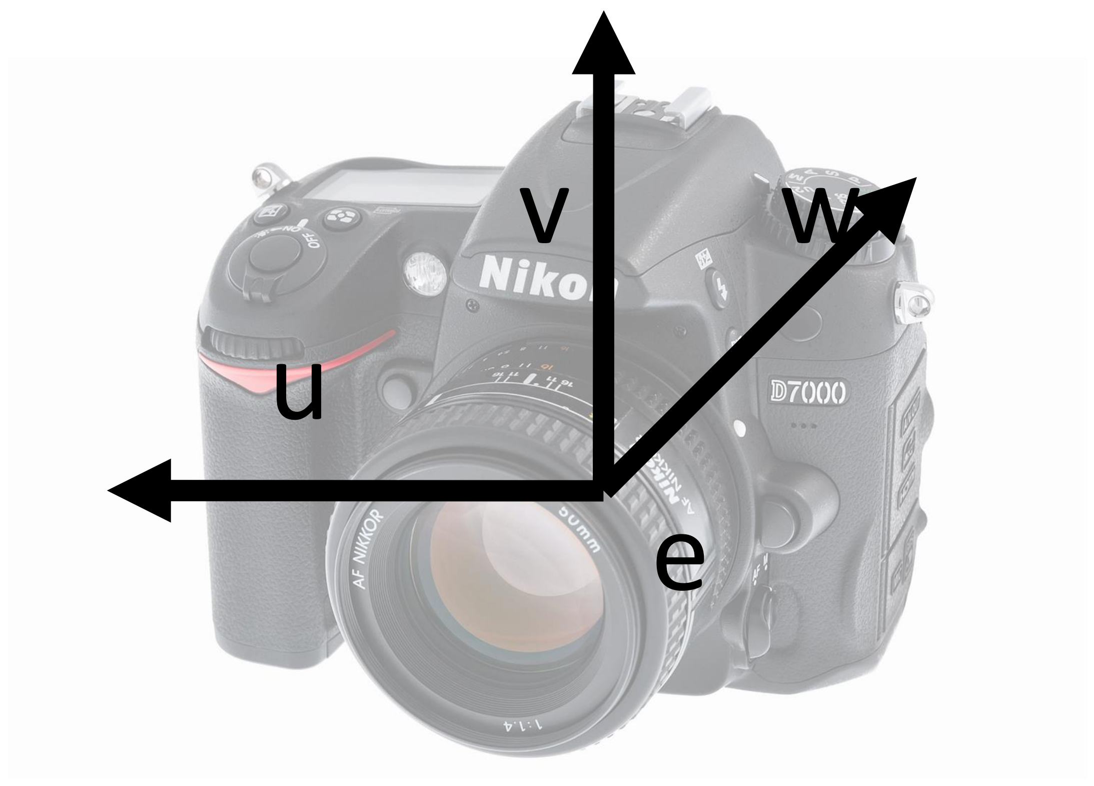


$$\mathbf{M}_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Camera Transformation



1. Construct the camera reference system given:
   1. The eye position e
   2. The gaze direction g
   3. The view-up vector t

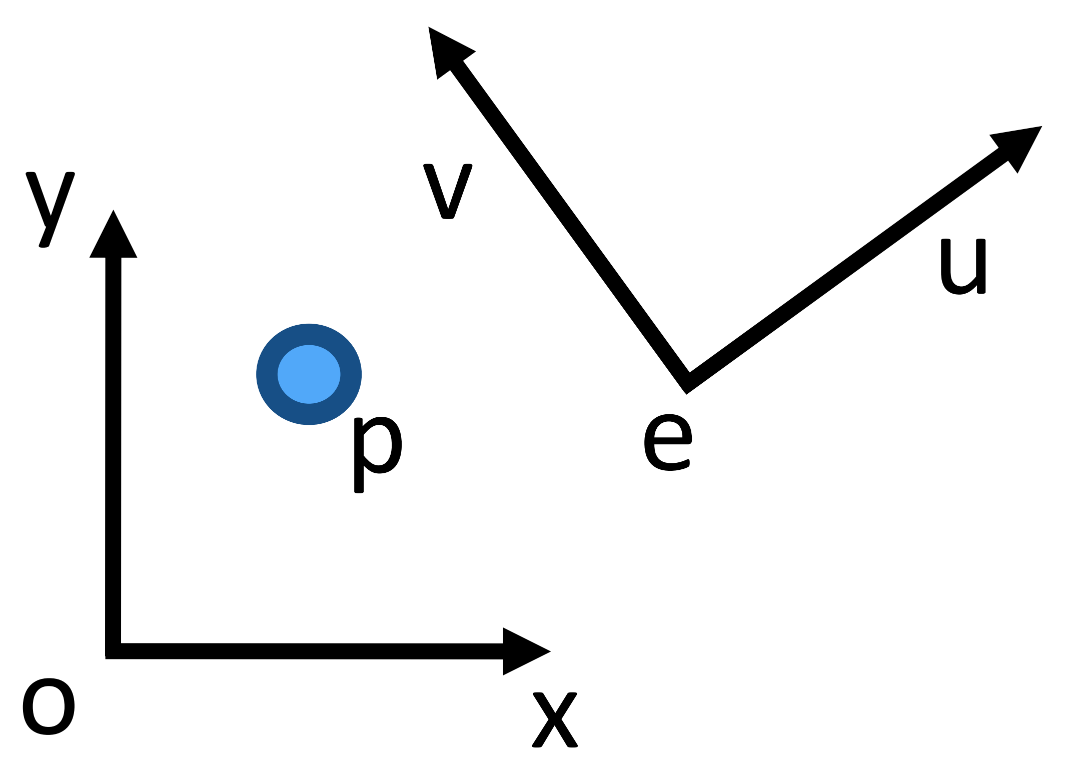


$$\mathbf{w} = -\frac{\mathbf{g}}{||\mathbf{g}||}$$

$$\mathbf{u} = \frac{\mathbf{t} \times \mathbf{w}}{||\mathbf{t} \times \mathbf{w}||}$$

$$\mathbf{v} = \mathbf{w} \times \mathbf{u}$$

Change of frame



$$\mathbf{p} = (p_x, p_y) = \mathbf{o} + p_x\mathbf{x} + p_y\mathbf{y}$$
$$\mathbf{p} = (p_u, p_v) = \mathbf{e} + p_u\mathbf{u} + p_v\mathbf{v}$$

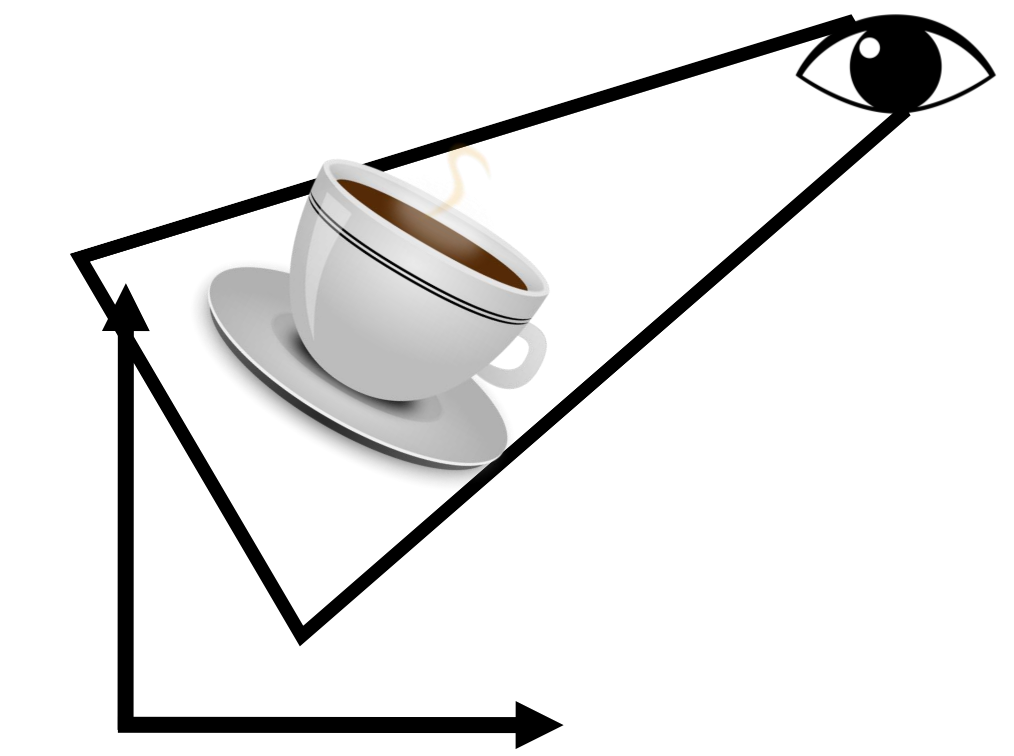
$$\begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & e_x \\ 0 & 1 & e_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_x & v_x & 0 \\ u_y & v_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ 1 \end{bmatrix} = \begin{bmatrix} u_x & v_x & e_x \\ u_y & v_y & e_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ 1 \end{bmatrix}$$

$$\mathbf{p}_{xy} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{e} \\ 0 & 0 & 1 \end{bmatrix} \mathbf{p}_{uv}$$

$$\mathbf{p}_{uv} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{e} \\ 0 & 0 & 1 \end{bmatrix}^{-1} \mathbf{p}_{xy}$$

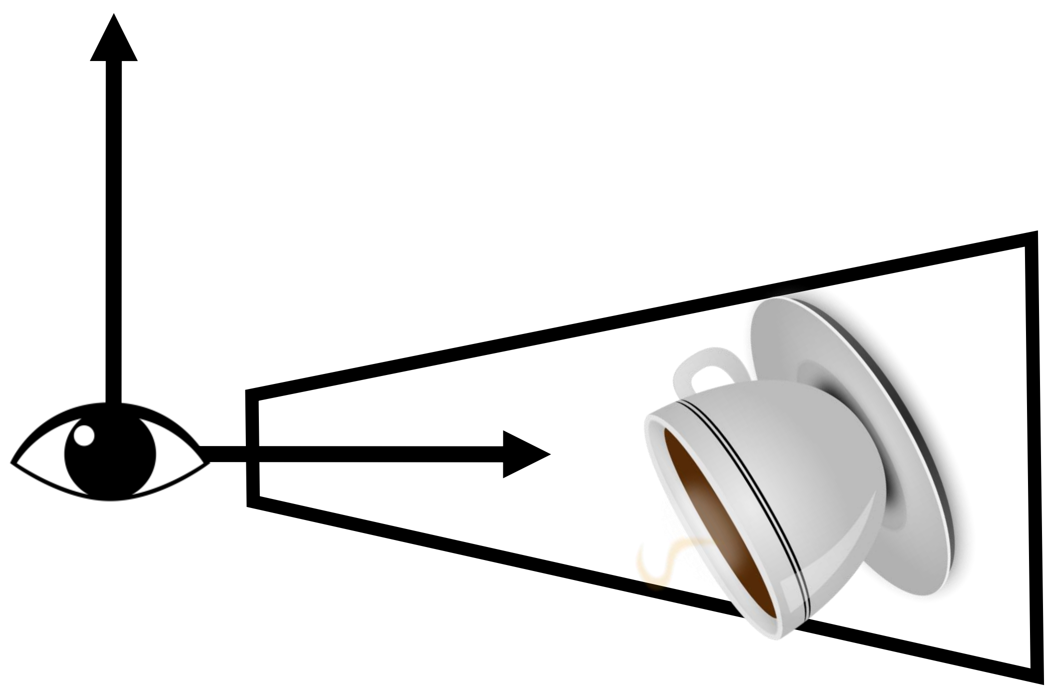
Can you write it directly without the inverse?

Camera Transformation



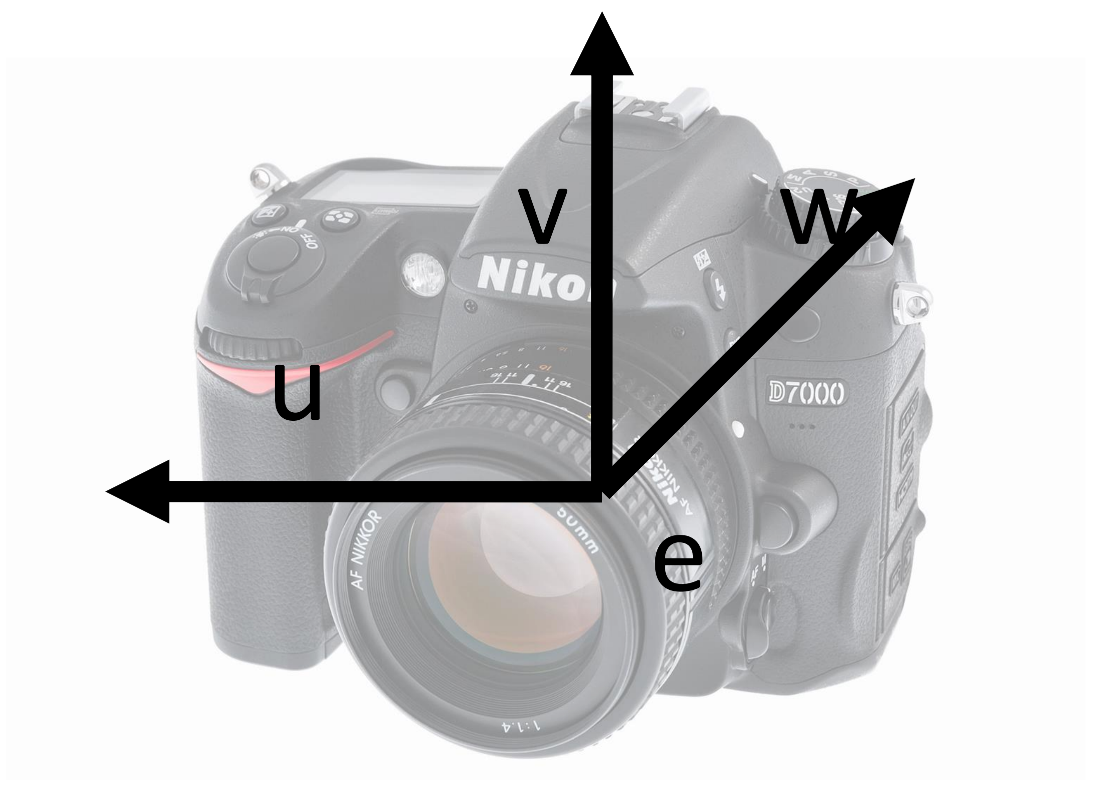
world space

camera



camera space

1. Construct the camera reference system given:
   1. The eye position e
   2. The gaze direction g
   3. The view-up vector t



$$\mathbf{w} = -\frac{\mathbf{g}}{||\mathbf{g}||}$$

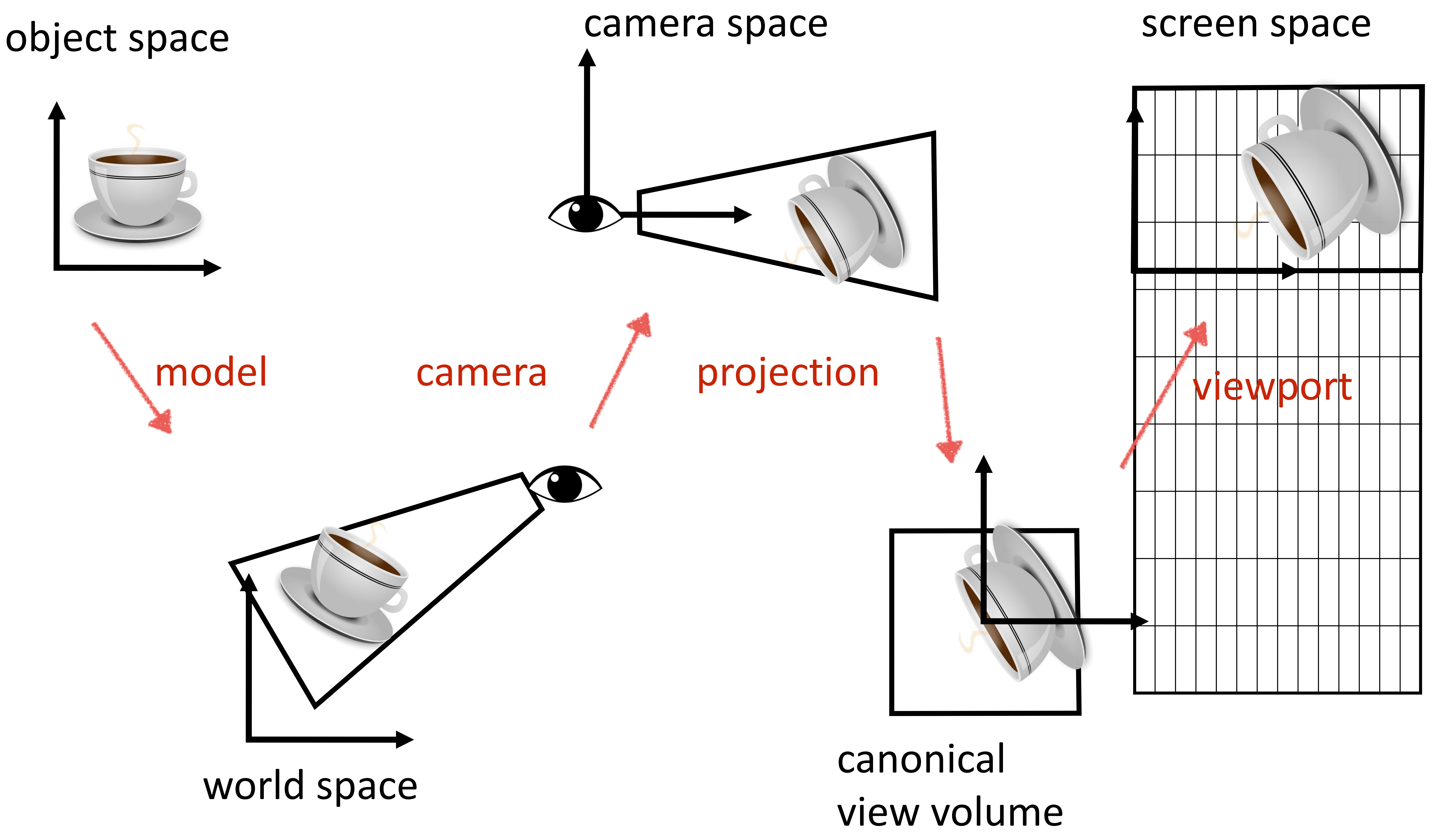
$$\mathbf{u} = \frac{\mathbf{t} \times \mathbf{w}}{||\mathbf{t} \times \mathbf{w}||}$$

$$\mathbf{v} = \mathbf{w} \times \mathbf{u}$$

2. Construct the unique transformations that converts world coordinates into camera coordinates

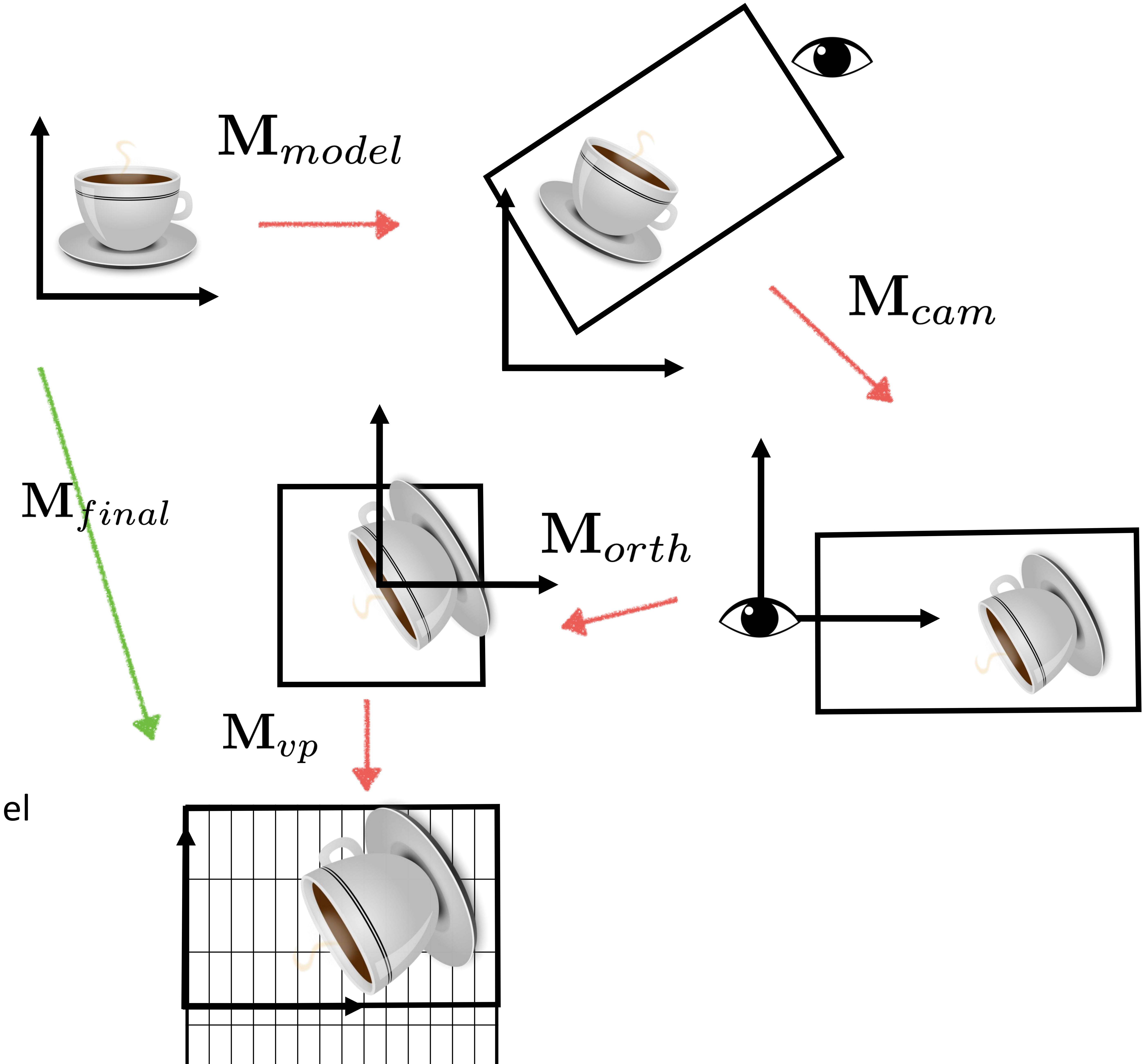
$$\mathbf{M}_{cam} = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{e} \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}$$

View Transformation



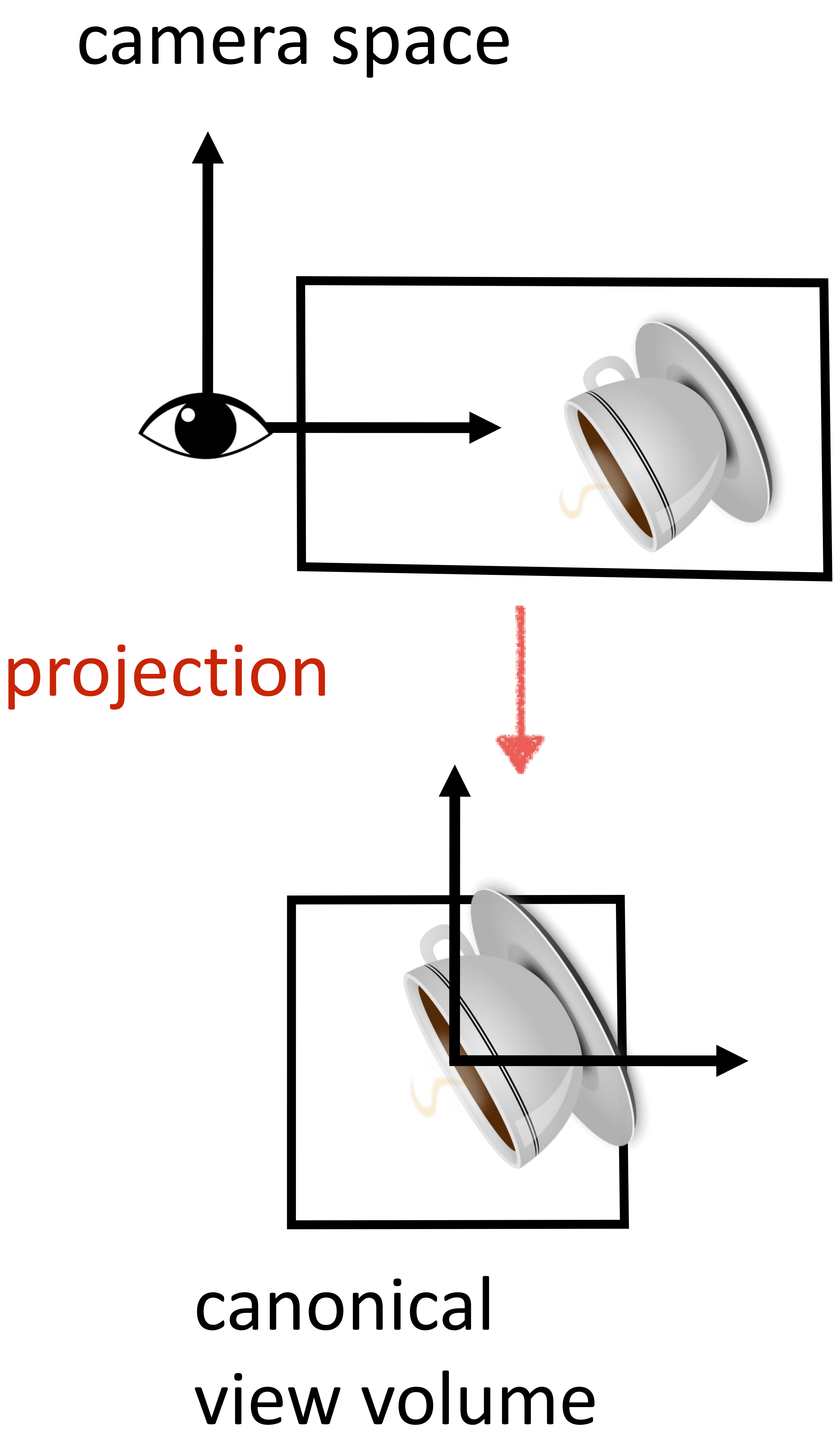
Algorithm

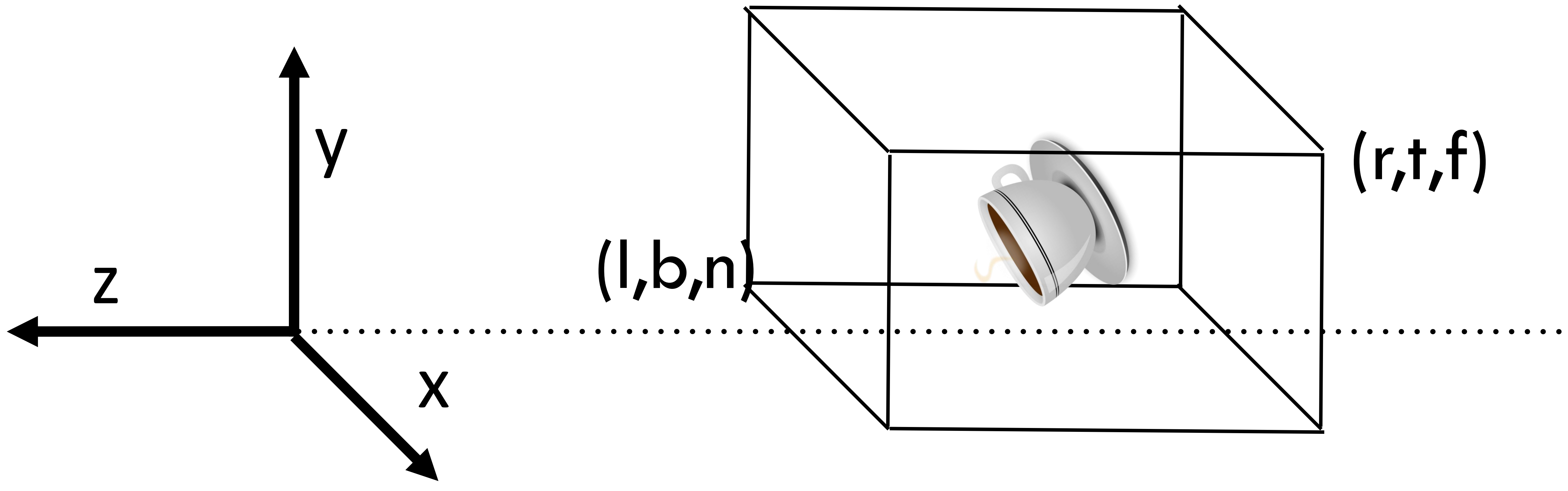
- Construct Viewport Matrix $\mathbf{M}_{vp}$
- Construct Projection Matrix $\mathbf{M}_{orth}$
- Construct Camera Matrix $\mathbf{M}_{cam}$
- $\mathbf{M} = \mathbf{M}_{vp}\mathbf{M}_{orth}\mathbf{M}_{cam}$
- For each model $\mathbf{M}_{model}$
  - Construct Model Matrix
  - $\mathbf{M}_{final} = \mathbf{M}\mathbf{M}_{model}$
  - For every point p in each primitive of the model
    - $\mathbf{p}_{final} = \mathbf{M}_{final}\mathbf{p}$
  - Rasterize the model



Perspective Projection

Orthographic Projection

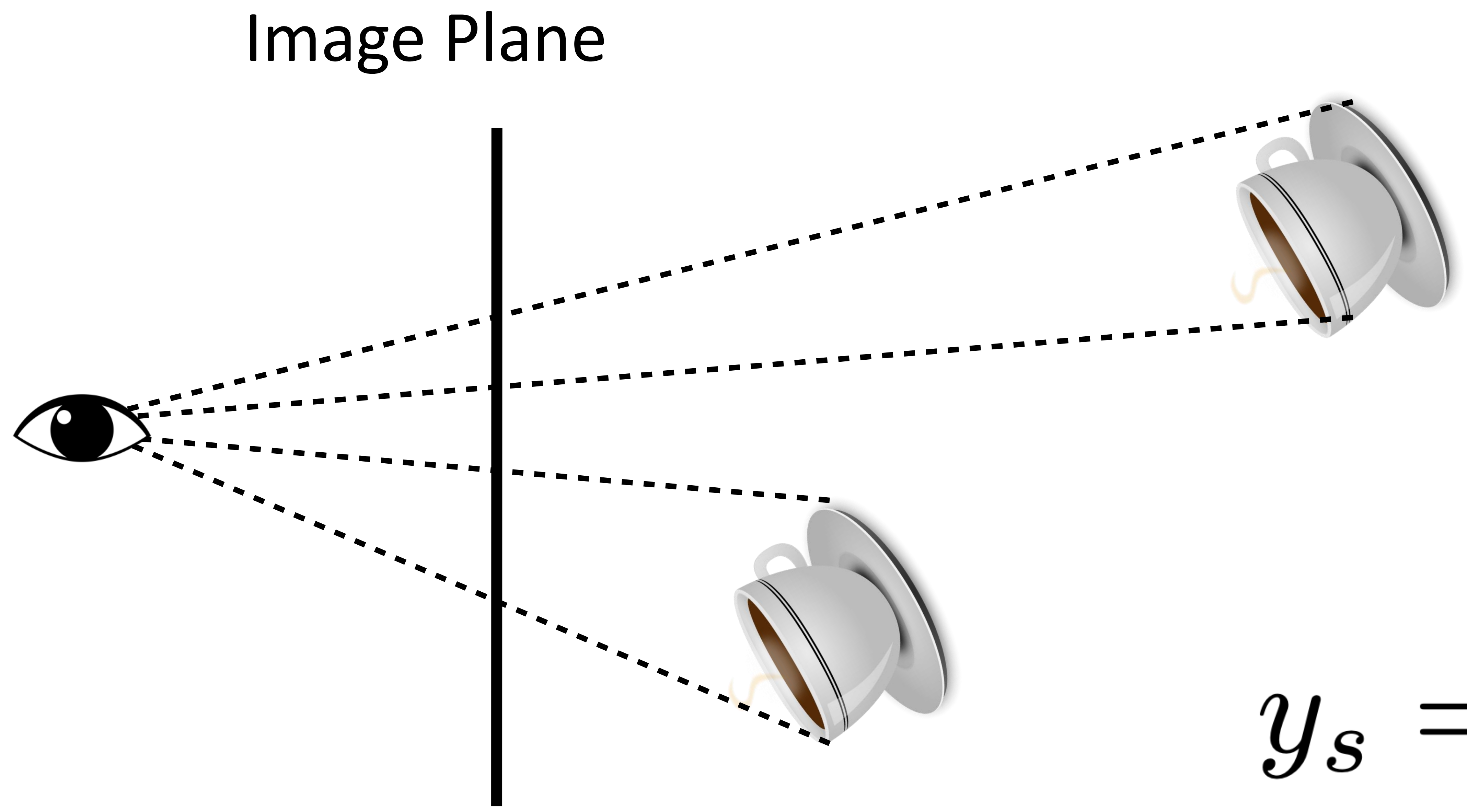




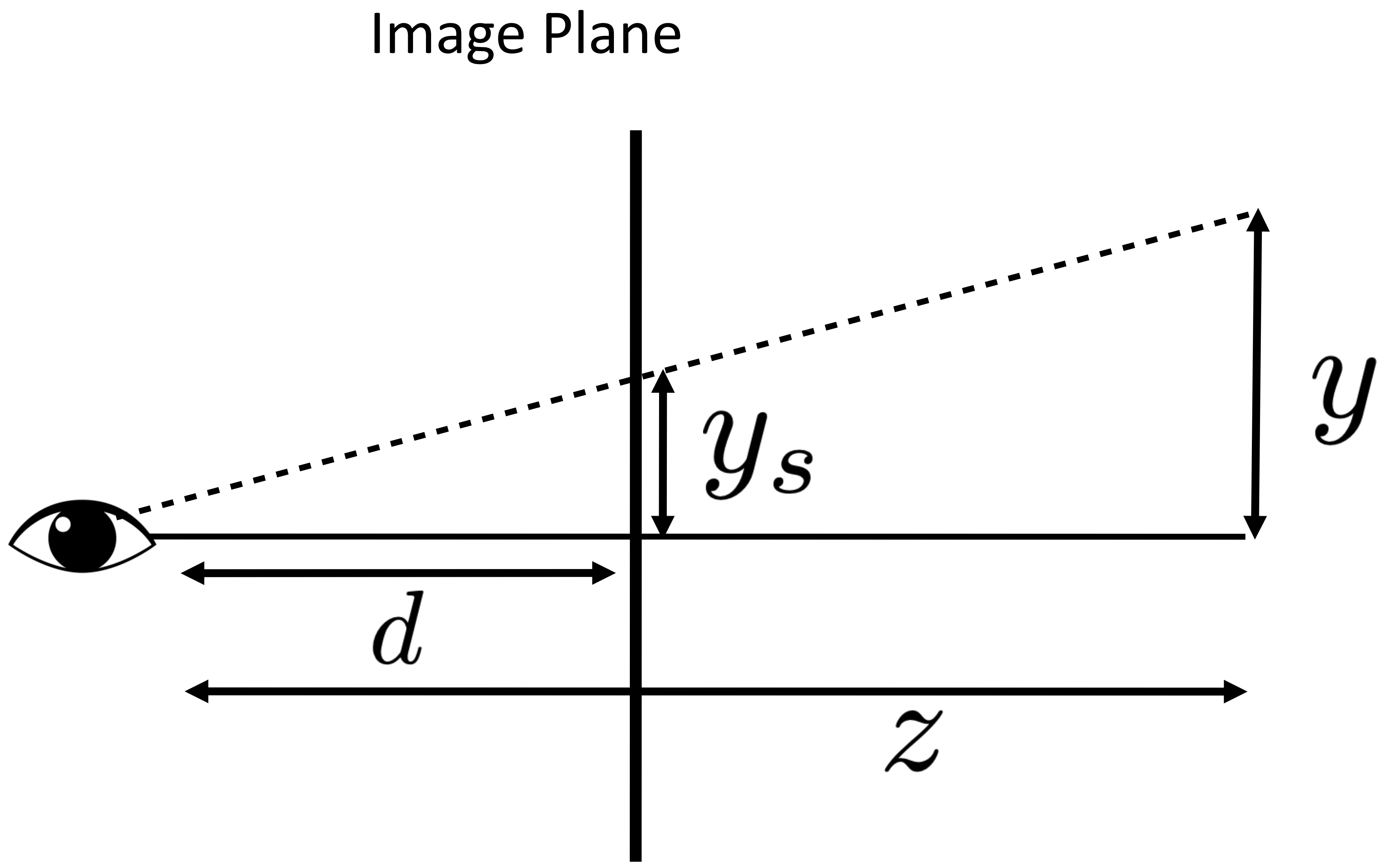
$$\mathbf{M}_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Perspective Projection

- In Orthographic projection, the size of the objects does not change with distance
- In Perspective projection, the objects that are far away look smaller



$$y_s = \frac{dy}{z}$$



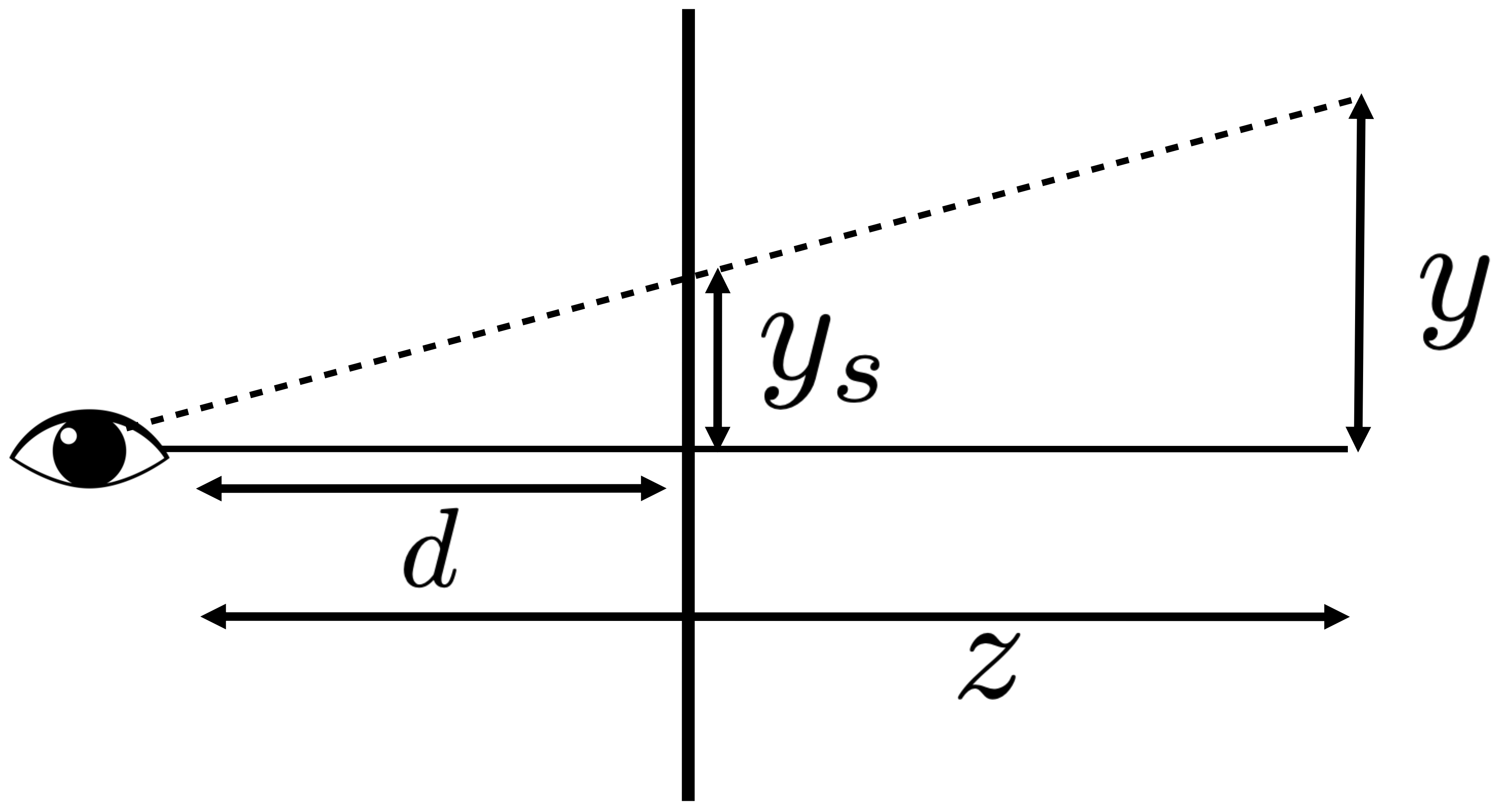
Divisions in Matrix Form

- How do we encode divisions?

$$y_s = \frac{dy}{z}$$

- We extend homogeneous coordinates

Image Plane



Until now…

- What do we have left?

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 y + c_1 \\ a_2 x + b_2 y + c_2 \\ 1 \end{pmatrix}$$

- Use the last row of the transformation:

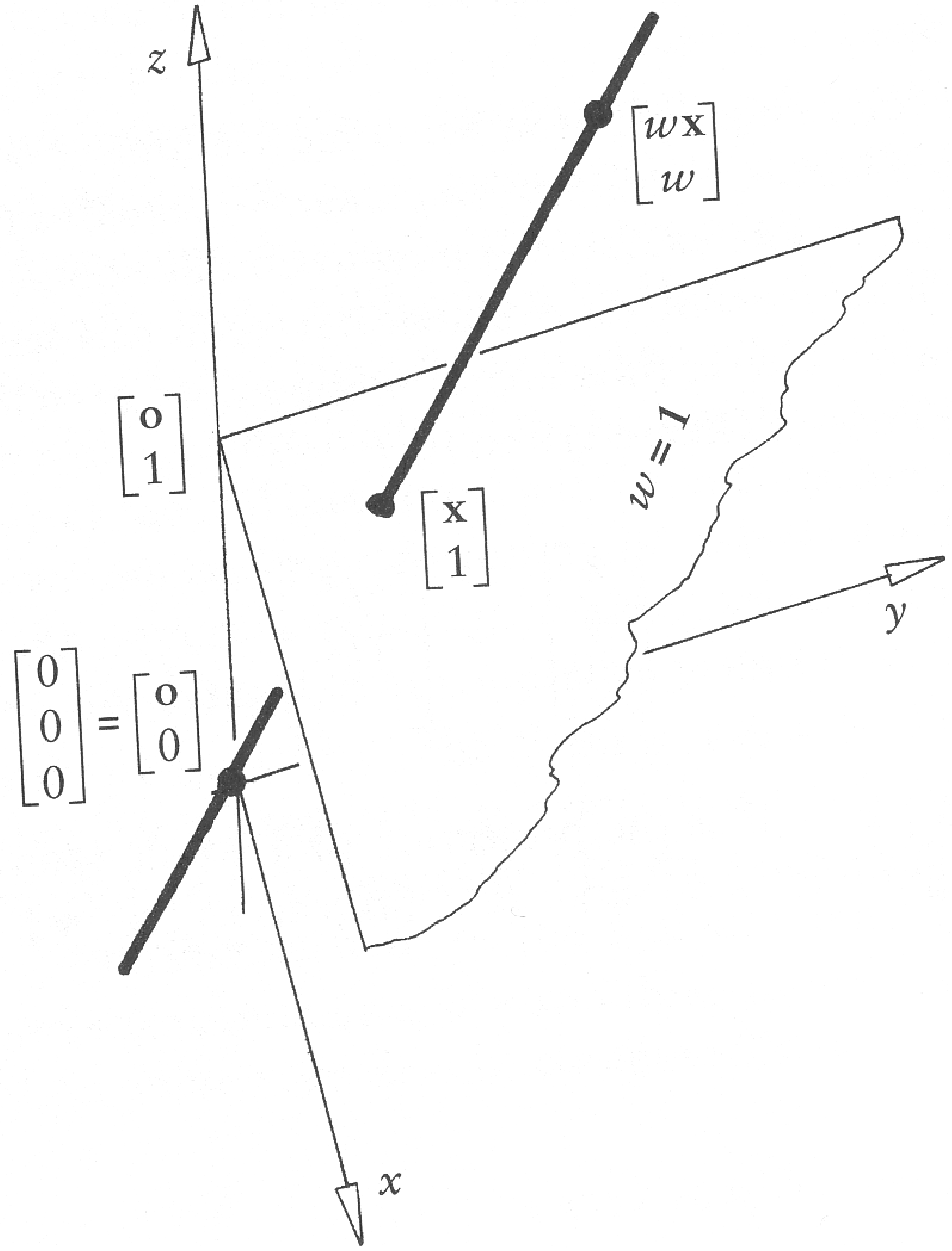
$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ e & f & g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 y + c_1 \\ a_2 x + b_2 y + c_2 \\ ex + fy + g \end{pmatrix} \sim \begin{pmatrix} \frac{a_1 x + b_1 y + c_1}{ex + fy + g} \\ \frac{a_2 x + b_2 y + c_2}{ex + fy + g} \\ 1 \end{pmatrix}$$

Intuition

- Purely algebraic:

$$\begin{pmatrix} x \\ y \\ w \end{pmatrix} \sim \begin{pmatrix} x/w \\ y/w \\ 1 \end{pmatrix}$$

- As a projection, each line is identified by a point on the plane z=1



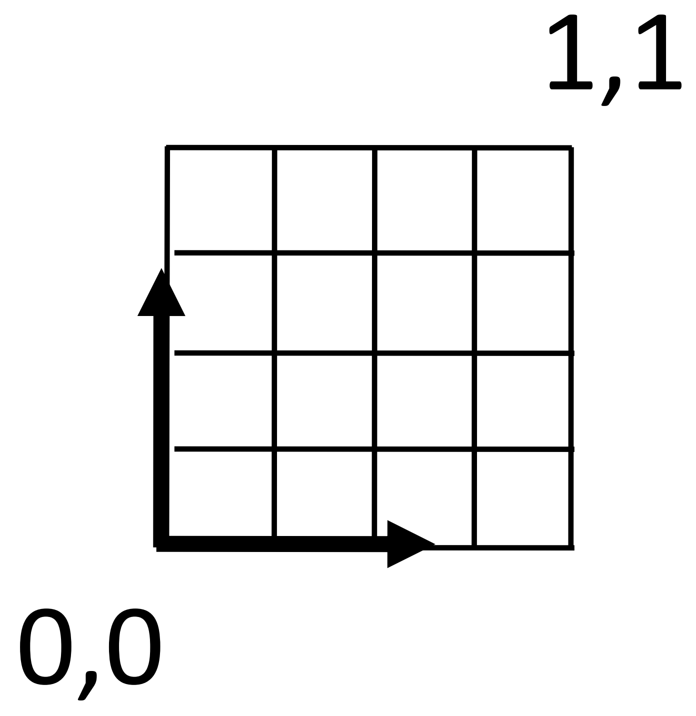
Projective Transformation

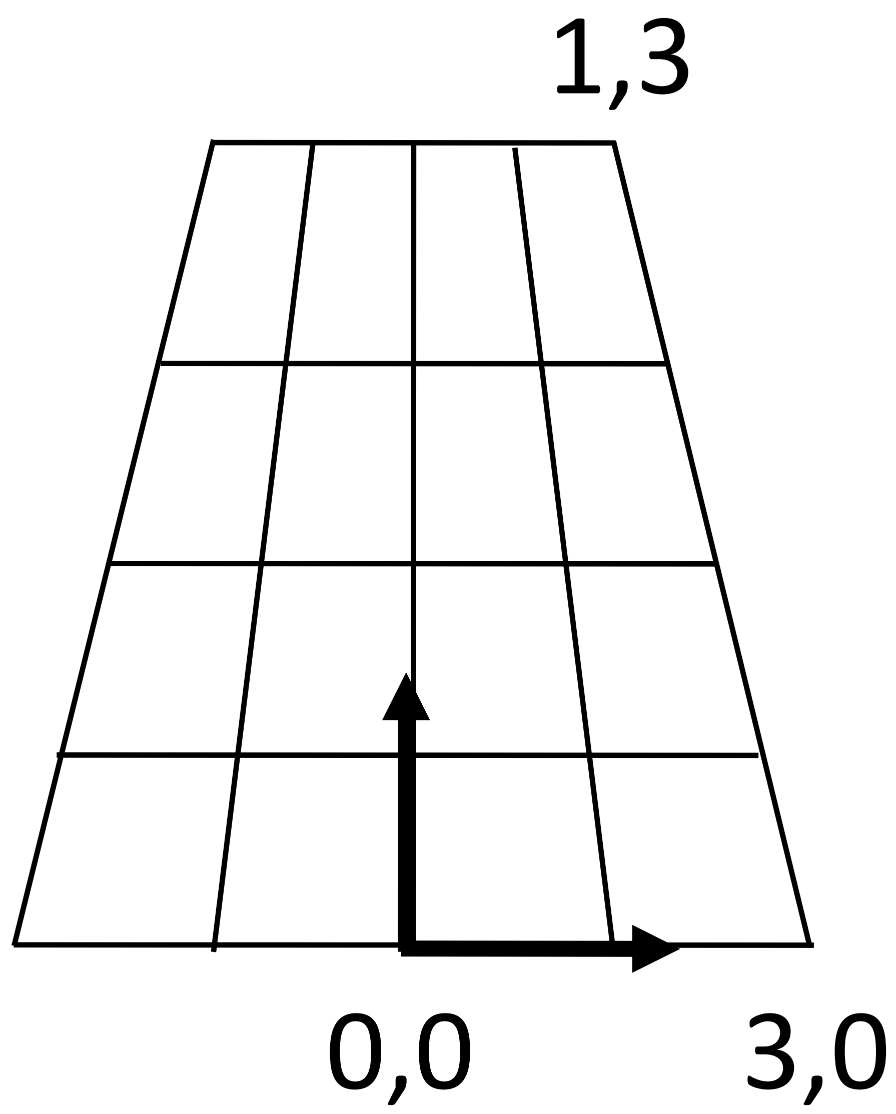
- A transformation of this form is called a projective transformation (or a homography)
- The points are represented in homogeneous coordinates

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ e & f & g \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} a_1 x + b_1 y + c_1 \\ a_2 x + b_2 y + c_2 \\ ex + fy + g \end{pmatrix} \sim \begin{pmatrix} \frac{a_1 x + b_1 y + c_1}{ex + fy + g} \\ \frac{a_2 x + b_2 y + c_2}{ex + fy + g} \\ 1 \end{pmatrix}$$

Example

$$\mathbf{M} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$





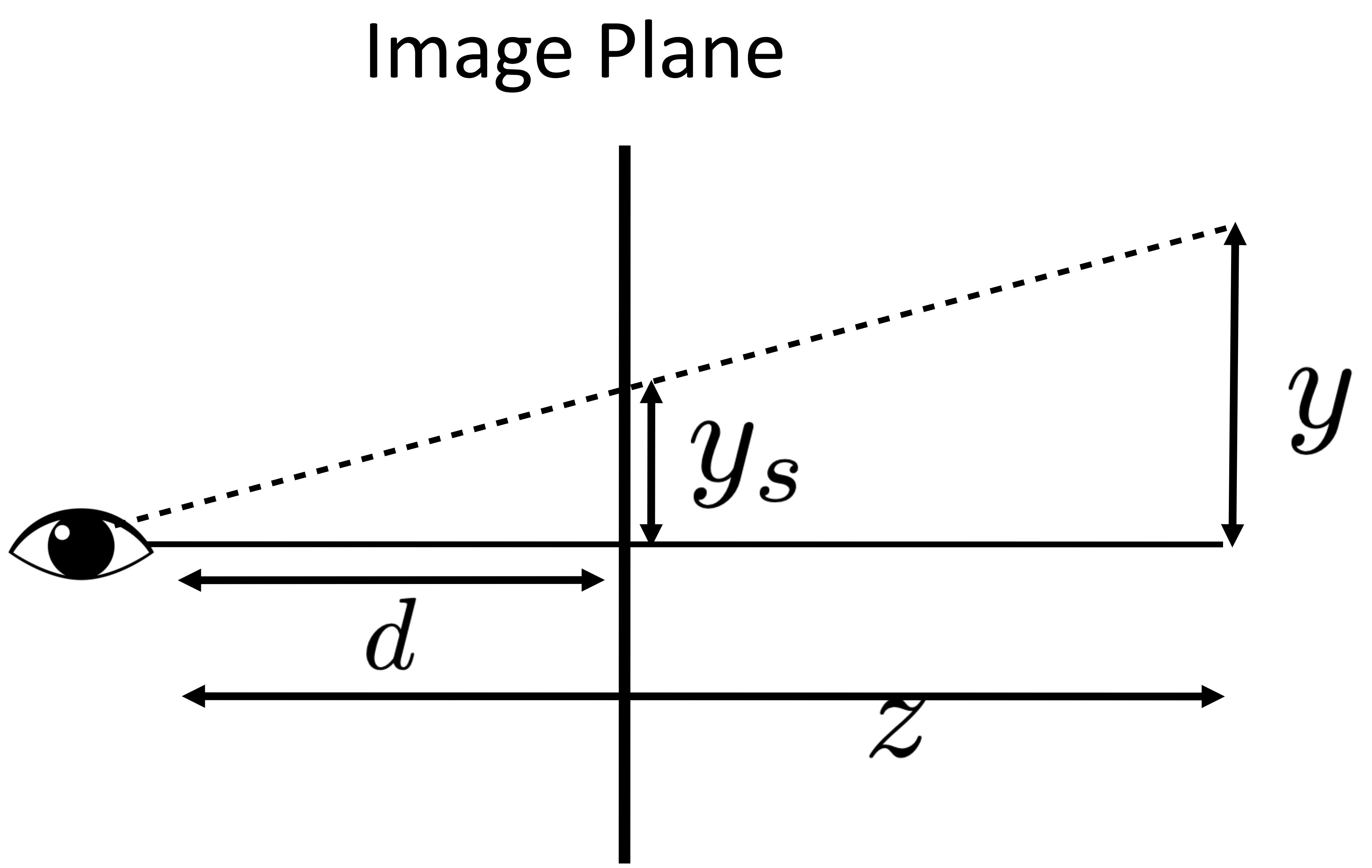
- It transforms a square into a quadrilateral — note that straight lines are preserved, but parallel lines are not!
- You can use homogeneous coordinates for as many transformations as you want, only when you need the cartesian representation you have to normalize

Perspective Projection

- Perspective projection is easily implementable using this machinery

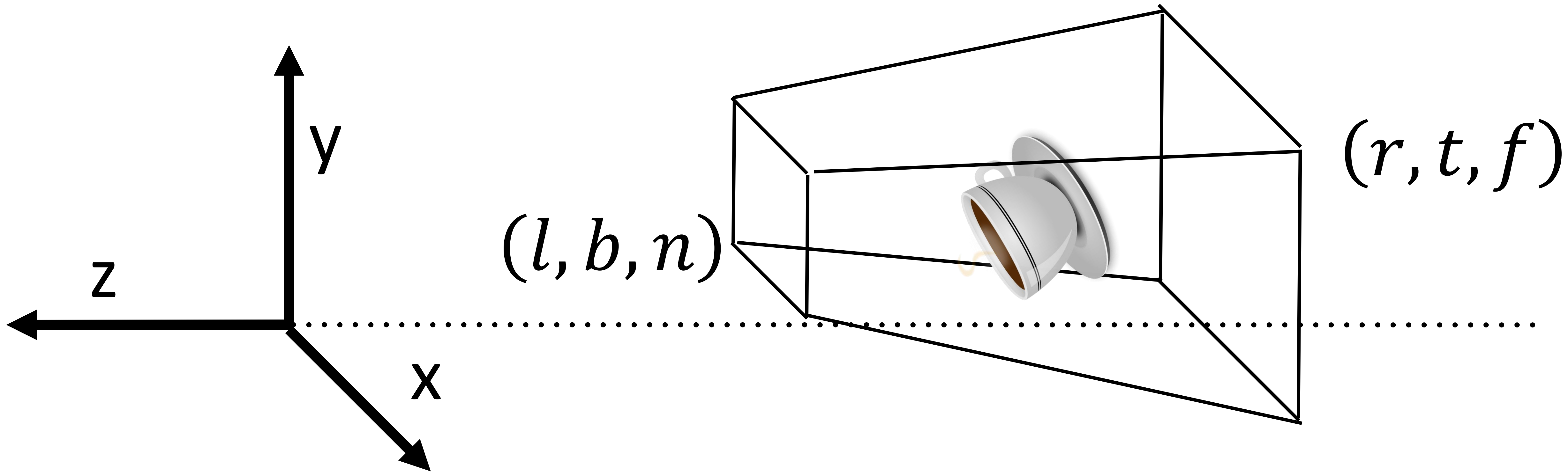
$$y_s = \frac{dy}{z}$$

$$\begin{pmatrix} y_s \\ 1 \end{pmatrix} \sim \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \\ 1 \end{pmatrix}$$



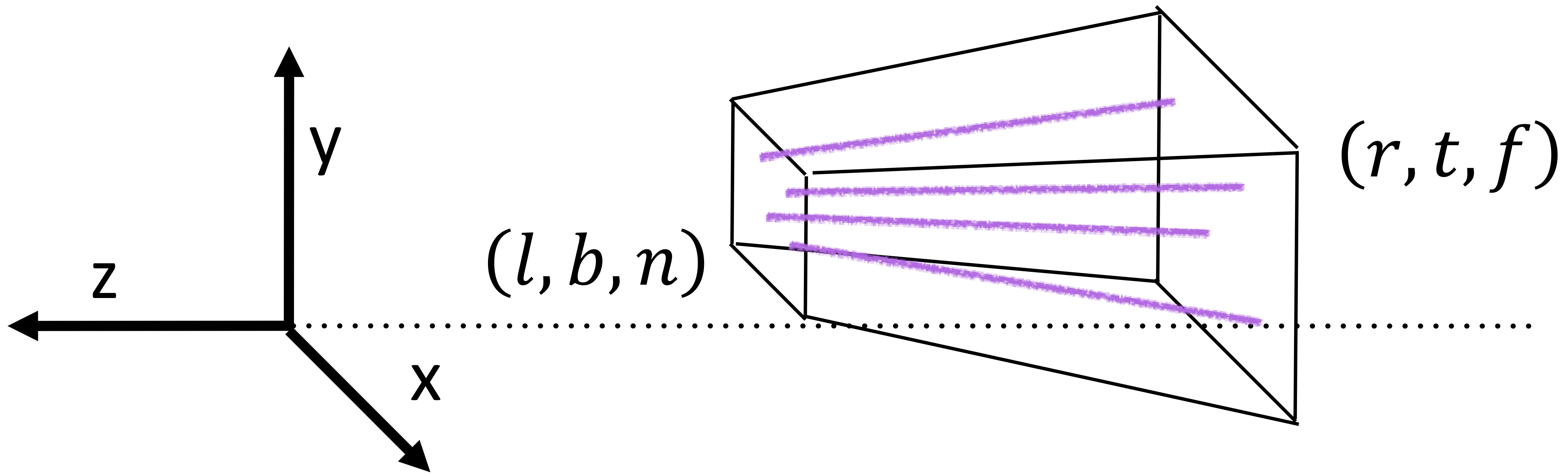
Perspective Projection

- We will use the same conventions that we used for orthographic:
  - Camera at the origin, pointing negative z
  - We scale x, y and “bring along” the z



$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

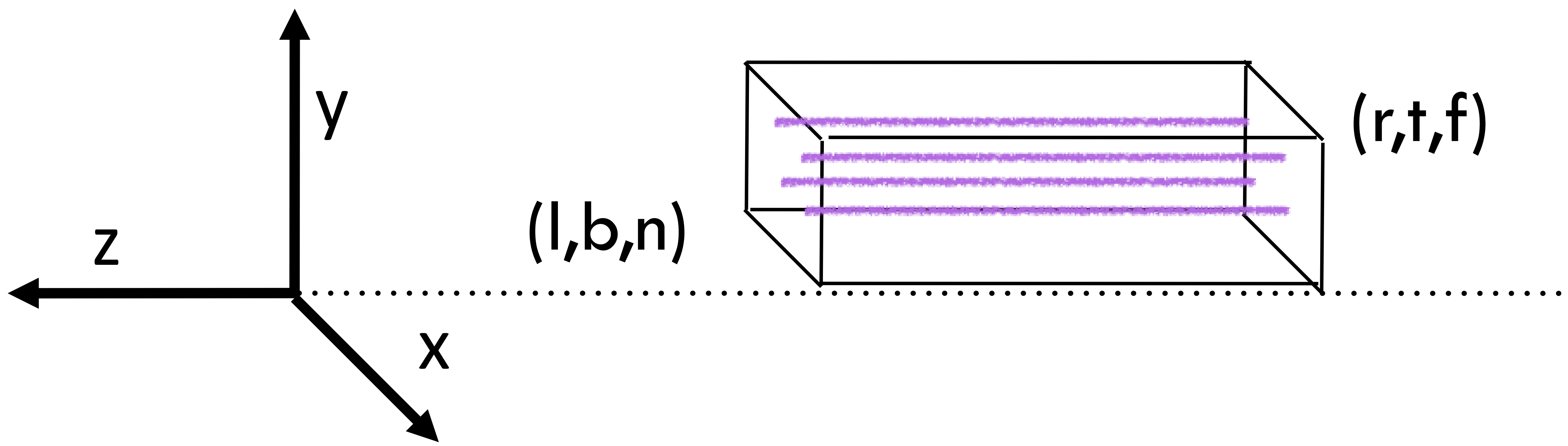
Effect on the points



$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n + f - \frac{fn}{z} \\ 1 \end{pmatrix}$$

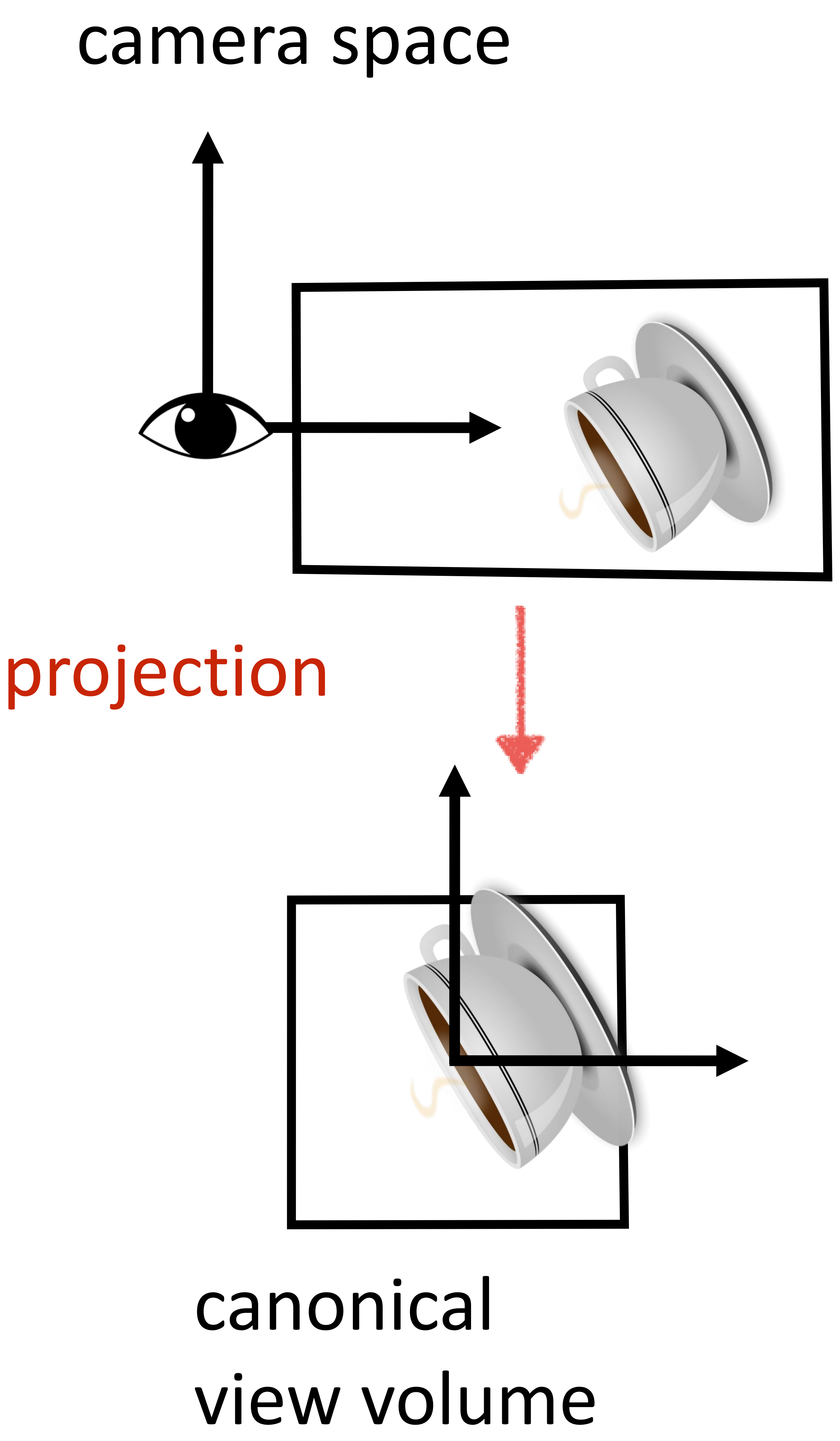
Effect on the points

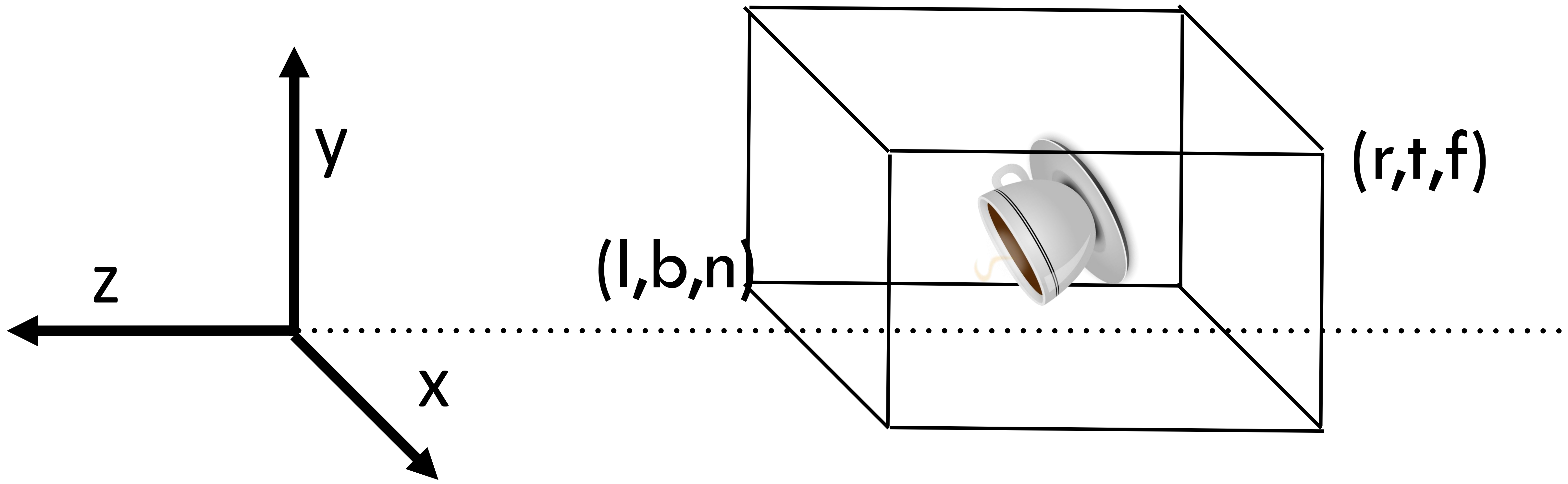


$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{P}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ (n+f)z - fn \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ n+f-\frac{fn}{z} \\ 1 \end{pmatrix}$$

Orthographic Projection



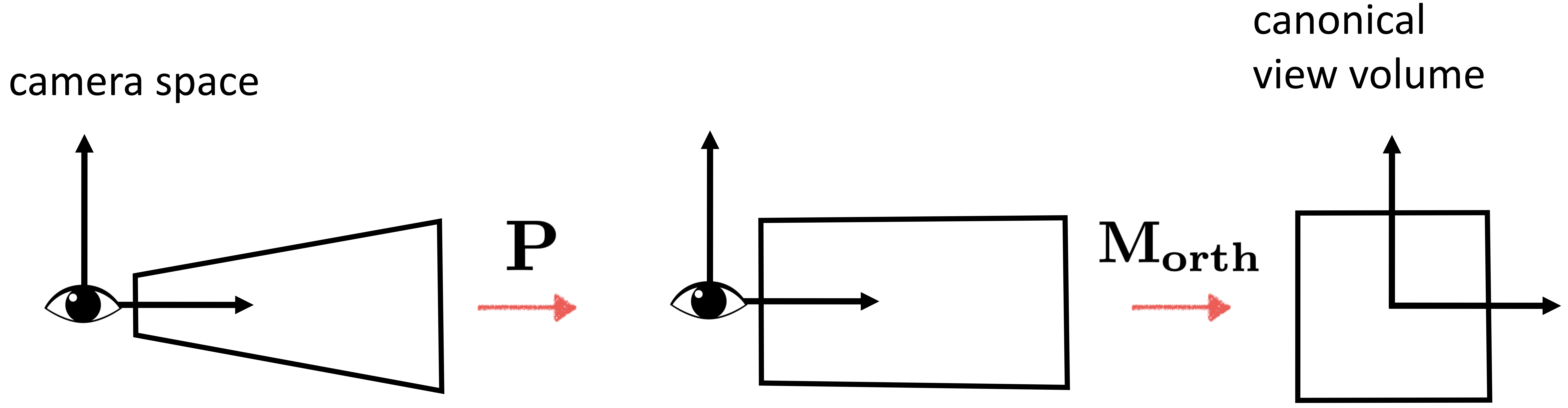


$$\mathbf{M}_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Complete Perspective Transformation

$$\mathbf{P} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{M}_{orth} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Thank you!

Questions?