中国科学技术大学
University of Science and Technology of China

GAMES 301: 第3讲

无翻转参数化方法
初始存在翻转

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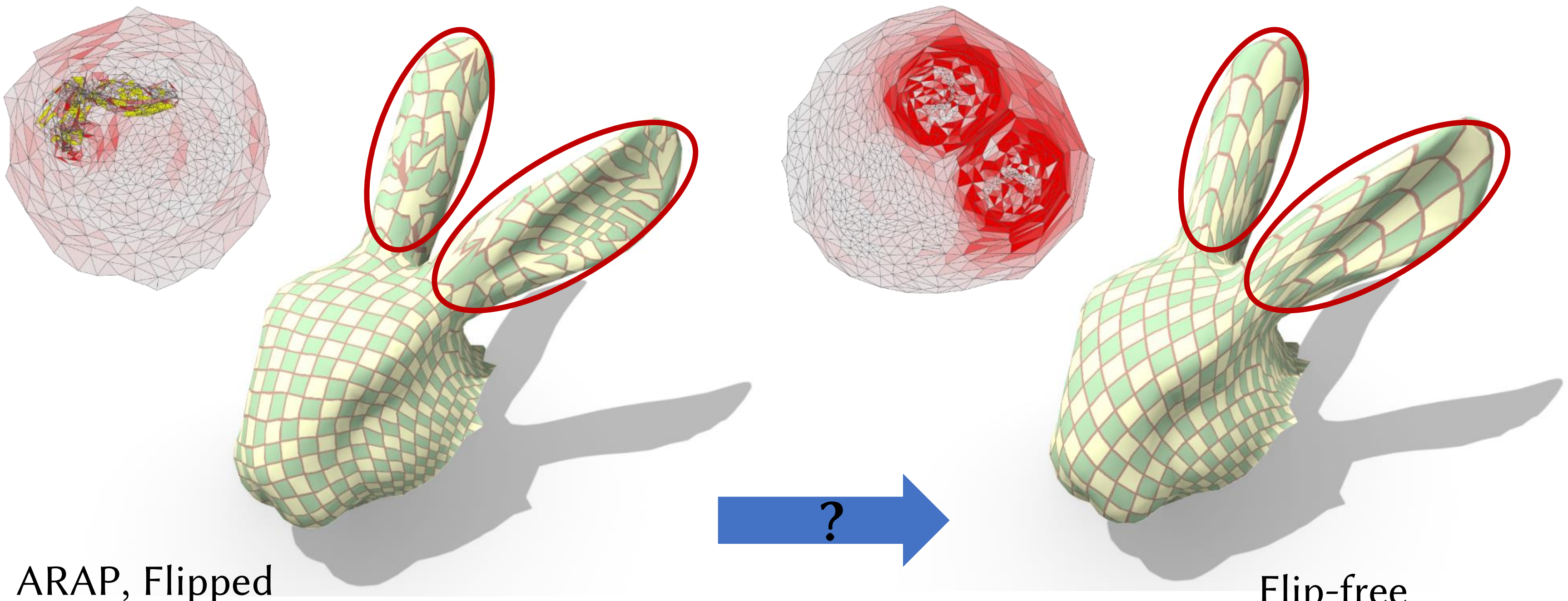
Computing flip-free mappings with flipped initializations

Problem definition



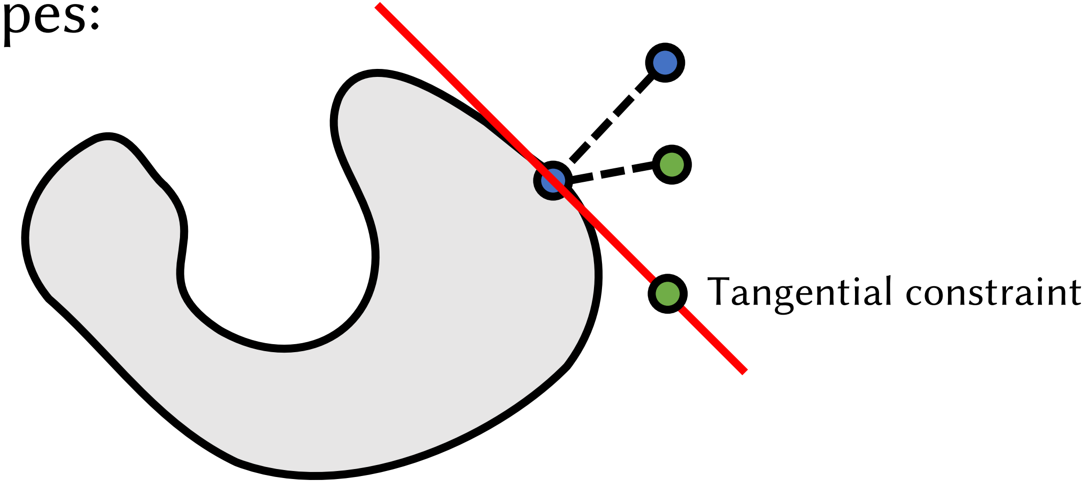
ARAP, Flipped

Flip-free

Projection

Key idea

- Project the flipped elements into the flip-free mapping space.
- An alternating algorithm
  - Local step: project each element onto the constraint set (flip-free space)
  - Global step: minimize the distance to the projected elements
- Common projection types:
  - Closest point projection
  - Tangential projection



Preliminaries

- Signed singular value decomposition: $J_i(\mathbf{u}) = U_i S_i V_i^T$
  - U_i and V_i are two rotation matrices
  - $S_i = \text{diag}(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i,3}), \sigma_{i,1} \geq \sigma_{i,2} \geq |\sigma_{i,3}|$
- Foldover-free constraints
  - $\det J_i(\mathbf{u}) > 0, i = 1, \cdots, N \Leftrightarrow \sigma_{i,3} > 0$
- Conformal distortion
  - $\tau(J_i(\mathbf{u})) = \sigma_{i,1}/\sigma_{i,3}$
- Bounded conformal distortion constraints
  - $1 \leq \tau(J_i(\mathbf{u})) \leq K$

Constraints

Flip-free constraints

$\det J_i(\mathbf{u}) > 0$

$\sigma_{i,3} > 0, \tau(J_i) = \sigma_{i,1}/\sigma_{i,3}$

$\sigma_{i,1} \geq \sigma_{i,2} \geq |\sigma_{i,3}|$

$K = \max_{i=1,\cdots,N} \tau(J_i)$

Bounded conformal distortion constraints

$1 \leq \tau(J_i(\mathbf{u}))$

$\tau(J_i(\mathbf{u})) \leq K$

$\tau(J_i) \geq 1, \sigma_{i,3} > 0, \sigma_{i,3} > 0$

Formulation

$$\min_{\mathbf{u}} \quad E_d = \sum_{i=1,\cdots,N} \|J_i(\mathbf{u}) - H_i\|_F^2,$$

$$s.t. \quad H_i \in \mathcal{H}_i, i = 1, \cdots, N,$$

$$A\mathbf{u} = b.$$

$\mathcal{H}_i = \{H_i | 1 \leq \tau(H_i) \leq K\}$: bounded conformal distortion space.

Su, Jian-Ping, Xiao-Ming Fu, and Ligang Liu. "Practical foldover-free volumetric mapping construction." *Computer Graphics Forum*. Vol. 38. No. 7. 2019.

Solver

- Local-global solver

Local step

Fix $\mathbf{u}$ and J_i, solve H_i

$$\min_{\mathbf{u}} \quad E_d = \sum_{i=1,\cdots,N} \|J_i(\mathbf{u}) - H_i\|_F^2 ,$$

$$s.t. \quad H_i \in \mathcal{H}_i, i = 1, \cdots, N,$$

Global step

Fix H_i, solve $\mathbf{u}$

$$\min_{\mathbf{u}} \quad E_d = \sum_{i=1,\cdots,N} \|J_i(\mathbf{u}) - H_i\|_F^2 ,$$

$$s.t. \quad A\mathbf{u} = b$$

Very slow convergence...

Anderson acceleration

- Fixed-point iteration

$$G = \phi \circ P$$

- Accelerated iteration $\mathbf{x}^{k+1} = G(\mathbf{x}^k)$

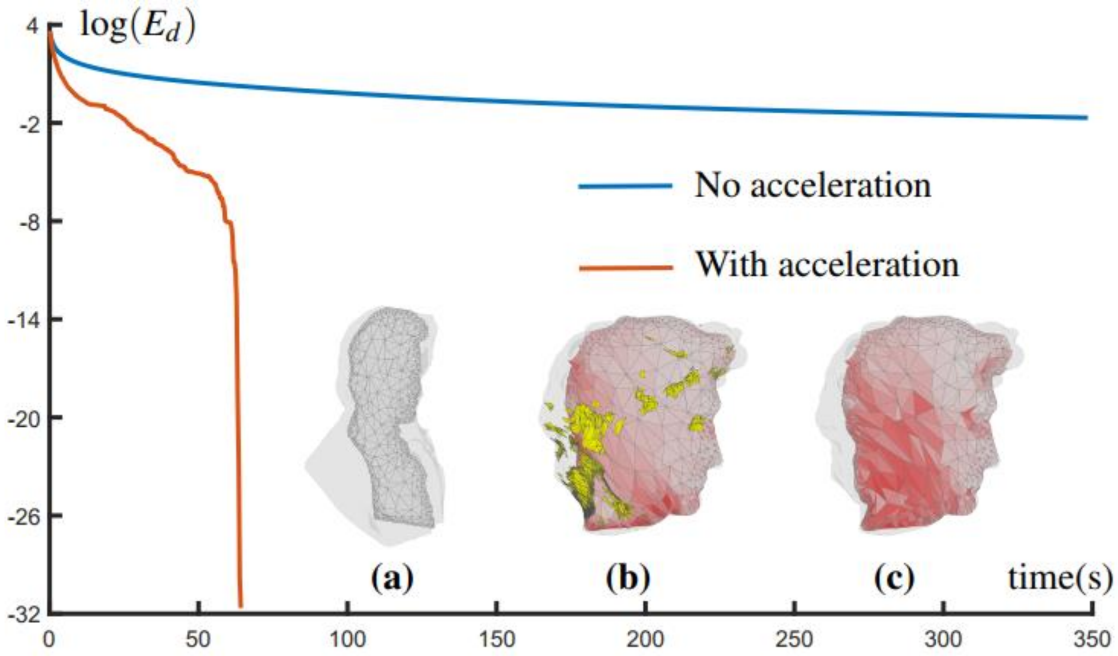
$$\mathbf{x}_{AA}^{k+1} = G(\mathbf{x}^k) - \sum_{j=1}^{m} \theta_j^* \left(G(\mathbf{x}^{k-j+1}) - G(\mathbf{x}^{k-j})\right)$$

where $(\theta_1^*, \cdots, \theta_m^*)$ is the solution to a linear least-squares problem:

$$\min_{(\theta_1, \cdots, \theta_m)} \left\| F^k - \sum_{j=1}^{m} \theta_j \left(F^{k-j+1} - F^{k-j}\right) \right\|^2, F^k = G(\mathbf{x}^k) - \mathbf{x}^k$$

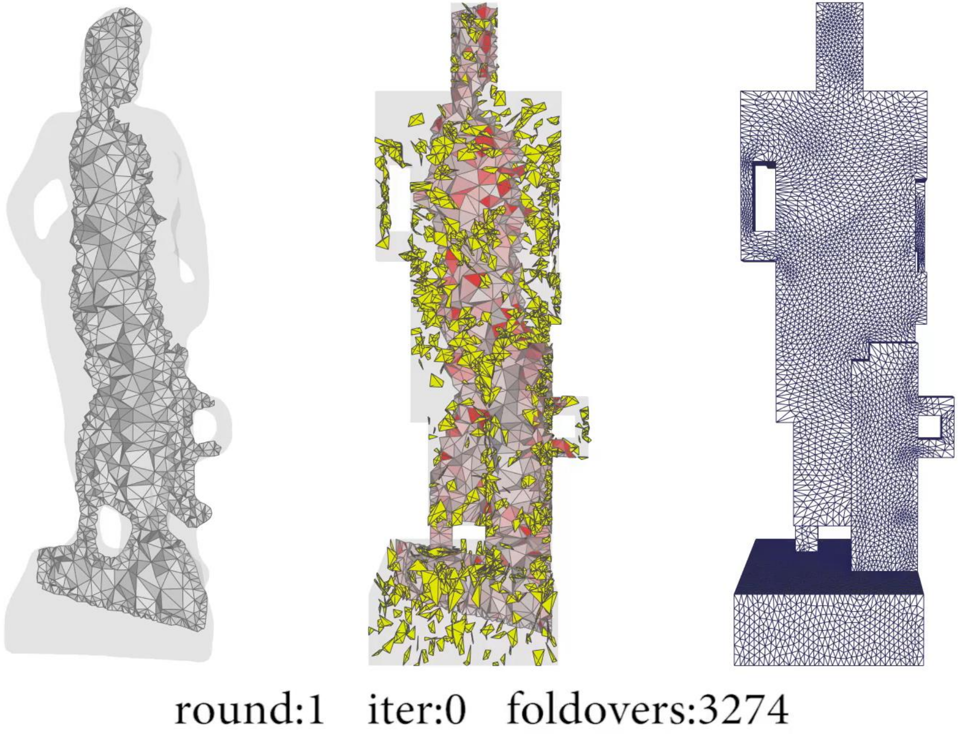
Peng, Yue, et al. "Anderson acceleration for geometry optimization and physics simulation." *ACM Transactions on Graphics (TOG)* 37.4 (2018): 1-14.

Anderson acceleration



Update bound K

- Projection cannot eliminate all foldovers



- $K^{new} = \beta K$
  - $\beta = 2$
  - initialize $K = 4$

Recap of the algorithm

Elimination Process :

alternates in each round:

- generate conformal distortion bound

- try to project the mapping into the bounded distortion space

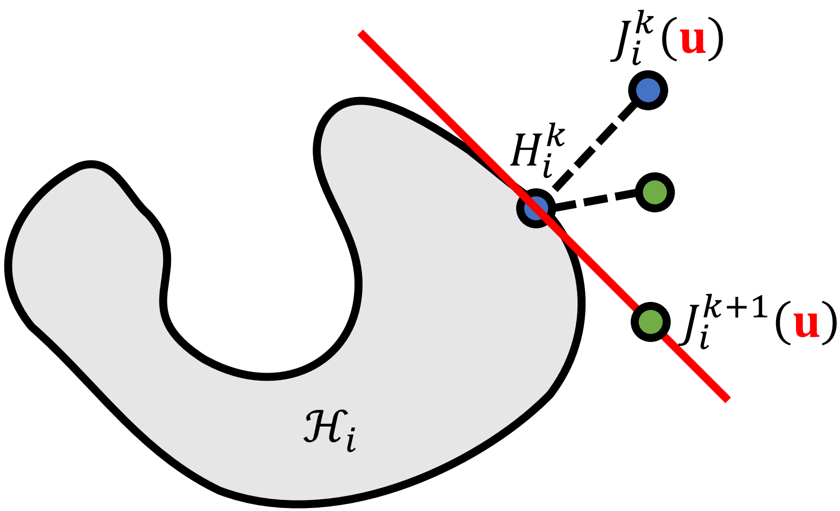
Tangential projection

- Global step: fix H_i, solve $\mathbf{u}$

$$\min_{\mathbf{u}} \quad E_d = \sum_{i=1,\cdots,N} \|J_i(\mathbf{u}) - H_i\|_F^2 \,,$$

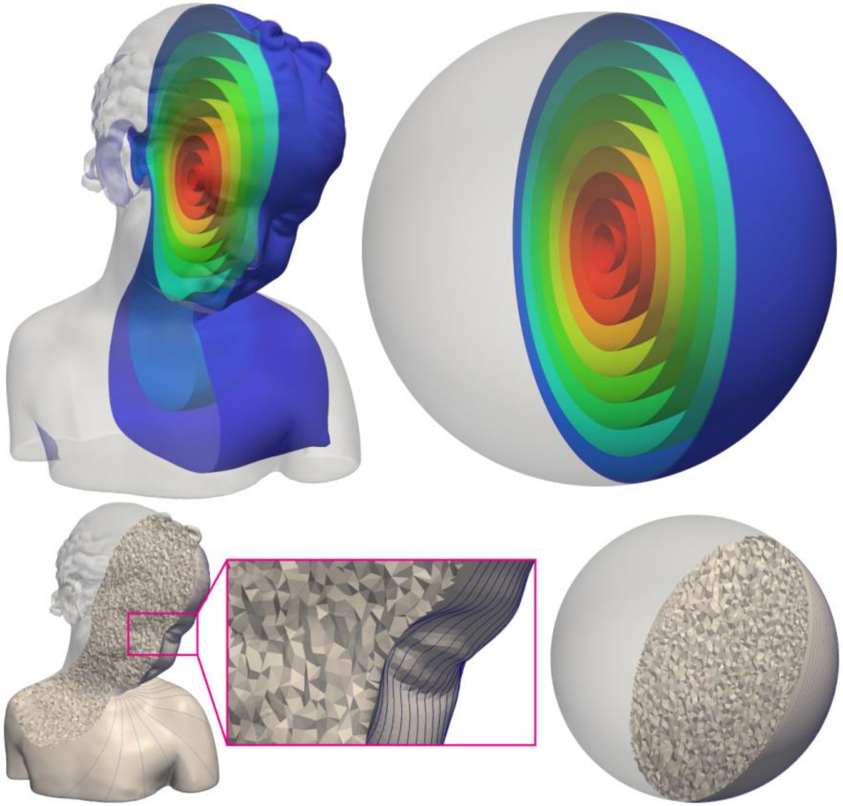
$$s.t. \quad A\mathbf{u} = b$$

$$(J_i^k(\mathbf{u}) - H_i^k) \perp (J_i^{k+1}(\mathbf{u}) - H_i^k)$$

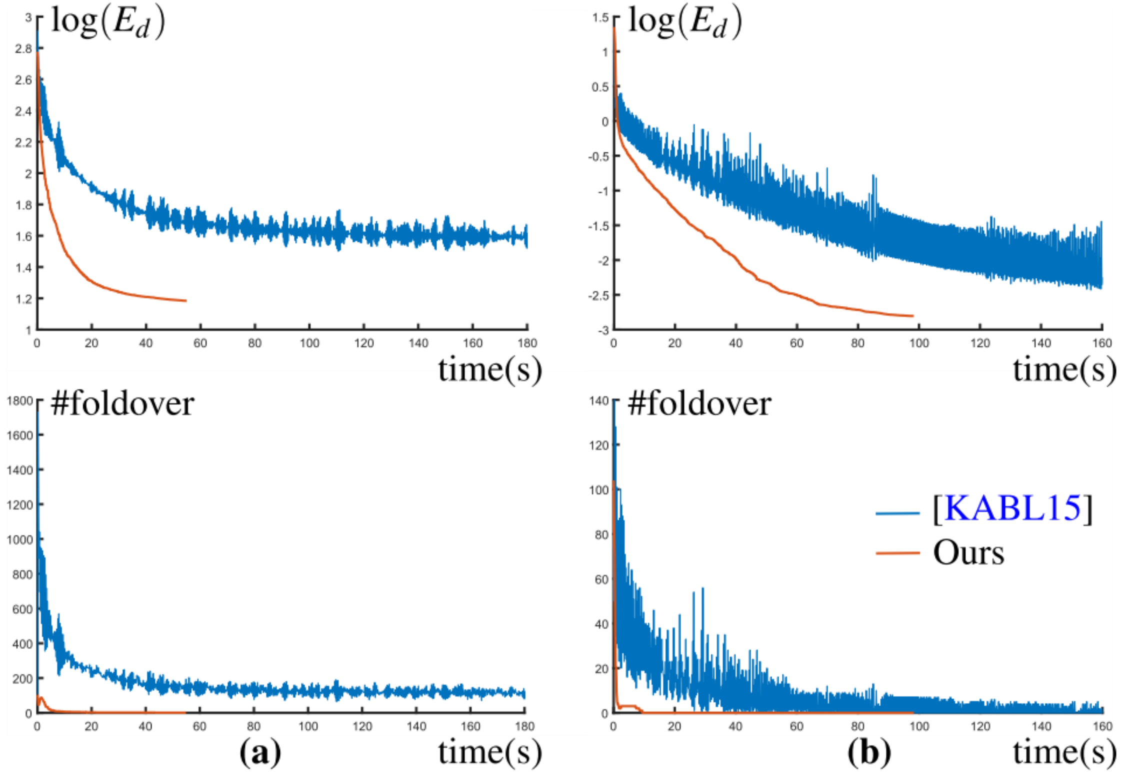


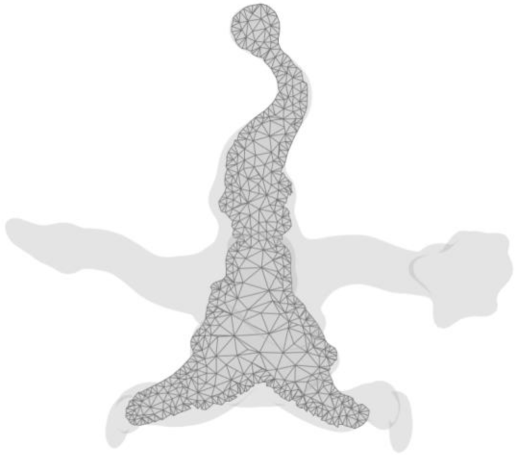
Kovalsky, Shahar Z., et al. "Large-scale bounded distortion mappings." *ACM Trans. Graph.* 34.6 (2015): 191-1.

Results



Comparison





Penalty

Overview

- Key idea: penalize the flipped elements via penalty functions

- Main properties:
  - it is very large to penalize flipped Jacobian matrices;
  - it is very small to accept flip-free Jacobian matrices.

- Solvers are then the challenges
  - Non-linear and non-convex

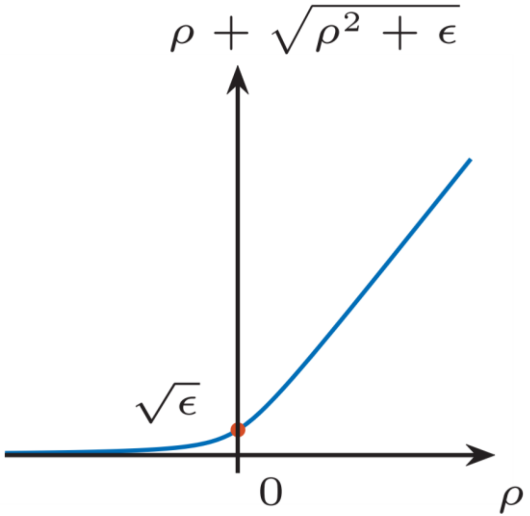
Common penalty functions

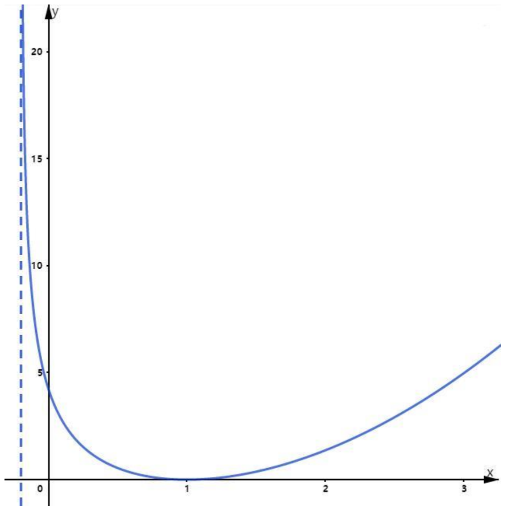
$$\sum_i \frac{\|J_i\|_F^d}{\det J_i + \sqrt{(\det J_i)^2 + \varepsilon}}$$

Key: the small positive ε

$$\sum_i (\det J_i - 1)^2 + \left(\log \frac{x - \varepsilon}{1 - \varepsilon}\right)^2$$

Key: parameter ε





Parameter ε

the regularization parameter ε^k. Starting from $\varepsilon^0 := 1$, we define the sequence as follows:

$$\varepsilon^{k+1} := \begin{cases} \left(1 - \frac{\sigma^k \sqrt{(D_-^{k+1})^2 + (\varepsilon^k)^2}}{|D_-^{k+1}| + \sqrt{(D_-^{k+1})^2 + (\varepsilon^k)^2}}\right) \varepsilon^k, & \text{if} \quad D_-^{k+1} < 0, \\ (1 - \sigma^k)\varepsilon^k, & \text{if} \quad D_-^{k+1} \geq 0, \end{cases}$$

where $D_-^{k+1} := \min\limits_{t \in 1 \ldots \#T} \det J_t^{k+1}$ is the minimum value of the Jacobian determinant over all cells of the mesh at the iteration $k + 1$, and $\sigma^k := \max\left(\frac{1}{10}, 1 - \frac{F(U^{k+1}, \varepsilon^k)}{F(U^k, \varepsilon^k)}\right)$.

Garanzha, Vladimir, et al. "Foldover-free maps in 50 lines of code." *ACM Transactions on Graphics (TOG)* 40.4 (2021): 1-16.

Solvers

- Block coordinate descent method
- Monotone preconditioned conjugate gradient method
- L-BFGS
- SGD
- Secord-order methods

Bounded distortion spaces

Bounded distortion space

- Goal: explicitly bound the conformal distortion
- Constraints: $\delta_t^{con} = \frac{\sigma_{max}}{\sigma_{min}} < K, \det J_t > 0$
- Non-linear and non-convex

$$J_t = \begin{pmatrix} a_t + c_t & d_t - b_t \\ d_t + b_t & a_t - c_t \end{pmatrix}$$

$$\sigma_{max} = \sqrt{a_t^2 + b_t^2} + \sqrt{c_t^2 + d_t^2}$$

$$\sigma_{min} = \left| \sqrt{a_t^2 + b_t^2} - \sqrt{c_t^2 + d_t^2} \right|$$

Lipman, Yaron. "Bounded distortion mapping spaces for triangular meshes." *ACM Transactions on Graphics (TOG)* 31.4 (2012): 1-13.

Reformulation

$$J_t = \begin{pmatrix} a_t + c_t & d_t - b_t \\ d_t + b_t & a_t - c_t \end{pmatrix}$$

$$\sigma_{max} = \sqrt{a_t^2 + b_t^2} + \sqrt{c_t^2 + d_t^2}$$

$$\sigma_{min} = \left| \sqrt{a_t^2 + b_t^2} - \sqrt{c_t^2 + d_t^2} \right|$$

$\delta_t^{con} < K, \det J_t > 0$

$\det J_t > 0$ → $\sqrt{c_t^2 + d_t^2} < \sqrt{a_t^2 + b_t^2}$

$\delta_t^{con} < K$ → $\sqrt{c_t^2 + d_t^2} \le \frac{K-1}{K+1}\sqrt{a_t^2 + b_t^2}$

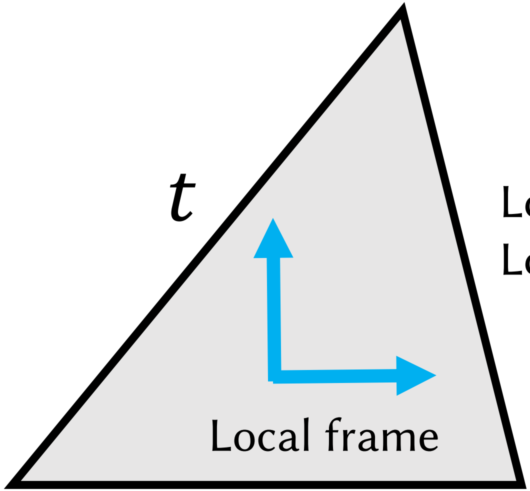
$r_t \le \sqrt{a_t^2 + b_t^2}$ Non-convex

$\sqrt{c_t^2 + d_t^2} \le \frac{K-1}{K+1} r_t$ Convex

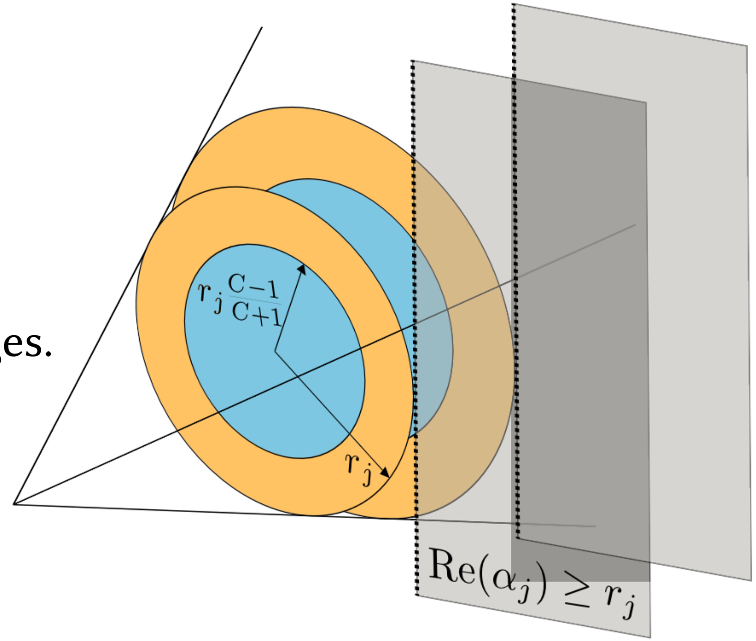
$r_t > 0$ Convex

Maximum convex subset

$r_t \leq \sqrt{a_t^2 + b_t^2}$ → $r_t \leq a_t$ Convex



Local frame changes, a_t changes.
Local frame is also a variable.



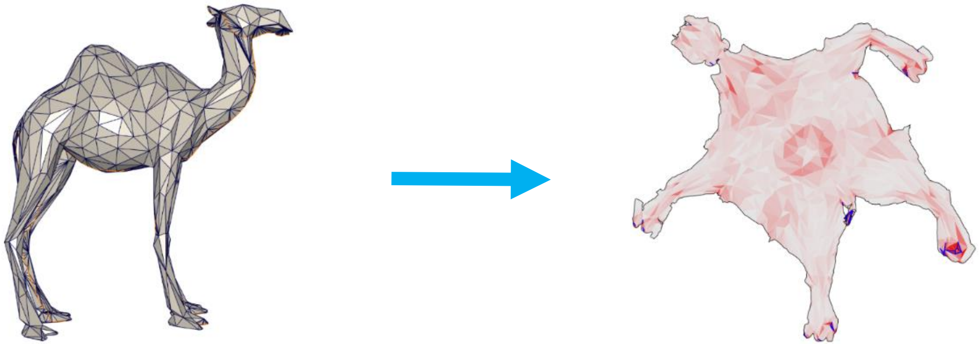
$\alpha_j = a_j + i \cdot b_j$

Optimization process

- Objective function:
  - LSCM: $E = \sum_t \text{Area}(t) \cdot (c_t^2 + d_t^2)$
  - ARAP: $E = \sum_t \text{Area}(t) \cdot \left((a_t - 1)^2 + b_t^2 + c_t^2 + d_t^2\right)$

$$J_t = \begin{pmatrix} a_t + c_t & d_t - b_t \\ d_t + b_t & a_t - c_t \end{pmatrix}$$

- Optimization:
  - Fix the local frame on each triangle: Second-Order Cone Programming;
  - Update local frame to let $b_t = 0$.



$\delta_{max}^{con} = 26.98$
Time: 4.03s

1. How to choose K?
2. The speed is slow.

Known U and V

- Signed singular value decomposition:
  - $J_i = U_i S_i V_i^T$
  - U_i and V_i are two rotation matrices
  - $S_i = \text{diag}(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i,3}), \sigma_{i,1} \geq \sigma_{i,2} \geq |\sigma_{i,3}|$
- Observation:
  - $S_i = U_i^T J_i V_i$
  - if U_i and V_i are known, S_i is a linear vector function w.r.t vertex positions.
  - Bounded conformal distortion constraint
    - $\tau(J_i) = \sigma_{i,1}/\sigma_{i,3} \leq K$ become linear inequality constraint.
- Problem: how to get U_i and V_i?

Aigerman, Noam, and Yaron Lipman. "Injective and bounded distortion mappings in 3D." *ACM Transactions on Graphics (TOG)* 32.4 (2013): 1-14.

Pipeline

Algorithm 1: Projection on $\mathcal{F}_K$

Input: Tetrahedral mesh $\mathbf{M} = (\mathbf{T}, \mathbf{V})$
Source map $\Phi \in \mathcal{F}$ to project
Distortion bound $K \geq 1$

Output: K-bounded-distortion map $\Pr(\Phi) \in \mathcal{F}_K$

$\Psi^0 = \Phi$;

while $\text{Dist}\left(\Psi^{n+1}, \Psi^n\right) > \varepsilon$ **do**

Compute SSVD $B_j^n = U_j \Sigma_j V_j^T$ for all $t_j \in \mathbf{T}$;

Update:

$$\begin{aligned} \Psi^{n+1} = & \operatorname{argmin} \text{Dist}\left(\Psi_{\boldsymbol{w}}, \Psi^n\right) \\ & \text{s.t.} \\ & \boldsymbol{w} \in \mathbb{R}^{3 \times n} \\ & \text{and for every } t_j \in \mathbf{T}, \\ & \text{the diagonal } \mathbf{x} = (x_1, x_2, x_3) \text{ of } U_j^T B_j(\boldsymbol{w}) V_j \\ & \text{satisfies } x_k \leq K x_\ell, \;\; 1 \leq k, \ell \leq 3 \end{aligned}$$

Return $\Pr(\Phi) = \Psi^{n+1}$;

Results

- A small number of iterations
  - < 10 iterations
- Quadratic programming is time-consuming
- Set K?

Area-based methods

Total unsigned area

- Signed area S_t of a triangle t
- Unsigned area U_t of a triangle t

- Facts for any 2D triangulation or 3D tetrahedron $\mathcal{T}$:
  - Total signed area $\sum_t S_t = \text{Area}(\mathcal{T})$
  - Total unsigned area $\sum_t U_t \geq \text{Area}(\mathcal{T})$
  - $\mathcal{T}$ is flip-free iff $\sum_t U_t = \text{Area}(\mathcal{T})$

- Optimizing total unsigned area for achieving flip-free mappings.

Xu, Yin, et al. "Embedding a triangular graph within a given boundary." *Computer Aided Geometric Design* 28.6 (2011): 349-356.

Failure reasons

- It is not smooth, considered as a function of the embedding
  - TUA exhibits a C^1 discontinuity as a vertex moves across the supporting line of its opposite edge in a triangle.
- While any injective embedding achieves the global minimum of TUA, the inverse is not true
  - the triangulation containing degenerate elements is a global minimum of TUA but a non-injective embedding
- TUA has zero gradients with respect to any vertex surrounded by a ring of consistently oriented triangles.

Du, Xingyi, et al. "Lifting simplices to find injectivity." *ACM Trans. Graph.* 39.4 (2020): 120.

Total lifted content

$$\sum_{t} \frac{1}{d!} \sqrt{\det(X^T X + \alpha \tilde{X}^T \tilde{X})}$$

- X: a $d \times d$ matrix whose column vectors are the edge vectors from one vertex to the other d vertices.
- Thus, $\sqrt{\det(X^T X)} = U_t$.
- $\tilde{X}$: similar to X, but from an auxiliary simplex, such as an equilateral triangles or tetrahedra of the same size as the input.
- TLC becomes TUA when $\alpha = 0$.

Properties of TLC

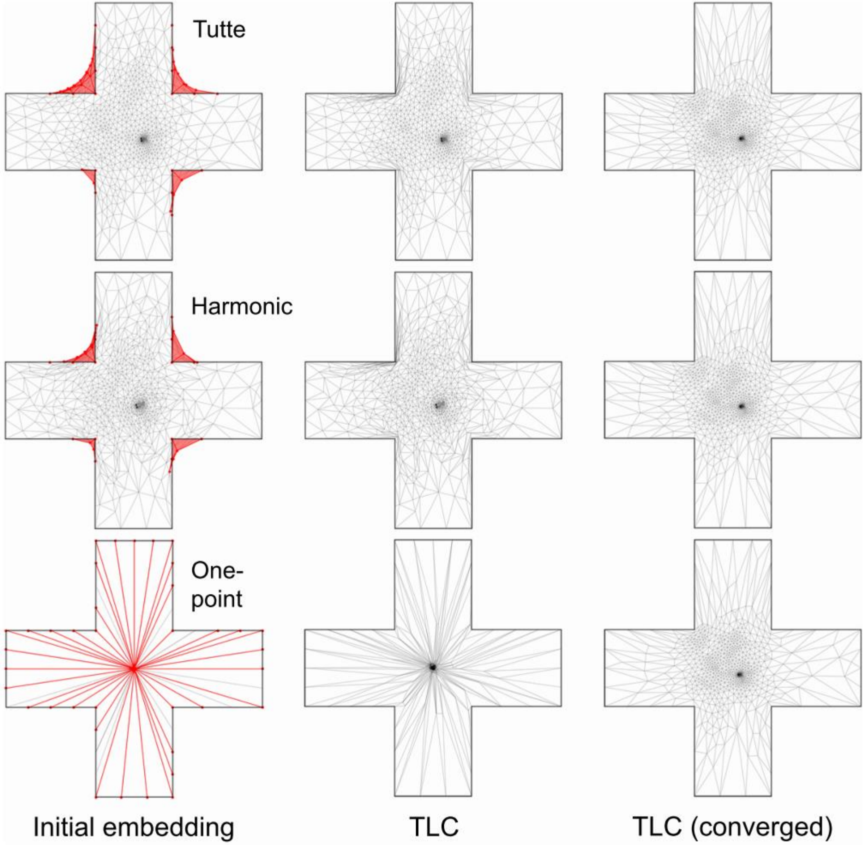
- TLC is smooth over the entire space

Corollary 4.2. *The Total Lifted Content,* $E_{\widetilde{T},\alpha}(T)$*, is strictly positive, greater than the total unsigned areas or volumes of* T*, and differentiable to any order over the embedding space of* T*, given any set of non-degenerate auxiliary simplices* $\widetilde{T}$ *and positive* α*.*

- TLC has only an injective global minimum for sufficiently small values of α.

Proposition 4.3. *Let* T_0 *be some* d*-dimensional* ($d = 2, 3$) *injective embedding into the target boundary and* $\widetilde{T}$ *be a set of non-degenerate auxiliary simplices. Then there exists some* $\beta > 0$ *such that* $E_{\widetilde{T},\alpha}(T) > E_{\widetilde{T},\alpha}(T_0)$ *for any non-injective embedding* T *and* $\alpha < \beta$*.*

Results

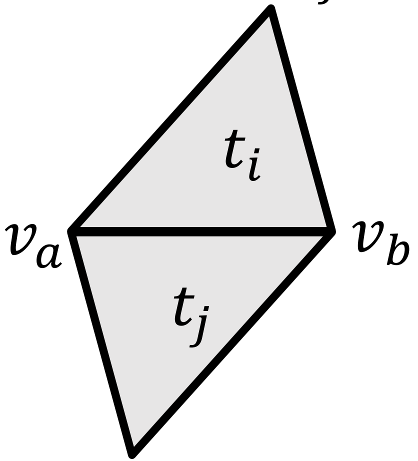

Tutte
Harmonic
One-
point
Initial embedding
TLC
TLC (converged)

Different representations

Affine transformation

- Key observation: the parameter space is a 2D triangulation, uniquely defined by all the AFFINE TRANSFORAMTIONS on the triangles.
- **Edge assembly constraints:**

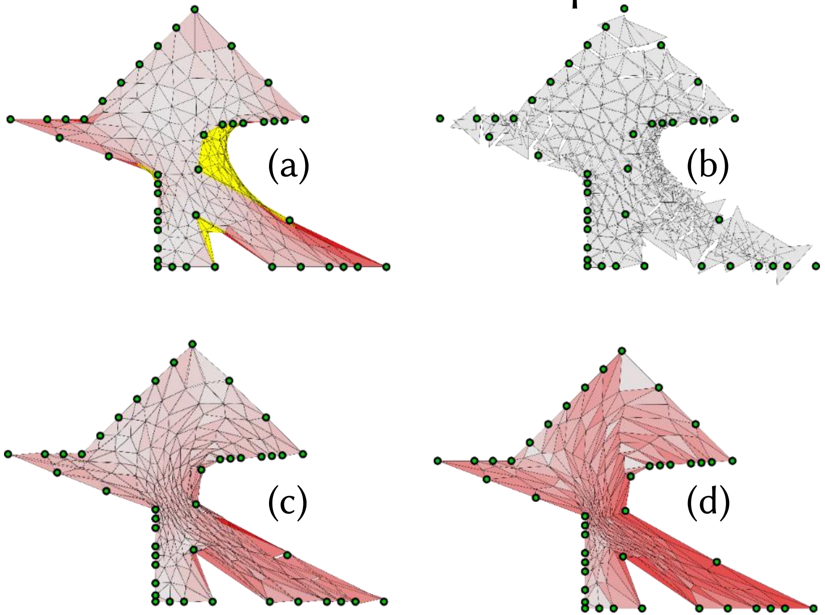
$$J_i(v_a - v_b) = J_j(v_a - v_b)$$

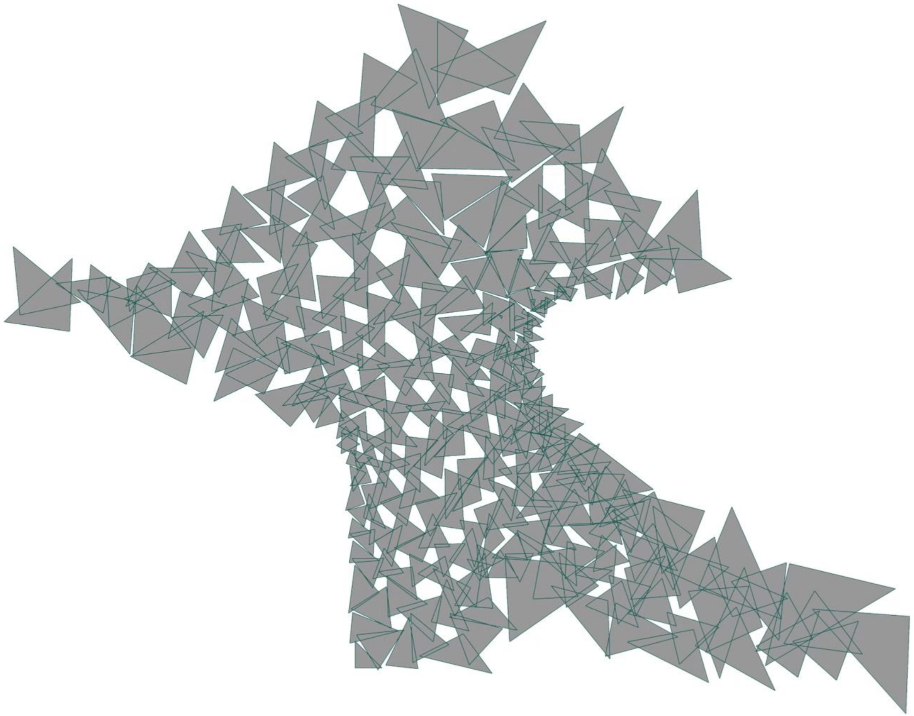


Fu, Xiao-Ming, and Yang Liu. "Computing inversion-free mappings by simplex assembly." *ACM Transactions on Graphics (TOG)* 35.6 (2016): 1-12.

Key idea

- Disassembly + Assembly
  - Treat affine transformation as variables
  - Unconstrained optimization





Unconstrained optimization problem

Disassembly: project initial A_i^0 into feasible space.

Assembly: unconstrained optimization.

$$\min_{\substack{J_1,\dots,J_N\\T_1,\dots,T_N}} \lambda E_{assembly} + E_C + \mu E_m$$

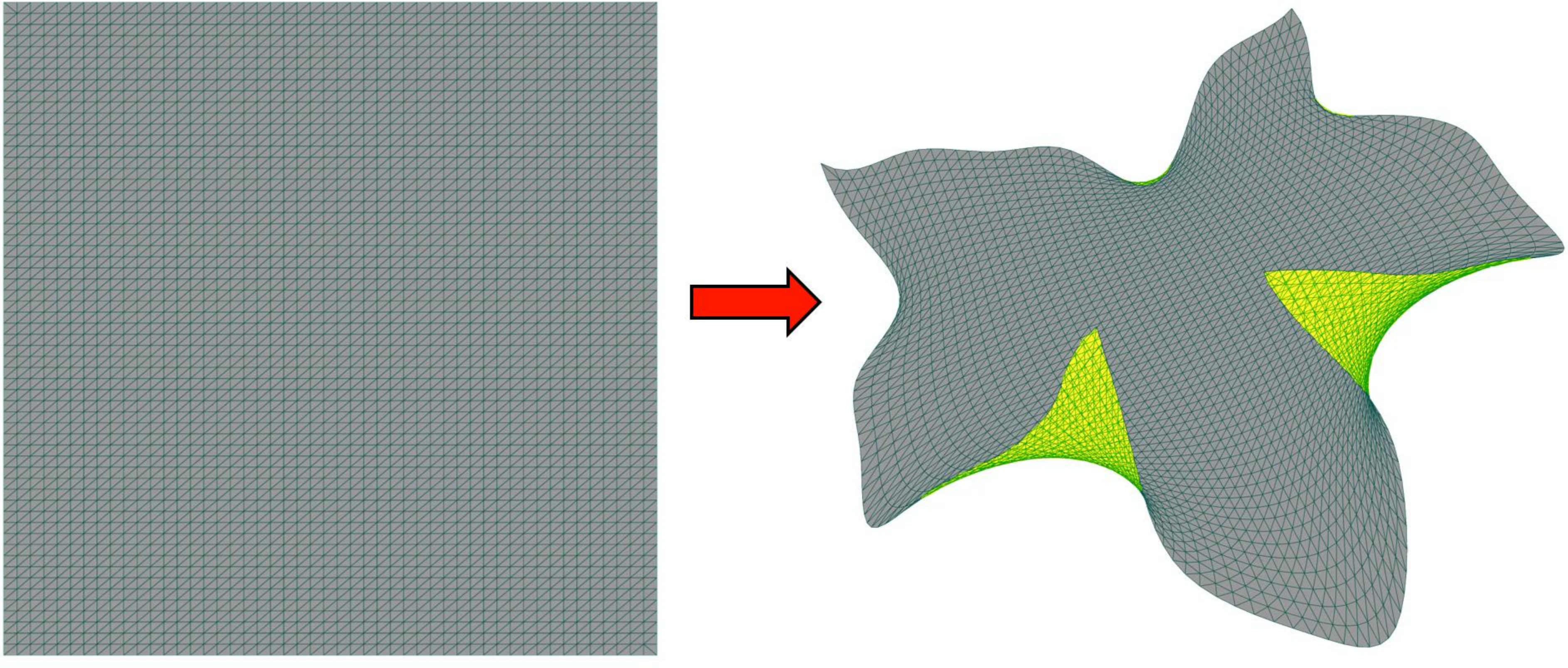
$E_{assembly}$: summation of squares of edge, assembly constraints.

E_C: Barrier function on distortion

E_m: users' designed energy

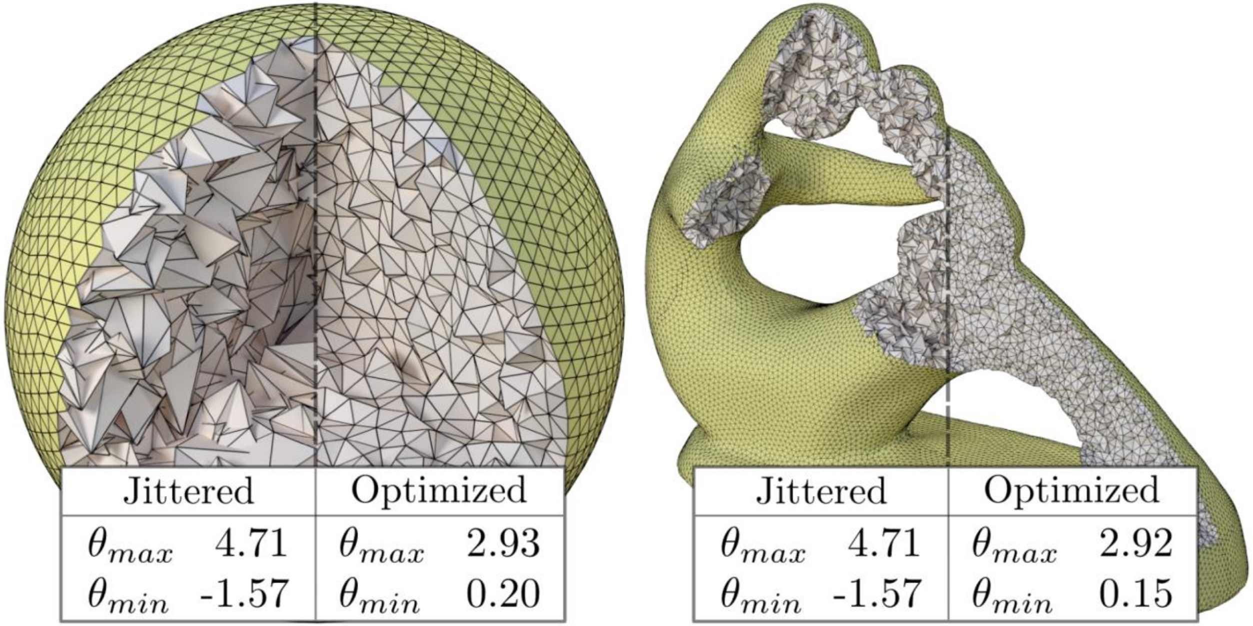
$$\lambda_{k+1} = \min\left(\lambda_{\min} \cdot \max\left(\frac{E_{C,k} + \mu E_{m,k}}{E_{assembly,k}}, 1\right), \lambda_{max}\right)$$

1. $E_{assembly}$ dominates the energy, approach zero;
2. λ_{max}: avoid large distortion.



Angles

- 2D: Angle-based flattening
- 3D: Dihedral Angle-based Maps of Tetrahedral Meshes



Paillé, Gilles-Philippe, et al. "Dihedral angle-based maps of tetrahedral meshes." *ACM Transactions on Graphics (TOG)* 34.4 (2015): 1-10.

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