中国科学技术大学
University of Science and Technology of China

GAMES 301: 第2讲

面向离散网格的参数化概述与传统方法介绍

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Introduction
Mesh-based mappings

Discrete meshes

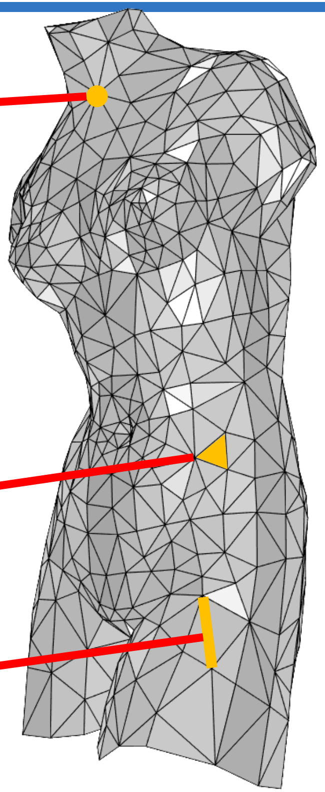
- Geometry
  - Vertex position
- Topology
  - Vertex
  - Edge
  - Facet
  - Element

$$\boldsymbol{v}_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} \in \mathcal{R}^3$$

$$V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{N_v}\}$$

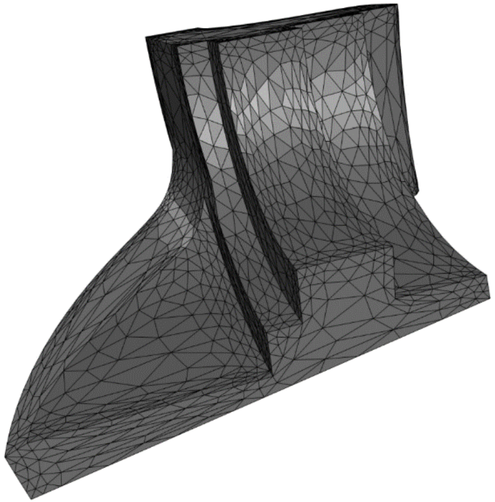
$$\boldsymbol{f}_i\!: \boldsymbol{v}_{i,1}, \boldsymbol{v}_{i,2}, \boldsymbol{v}_{i,3}$$

$$\boldsymbol{e}_j\!: \boldsymbol{v}_{j,1}, \boldsymbol{v}_{j,2}$$

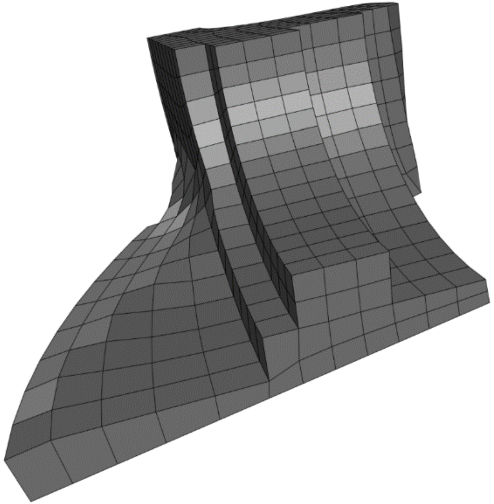


Triangle mesh

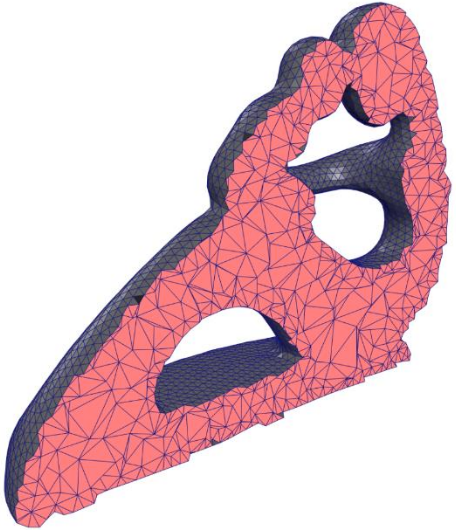
Discrete meshes



Triangle



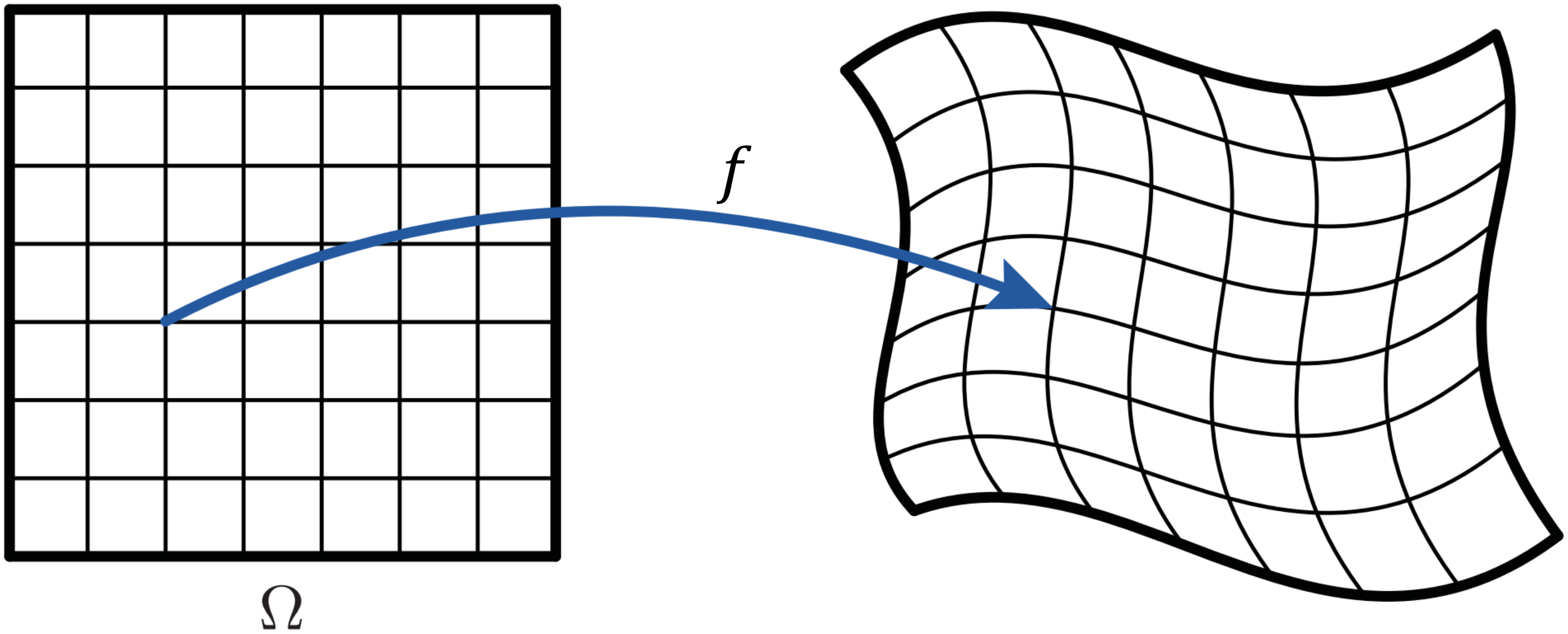
Quad



Tet

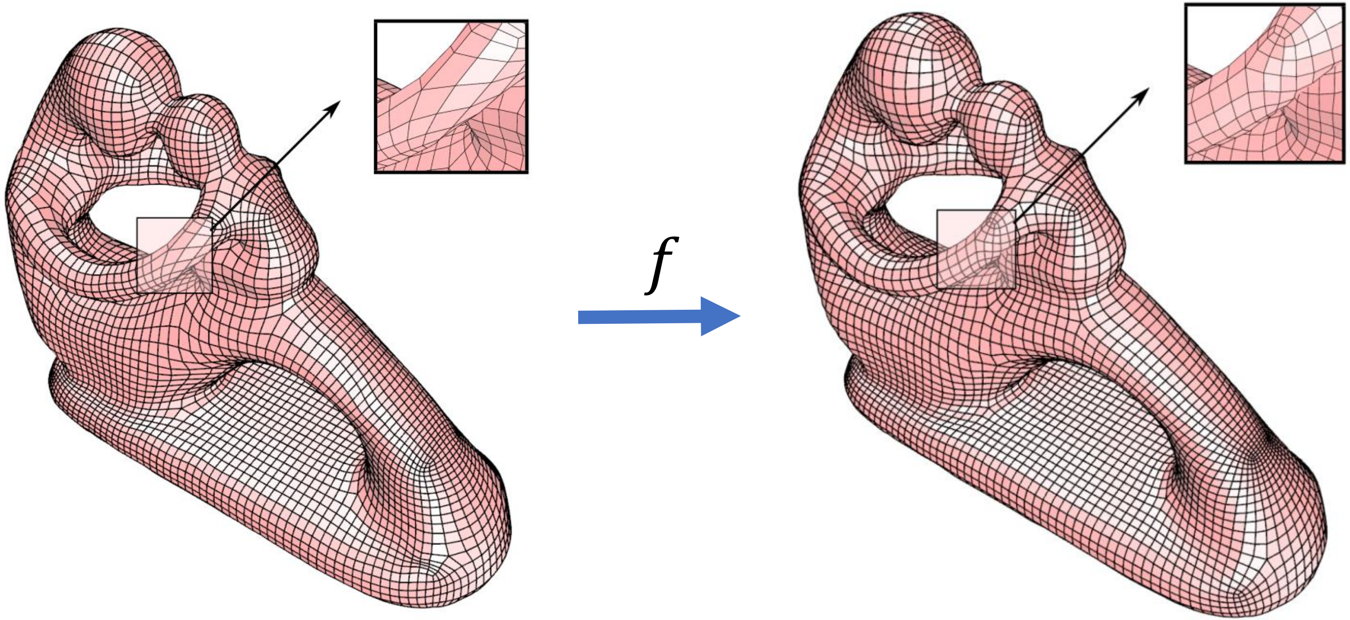
Hex

Mappings

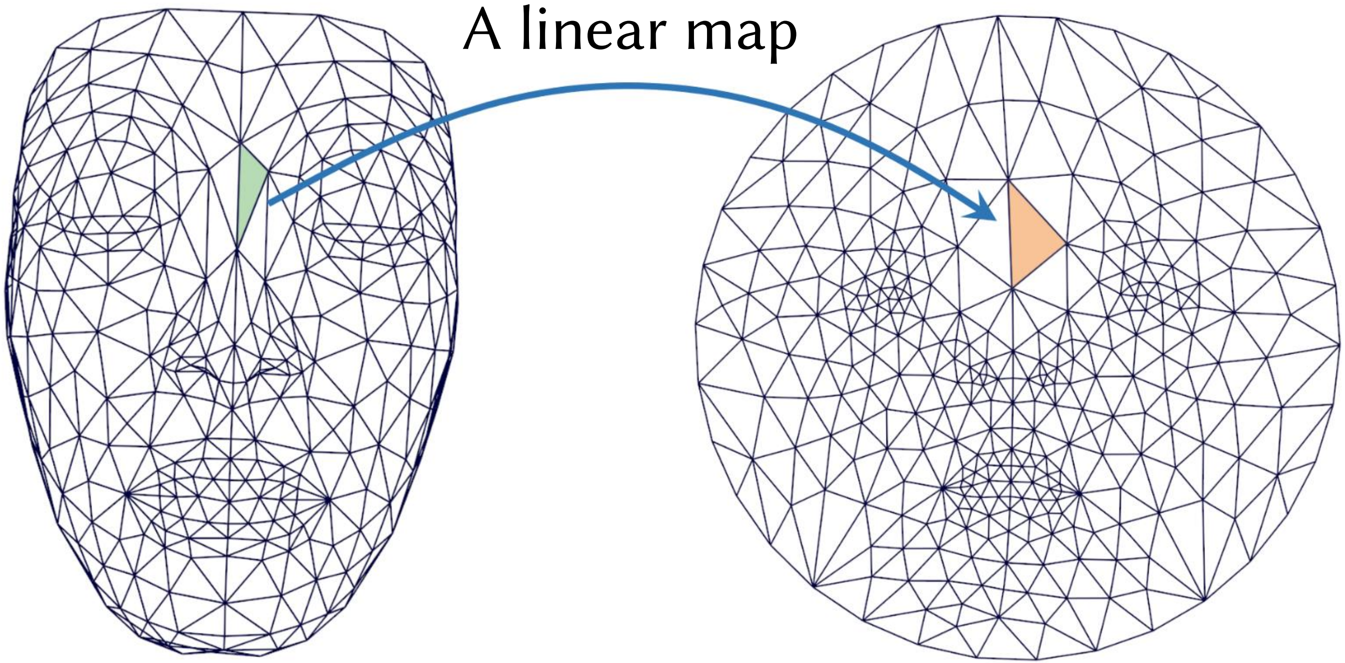


Variables

- Geometry
  - Vertex positions

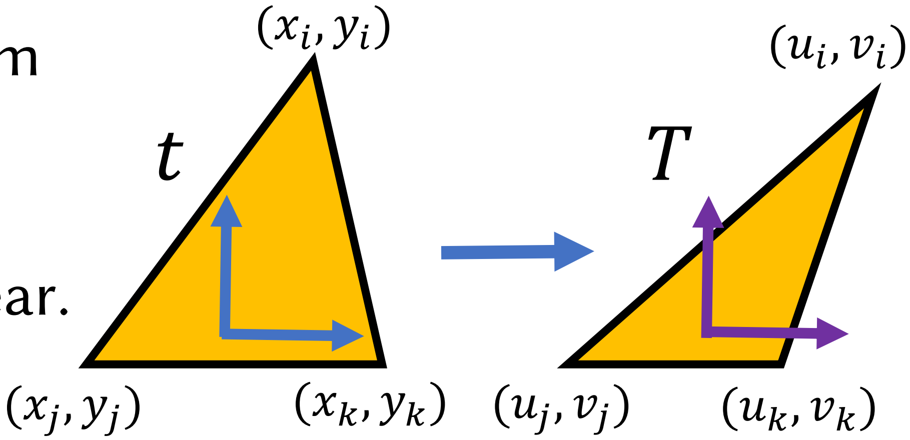


Piecewise mappings



Jacobian matrix

- Build a local coordinate system on input triangle t.
- The mapping is piecewise linear.
- J_t is 2×2.



$$f_t(\boldsymbol{x}) = J_t \boldsymbol{x} + \boldsymbol{b}_t$$

$$J_t = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} u_j - u_i & u_k - u_i \\ v_j - v_i & v_k - v_i \end{pmatrix} \begin{pmatrix} x_j - x_i & x_k - x_i \\ y_j - y_i & y_k - y_i \end{pmatrix}^{-1}$$

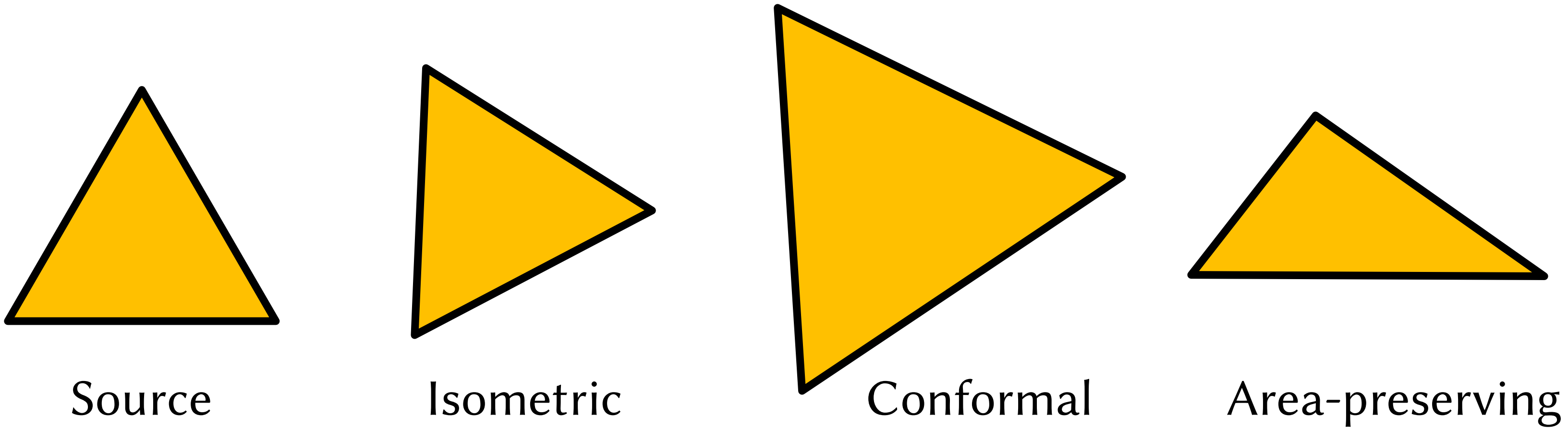
Jacobian matrix

- Input tet: $(x_i, y_i, z_i), (x_j, y_j, z_j), (x_k, y_k, z_k), (x_l, y_l, z_l)$
- Output tet: $(u_i, v_i, w_i), (u_j, v_j, w_j), (u_k, v_k, w_k), (u_l, v_l, w_l)$

$$\begin{pmatrix} u_j - u_i & u_k - u_i & u_l - u_i \\ v_j - v_i & v_k - v_i & v_l - v_i \\ w_j - w_i & w_k - w_i & w_l - w_i \end{pmatrix} \begin{pmatrix} x_j - x_i & x_k - x_i & x_l - x_i \\ y_j - y_i & y_k - y_i & y_l - y_i \\ z_j - z_i & z_k - z_i & z_l - z_i \end{pmatrix}^{-1}$$

Distortion types

- Isometric mapping: rotation + translation
- Conformal mapping: similarity + translation
- Area-preserving mapping: area-preserving + translation
- Conformal + Area-preserving $\Leftrightarrow$ Isometric



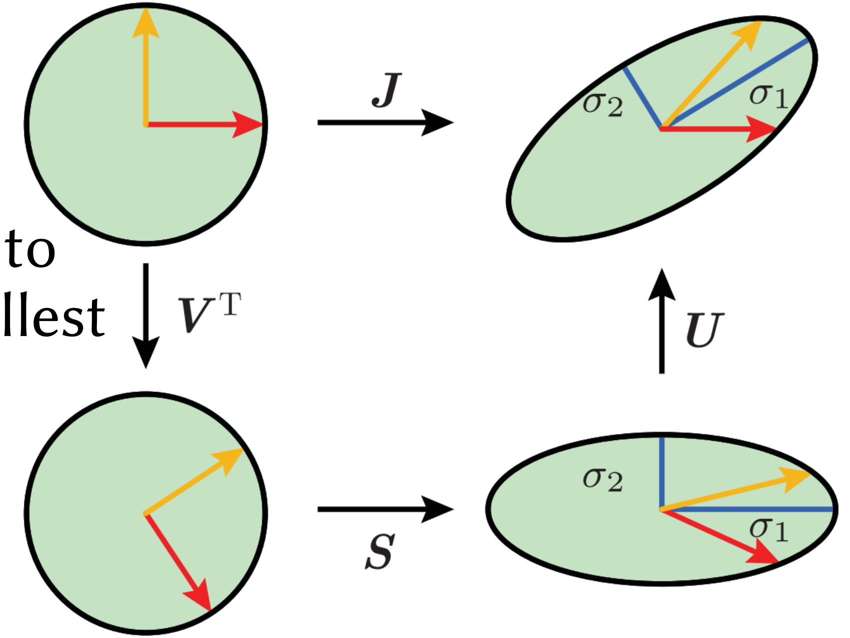
Signed singular value decomposition

- Signed singular value decomposition (SSVD):

$$J_t = U_t S_t V_t^T$$

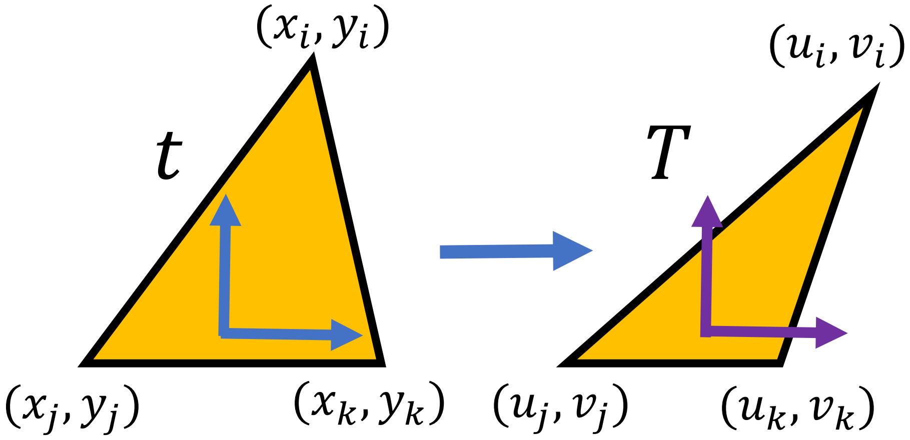
If $\det J_t > 0$, SSVD is SVD.

If $\det J_t \leq 0$, modifying U_t and V_t to be rotations matrices and the smallest singular value becomes negative.



Singular values

- Isometric mapping
  - $J_t \Longrightarrow$ rotation matrix
  - $\sigma_1 = \sigma_2 = 1$
- Conformal mapping
  - $J_t \Longrightarrow$ similar matrix
  - $\sigma_1 = \sigma_2$
- Area-preserving mapping
  - $\det J_t = 1$
  - $\sigma_1 \sigma_2 = 1$



$$f_t(\boldsymbol{x}) = J_t \boldsymbol{x} + \boldsymbol{b}_t$$

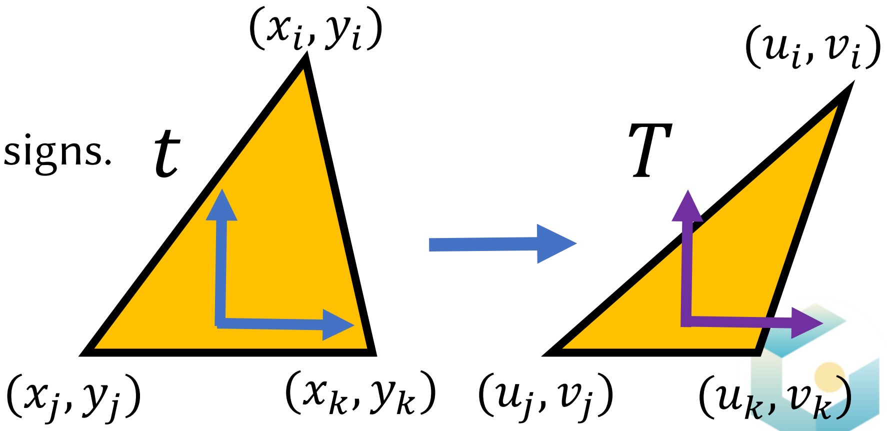
σ_1, σ_2 are the two singular values of J_t.

Common distortion metrics

- Conformal distortion
  - LSCM: $\sum_t \text{Area}(t)(\sigma_1 - \sigma_2)^2$
  - MIPS: $\sum_t \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$
- Isometric distortion
  - ARAP: $\sum_t \text{Area}(t)\left((\sigma_1 - 1)^2 + (\sigma_2 - 1)^2\right)$
  - AMIPS: $\sum_t \left(\left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right) + \left(\frac{1}{\sigma_2\sigma_1} + \sigma_2\sigma_1\right)\right)$
  - Symmetric Dirichlet: $\sum_t \text{Area}(t)(\sigma_1^2 + \sigma_1^{-2} + \sigma_2^2 + \sigma_2^{-2})$
- Area-preserving distortion:
  - $\sum_t \left(\frac{1}{\sigma_2\sigma_1} + \sigma_2\sigma_1\right)$

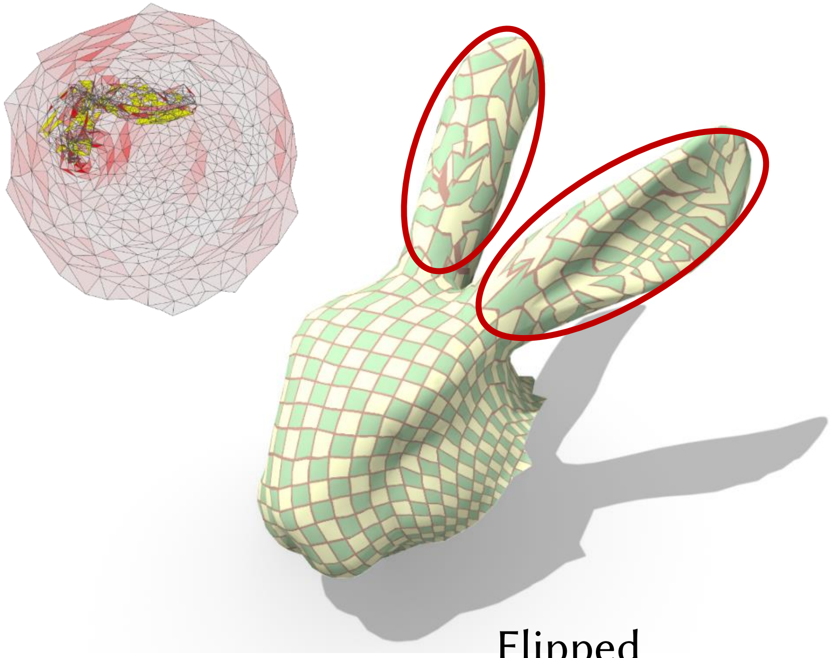
Constraints – Flip-free

- Motivations
  - No realistic material can be compressed to zero or even negative volume.
  - Flipped elements correspond to physically impossible deformation.
  - Inverted elements lead to invalidity for following applications, for example, remeshing.
- Formulation
  - Requirement:
    - Area(T) and Area(t) have the same signs.
  - $\det J_t > 0$
    - $\det J_t = \text{Area}(T)/\text{Area}(t)$

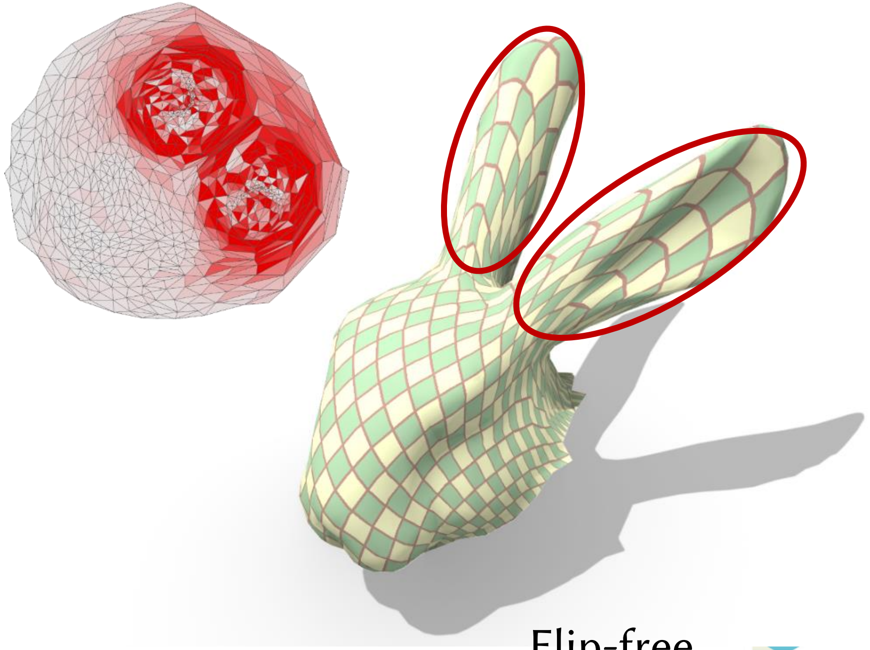


$$f_t(\boldsymbol{x}) = J_t\boldsymbol{x} + \boldsymbol{b}_t$$

Constraints – Flip-free



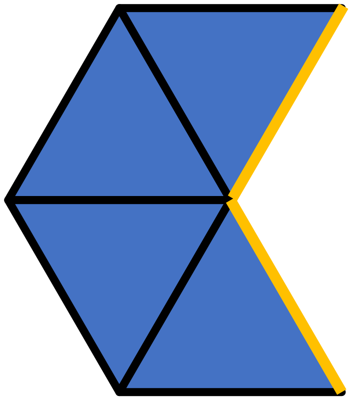
Flipped



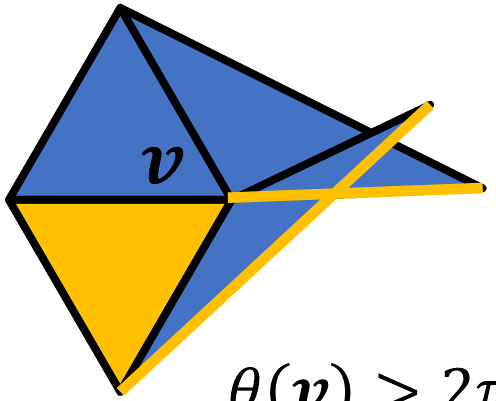
Flip-free

Constraints – Locally injective

- Flip-free condition.
- For boundary vertex, the mapping is locally bijective $\rightarrow$ $\theta(\boldsymbol{v}) < 2\pi$.



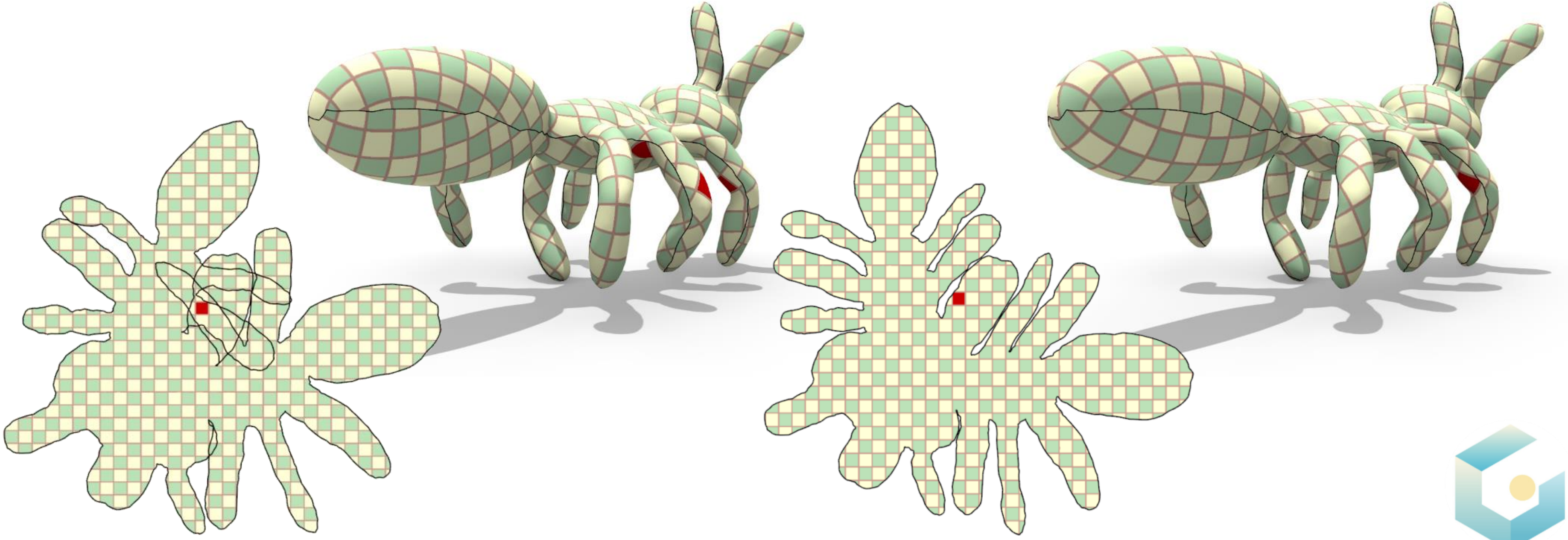
$\theta(\boldsymbol{v}) < 2\pi$



$\theta(\boldsymbol{v}) > 2\pi$

Constraints – globally injective

- The mapped mesh does not self-intersect.
- Flip-free condition.



Intersected

Intersection-free

Formulation

- Constrained optimization problem
- Objectives: distortion + specific metrics
  - Close to a reference mesh
  - Close to the ideal geometric measurements
  - ……
- Constraints: basic requirements + specific constraints
  - Positional constraints
  - Boundary-aligned constraints
  - Seamless conditions
  - …..

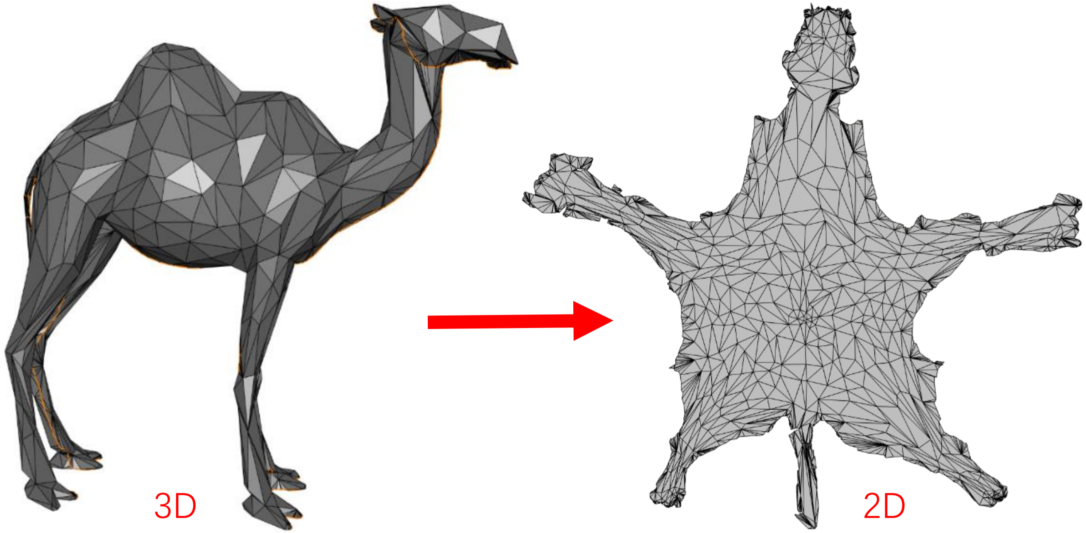
Challenges

- Non-convex and nonlinear
- Constraints
  - Flip-free, $\det J_t > 0$
  - Assume $J_t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c, d are linear functions of positions.
  - $\det J_t > 0 \ \rightarrow ad - bc > 0$, non-convex
- Objectives
  - MIPS: $\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1\sigma_2} = \frac{\|J_t\|_F^2}{\det J_t}$
  - $\|J_t\|_F^2$ and $\det J_t$: quadratic polynomials.
  - MIPS: rational polynomial.

Introduction
Mesh parameterizations

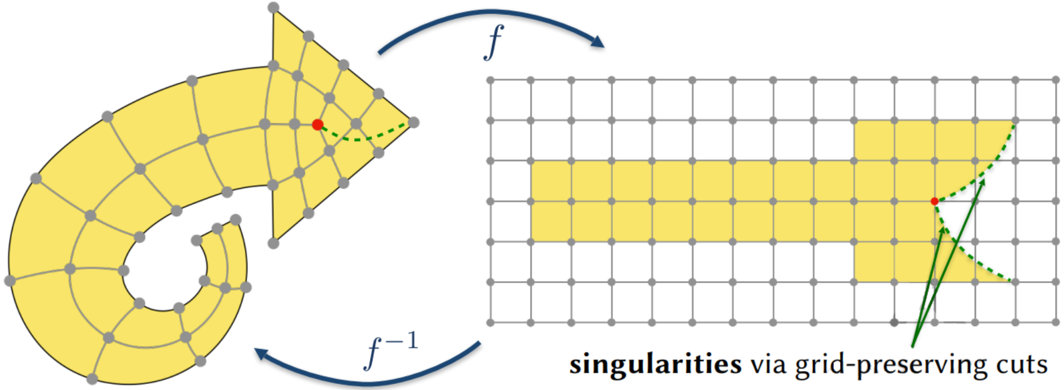
Definition

- A function that puts input surface in one-to-one correspondence with a 2D domain.
- Parameterization of a triangulated surface
  - all (u_i, v_i) coordinates associated with each vertex $\boldsymbol{v}_i = (x_i, y_i, z_i)^T$



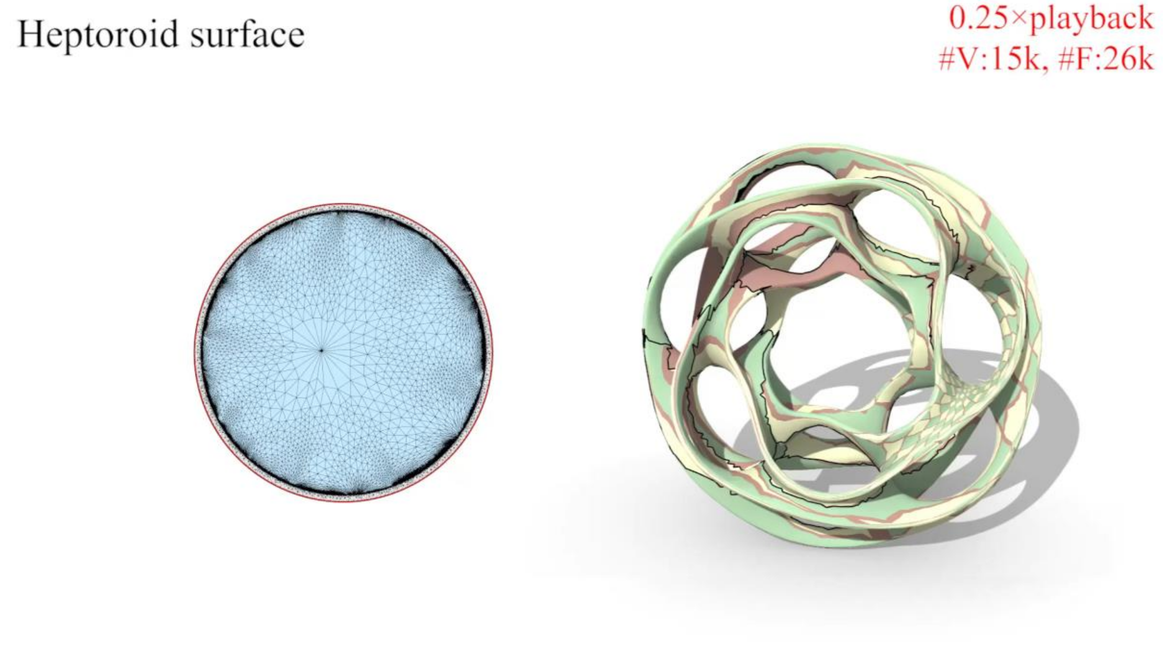
Applications

- Texture mapping
- Surface correspondence
- Remshing
- Attribute transfer
- Material design
- Computational art design
- ……



Formulation

- Objective: low distortion
- Constraints: globally injective



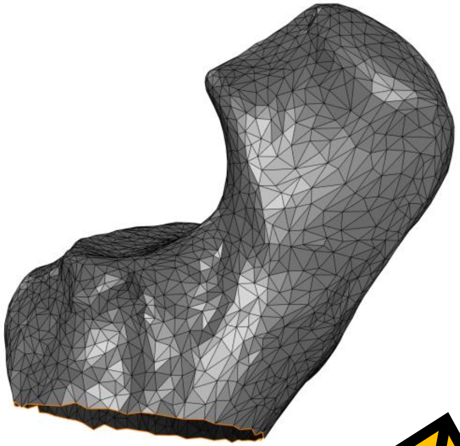
Tutte's embedding

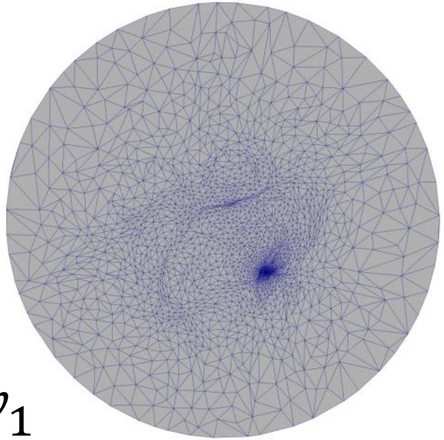
- 2D mappings on disk-topology meshes

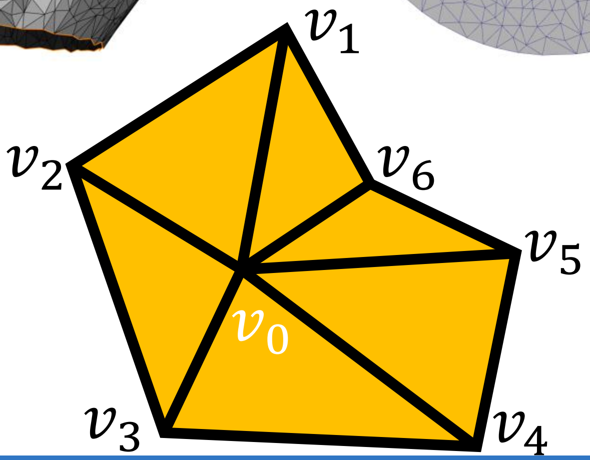
Given a triangulated surface homeomorphic to a disk, if the (u, v) coordinates at the boundary vertices lie on a convex polygon in order, and if the coordinates of the internal vertices are a convex combination of their neighbors, then the (u, v) coordinates form a valid parameterization (without self-intersections, globally injective).

Tutte's embedding

- Homeomorphic to a disk.
- A convex polygon
  - circle, square,……
- A convex combination
  - $\sum_{i=1}^{k} \lambda_i v_i = v_0, \sum_{i=1}^{k} \lambda_i = 1$
  - Uniform Laplacian, mean value coordinate
- Solver: linear equation.







Representative methods

Only considering low distortion

Angle-based flattening (ABF)

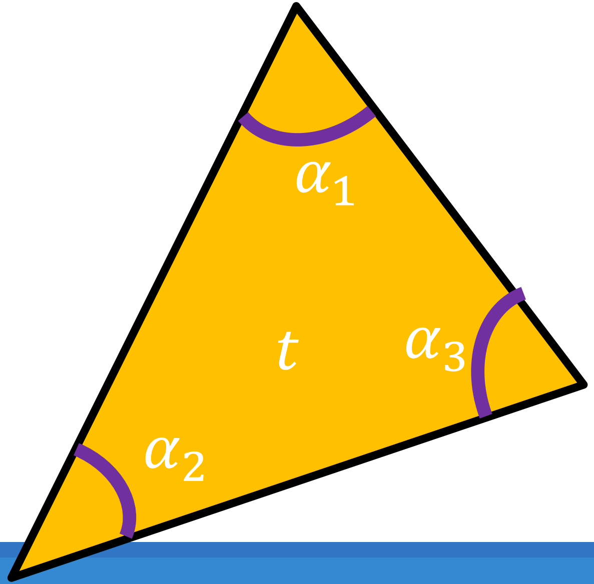
Sheffer A, de Sturler E. Parameterization of faceted surfaces for meshing using angle-based flattening[J]. Engineering with computers, 2001, 17(3): 326-337.

Sheffer A, Lévy B, Mogilnitsky M, et al. ABF++: fast and robust angle based flattening[J]. ACM Transactions on Graphics (TOG), 2005, 24(2): 311-330.

Zayer R, Lévy B, Seidel H P. Linear angle based parameterization[C]//Fifth Eurographics Symposium on Geometry Processing-SGP 2007. Eurographics Association, 2007: 135-141.

Angle-Based Flattening (ABF)

- Key observation: the parameterized triangles are uniquely defined by all the angles at the corners of the triangles.
  - Find angles instead of (u_i, v_i) coordinates.
  - Use angles to reconstruct (u_i, v_i) coordinates.



Objective

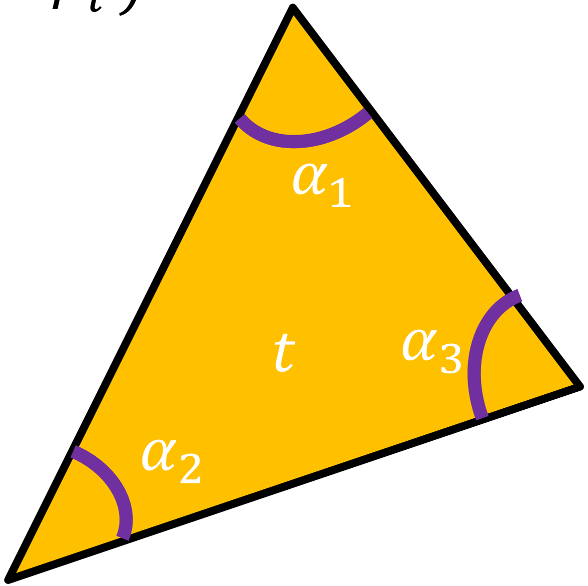
- Optimization goal:

$$E_{ABF} = \sum_{t} \sum_{i=1}^{3} \omega_i^t \left(\alpha_i^t - \beta_i^t\right)^2$$

β_i^t: Optimal angles for α_i^t.

$$\beta_i^t = \begin{cases} \dfrac{\tilde{\beta}_i^t \cdot 2\pi}{\sum_i \tilde{\beta}_i^t}, \text{Interior vertex} \\ \tilde{\beta}_i^t, \text{Boundary verterx} \end{cases}$$

$\omega_i^t = \left(\beta_i^t\right)^{-2}$.



Constraints

- Positive resulting angles:

$$\alpha_i^t > 0$$

- The three triangle angles have to sum to π:

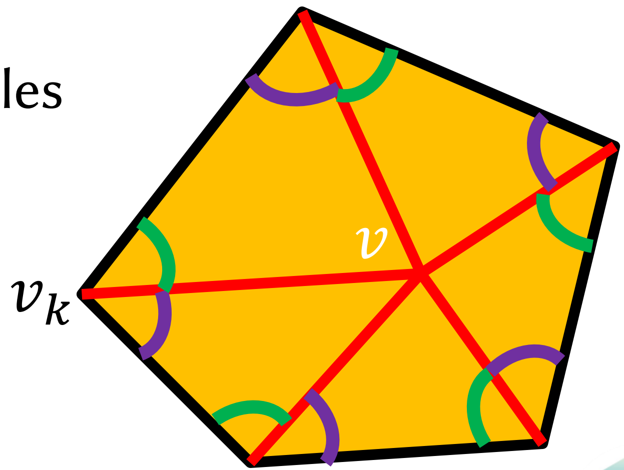
$$\alpha_i^t + \alpha_2^t + \alpha_3^t = \pi$$

- For each internal vertex, the incident angles have to sum to 2π:

$$\sum_{t \in \Omega(v)} \alpha_k^t = 2\pi$$

- Reconstruction constraints:

$$\prod_{t \in \Omega(v)} \sin \alpha_{k \oplus 1}^t = \prod_{t \in \Omega(v)} \sin \alpha_{k \ominus 1}^t$$



Linear ABF

- Reconstruction constraints are nonlinear and hard to solve.
- Initial estimation + estimation error
  - $\alpha_i^t = \gamma_i^t + e_i^t$

$$\log\left(\prod_{t\in\Omega(v)} \sin \alpha_{k\oplus 1}^t\right) = \log\left(\prod_{t\in\Omega(v)} \sin \alpha_{k\ominus 1}^t\right)$$

$$\sum_{t\in\Omega(v)} \log(\sin \alpha_{k\oplus 1}^t) = \sum_{t\in\Omega(v)} \log(\sin \alpha_{k\ominus 1}^t)$$

- Taylor expansion:

$$\log(\sin \alpha_{k\oplus 1}^t) = \log(\sin \gamma_{k\oplus 1}^t + e_{k\oplus 1}^t)$$
$$= \log(\sin \gamma_{k\oplus 1}^t) + e_{k\oplus 1}^t \cot \gamma_{k\oplus 1}^t + \cdots$$

It is linear with estimation error.

Solver

- Set $\gamma_i^t = \beta_i^t$
- Problem:

$$\min_e E_{ABF} = \sum_t \sum_{i=1}^{3} \omega_i^t \left(e_i^t\right)^2$$

$$\text{subject to} \quad Ae = b$$

$$\Rightarrow \begin{pmatrix} D & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\Rightarrow e = D^{-1} A^T \left(A D^{-1} A^T\right)^{-1} b$$

Reconstruct parameterization

- Greedy method.
  - Construct the triangles one by one using a depth-first traversal.
  - Key: for each triangle, given the position of two vertices and the angles, the position of the third vertex can be uniquely derived.
- Least squares method.
  - An angle-based least squares formulation
    - Solving a set of linear equations relating angles to coordinates.

Greedy method

- Choose a mesh edge $e^1 = (v_a^1, v_b^1)$.
- Project v_a^1 to (0,0,0) and v_b^1 to $(\|e^1\|, 0,0)$.
- Push e^1 on the stack S.
- While S not empty, pop an edge $e = (v_a, v_b)$. For each face $f_i = (v_a, v_b, v_c)$ containing e:
  - If f_i is marked as set, continue.
  - If v_c is not projected, compute its position based on v_a, v_b and the face angles of f_i.
  - Mark f_i as set, push edge (v_b, v_c) and (v_a, v_c) on the stack.

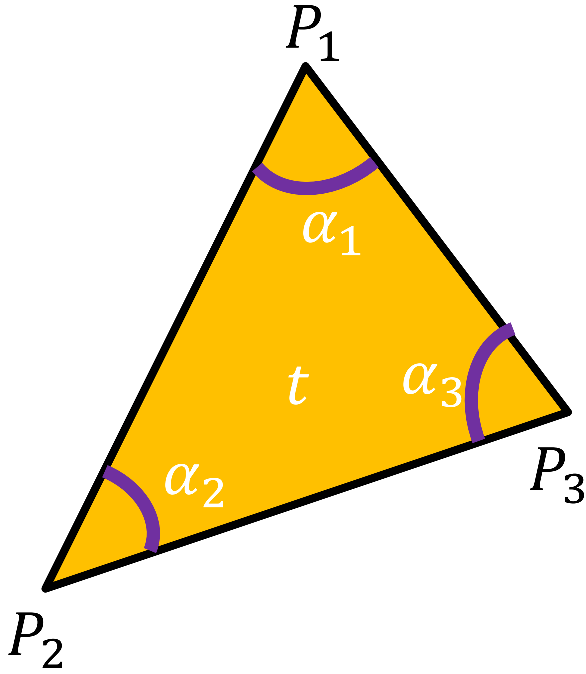
- Accumulate numerical error.

Least squares method

- The ratio of triangle edge lengths $\left\|\overrightarrow{P_1P_3}\right\|$ and $\left\|\overrightarrow{P_1P_2}\right\|$ is

$$\frac{\left\|\overrightarrow{P_1P_3}\right\|}{\left\|\overrightarrow{P_1P_2}\right\|} = \frac{\sin\alpha_2}{\sin\alpha_3}$$

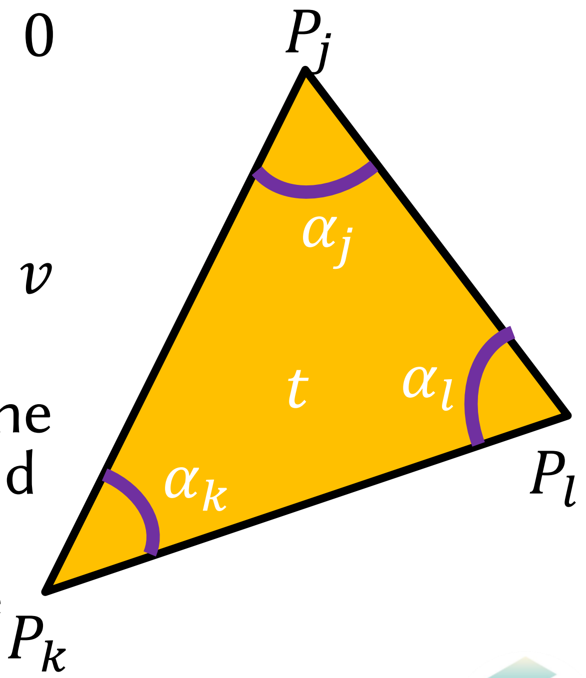
$$\Rightarrow \overrightarrow{P_1P_3} = \frac{\sin\alpha_2}{\sin\alpha_3}\begin{pmatrix}\cos\alpha_1 & -\sin\alpha_1 \\ \sin\alpha_1 & \cos\alpha_1\end{pmatrix}\overrightarrow{P_1P_2}$$



Least squares method

$$\forall t = (j,k,j), \qquad M^t\left(P_k - P_j\right) + P_j - P_l = 0$$

$$M^t = \frac{\sin\alpha_k}{\sin\alpha_l}\begin{pmatrix}\cos\alpha_j & -\sin\alpha_j \\ \sin\alpha_j & \cos\alpha_j\end{pmatrix}$$



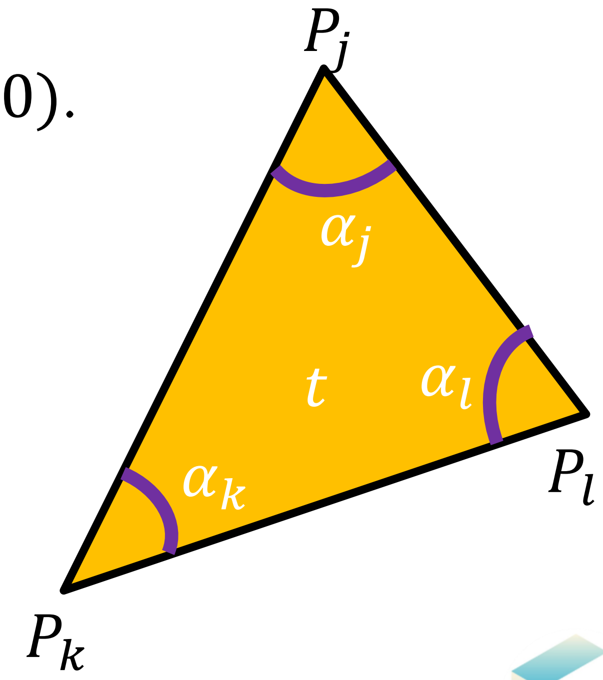
1. Two equations per triangle for the u and v coordinates of the vertices.
2. The angles of a planar triangulation define it uniquely up to to rigid transformation and global scaling.
   - Introduce four constraints which eliminate these degrees of freedom.
   - Fix two vertices sharing a common edge.

Least squares method

- Choose one edge $e^1 = (v_a^1, v_b^1)$.
- Project v_a^1 to $(0,0,0)$ and v_b^1 to $(\|e^1\|, 0,0)$.

- Solve following energy to compute positions of other vertices:

$$E = \sum_t \left\| M^t (P_k - P_j) + P_j - P_l \right\|^2$$



Least-Square conformal mapping (LSCM)

Lévy B, Petitjean S, Ray N, et al. Least squares conformal maps for automatic texture atlas generation[J]. ACM transactions on graphics (TOG), 2002, 21(3): 362-371.

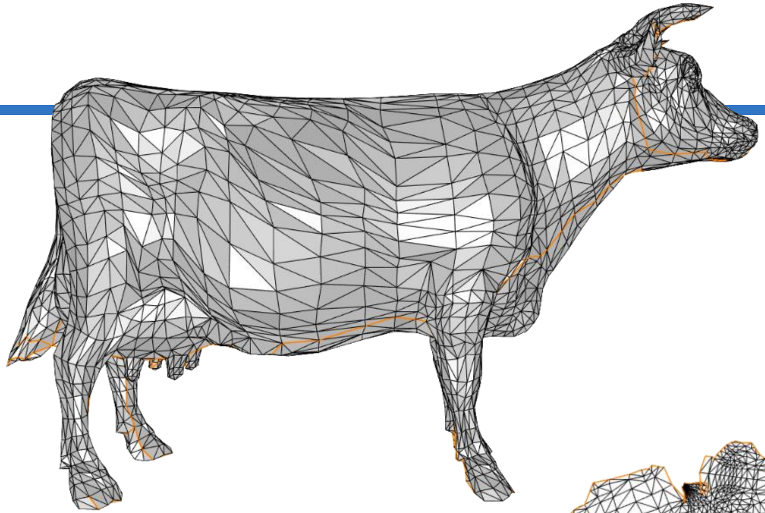
Similar transforms

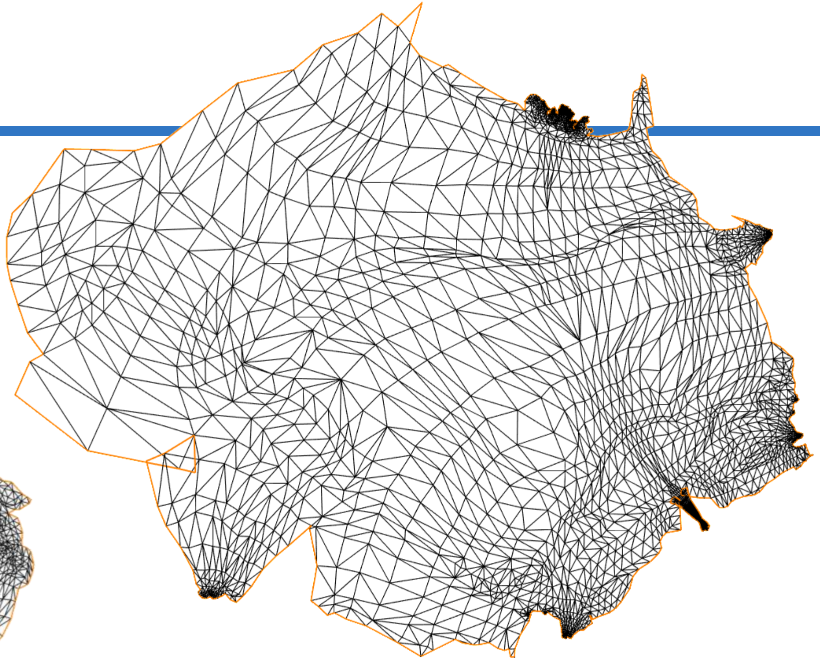
- 2D case: for one triangle t
  - $J_t = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = s \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
  - $\Longrightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$
  - Cauchy-Riemann Equations.

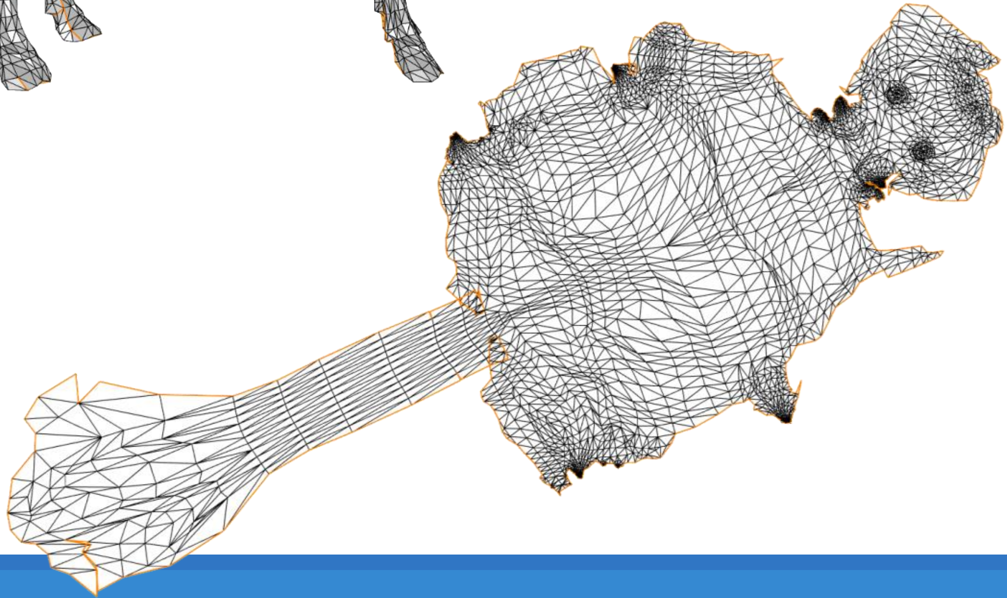
$$J_t = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

LSCM (As-similar-as-possible)

- Energy
  - $E_{LSCM} = \sum_t A_t \left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right)$
  - Measuring non-conformality
  - It is invariant with respect to arbitrary translations and rotations.
  - E_{LSCM} does not have a unique minimizer.
  - Fixing at least two vertices. Significantly affect the results.







As-rigid-as-possible (ARAP)

Liu L, Zhang L, Xu Y, et al. A local/global approach to mesh parameterization[C]//Computer Graphics Forum. Oxford, UK: Blackwell Publishing Ltd, 2008, 27(5): 1495-1504.

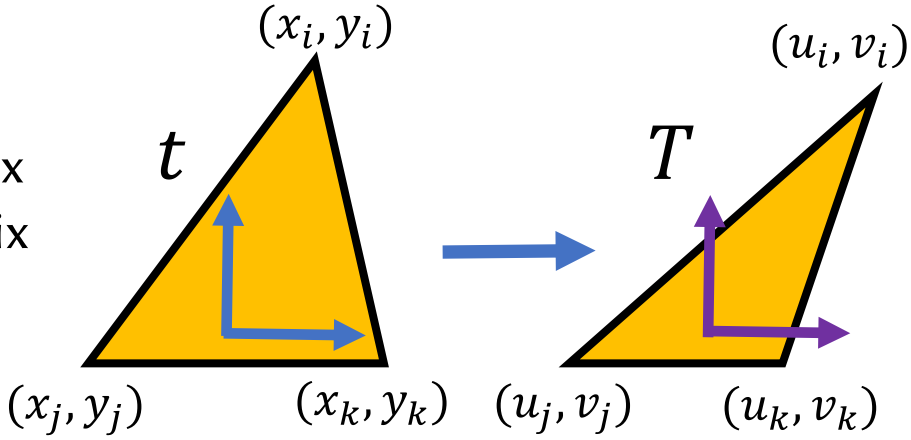
Formulation

$$E(u,L)=\sum_t A_t\|J_t-L_t\|_F^2$$

L_t: target transformation

- Isometric mapping: rotation matrix
- Conformal mapping: similar matrix

- Variables:
  - 2D parameterization coordinate
  - Target transformations



$$f_t(\boldsymbol{x})=J_t\boldsymbol{x}+\boldsymbol{b}_t$$

Local-global solver

- Alternatively optimization
  - Local step:
    - Fix 2D parameterization coordinates, optimize target transformations.
  - Global step:
    - Fix target transformations, optimize 2D parameterization coordinates.
- **Global step**:
  - $E(u, L) = \sum_t A_t \|J_t - L_t\|_F^2$, quadratic energy
  - Linear system

Local step: Procrustes analysis

- Approximating one 2×2 matrix J_t as best we can by another 2×2 matrix L_t.
- $d(J_t, L_t) = \|J_t - L_t\|_F^2 = \mathrm{trace}\left((J_t - L_t)^T (J_t - L_t)\right)$
- Minimizing $d(J_t, L_t)$ through Singular Value Decomposition (SVD)
  - $J_t = U \Sigma V^T, \quad \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$
  - Signed SVD: U and V are rotation matrices, σ_2 may be negative
  - Best rotation: UV^T
  - Best similar matrix: $U \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} V^T$, $s = \frac{\sigma_1 + \sigma_2}{2}$

Local/Global Approach summary

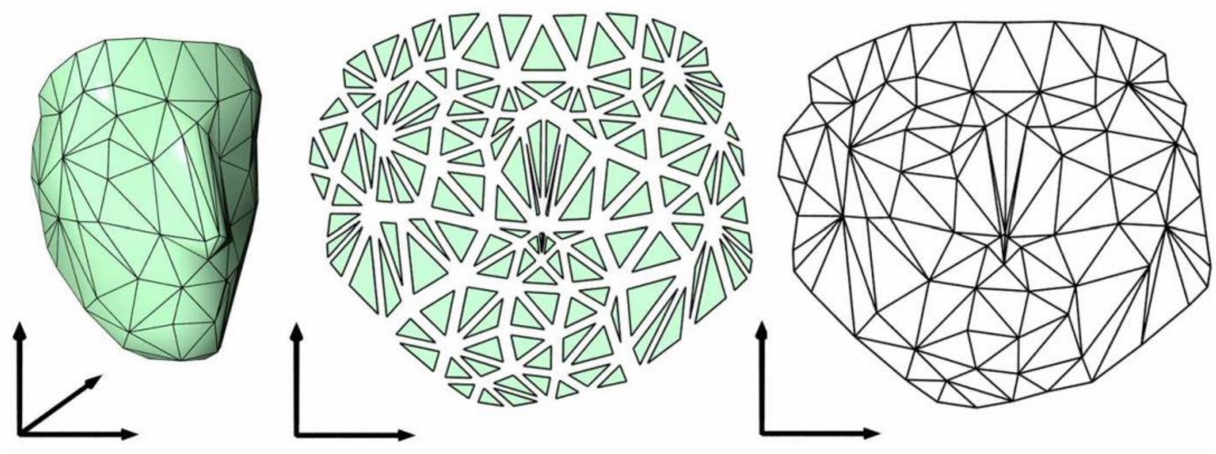


Figure 2: *Parameterizing a mesh by aligning locally flattened triangles. (Left) Original 3D mesh; (middle) flattened triangles; (right) 2D parameterization.*

Connection to singular values

- $E(u,L) = \sum_t A_t \|J_t - L_t\|_F^2$ and σ_t^1, σ_t^2 are the two singular values of J_t.

- Conformal

$$E(u) = \sum_t A_t (\sigma_t^1 - \sigma_t^2)^2$$

- Isometric

$$E(u) = \sum_t A_t \left((\sigma_t^1 - 1)^2 + (\sigma_t^2 - 1)^2 \right)$$

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谢 谢 ！

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