中国科学技术大学
University of Science and Technology of China

# GAMES 301: 第7讲

# 参数化应用
# – 网格生成

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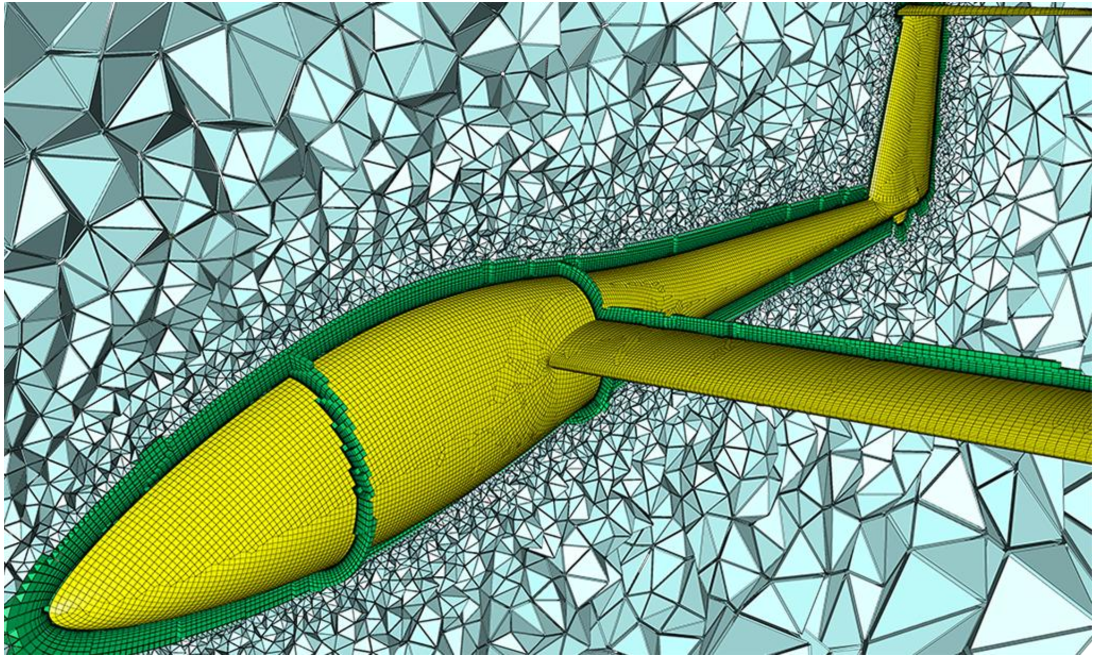
# Meshing

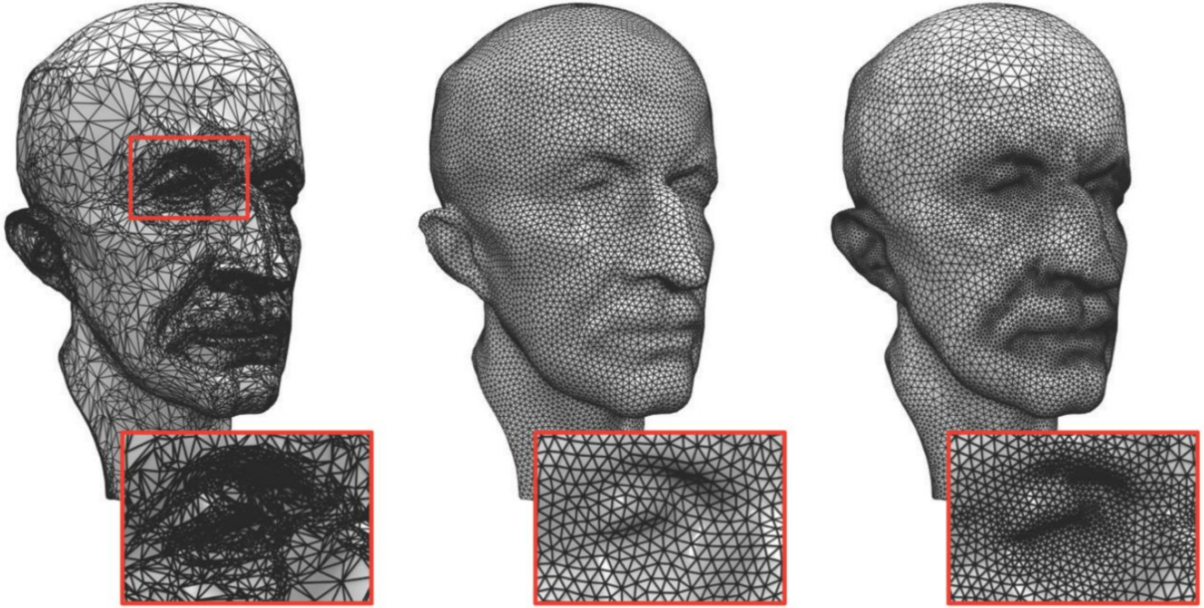
# Meshing

- Polygons or polyhedrons that connect in a series of lines and points to approximate a digital model's geometry.
- FEM

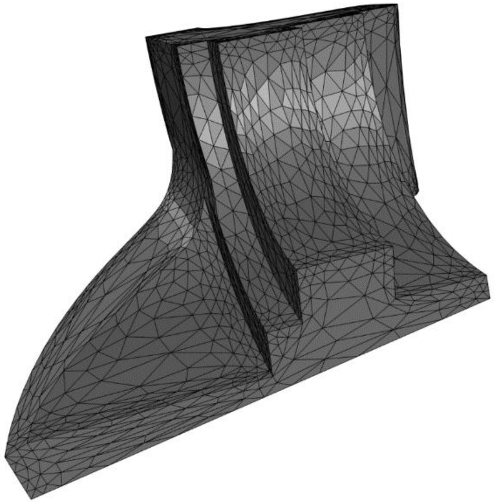


# Remeshing

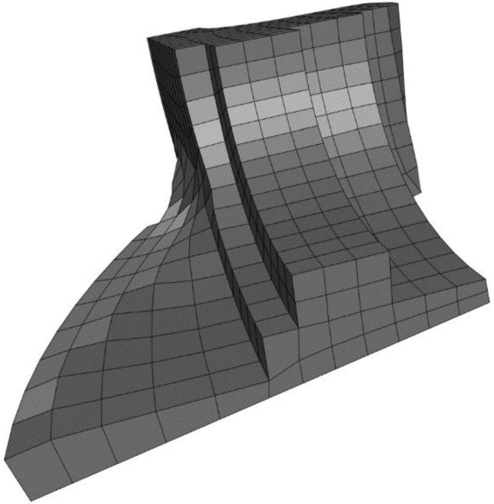
- Given a 3D mesh, compute another mesh, whose elements satisfy some quality requirements, while approximating the input acceptably.



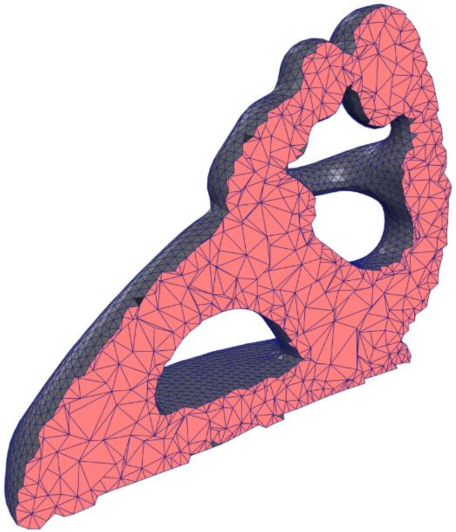
# Target mesh types



Triangle



Quad



Tet

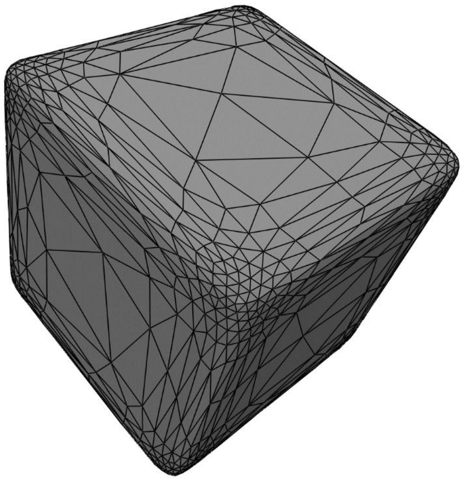
Hex

# Quality metrics

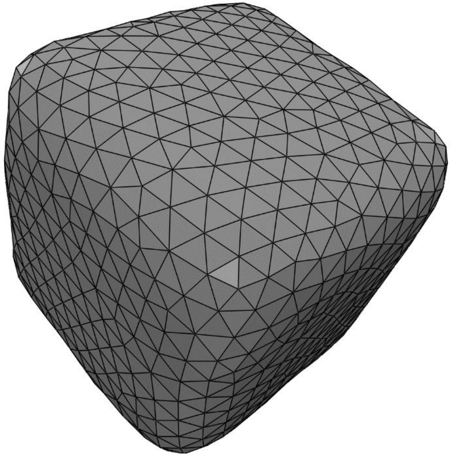
- Different applications imply different quality criteria and requirements.
- Mesh quality
  - Sampling density
  - Regularity
  - Size
  - Orientation
  - Alignment
  - Shape of the mesh elements.
  - Non-topological issues (mesh repair)

# Local Structure

- Element shape
  - Isometric
  - Anisotropic



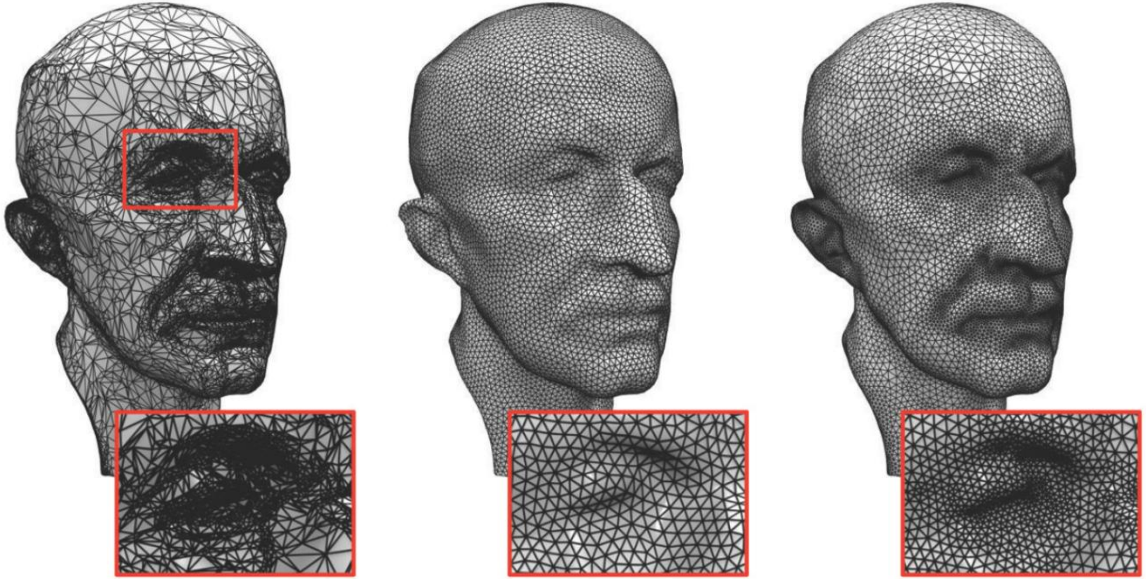
**Anisotropic**



**Isotropic**

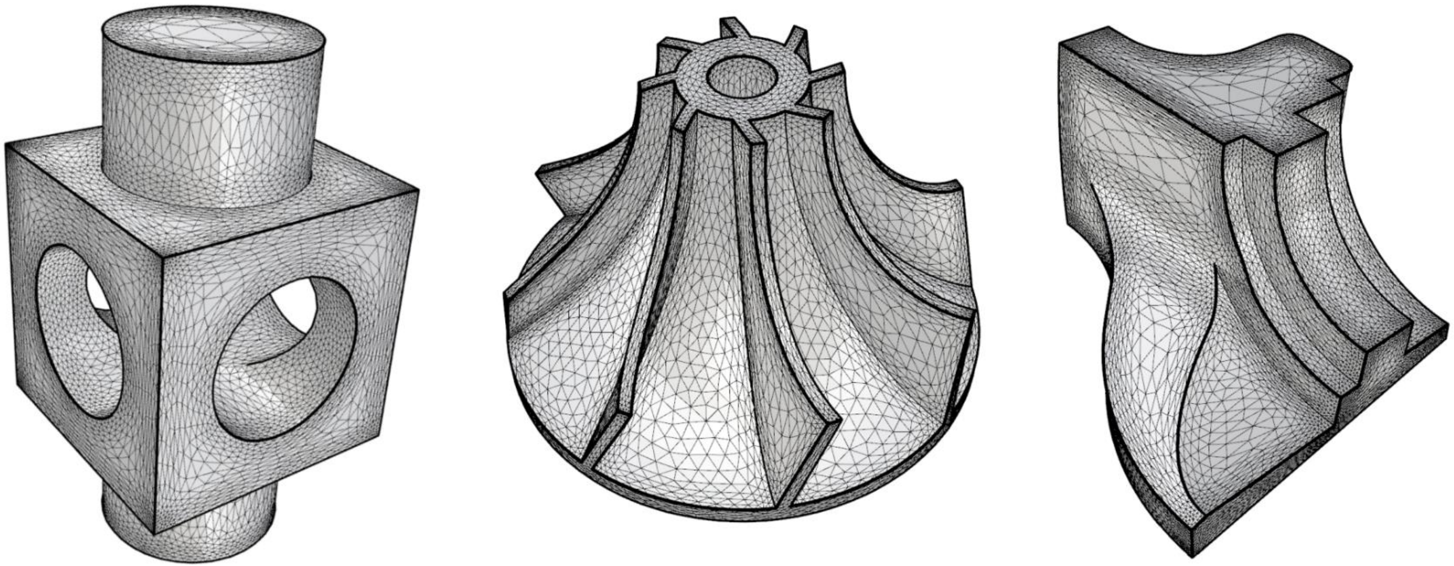
# Local Structure

- Element density
  - uniform VS. nonuniform or adaptive



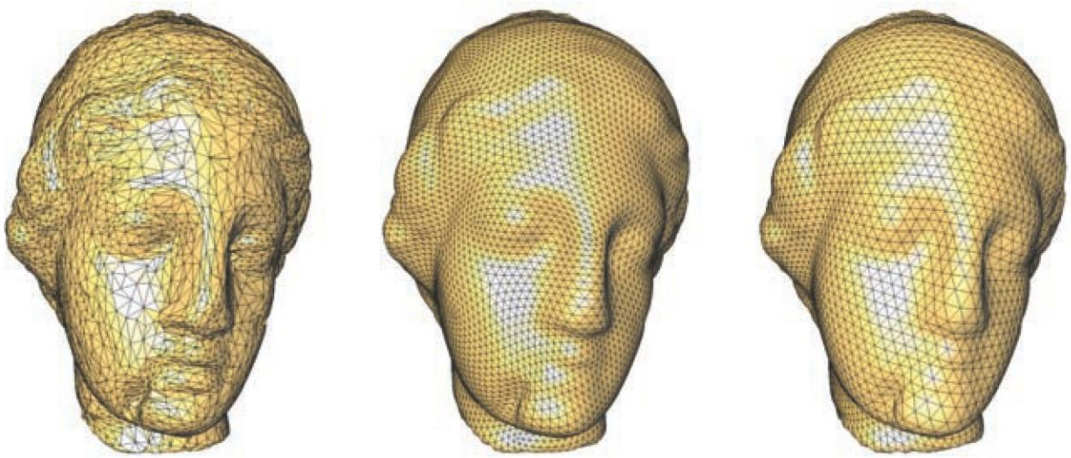
# Local Structure

- Element alignment and orientation
  - elements should align to sharp features
  - orientation of anisotropic elements



# Global structure

- Vertex
  - Regular
    - Valence = 6 for triangle mesh
    - Valence = 4 for quad mesh
  - Irregular (singular)
- Global
  - Irregular
  - Semiregular
    - regular subdivision of a c
  - Highly regular
    - most vertices are regular
  - Regular
    - all vertices are regular



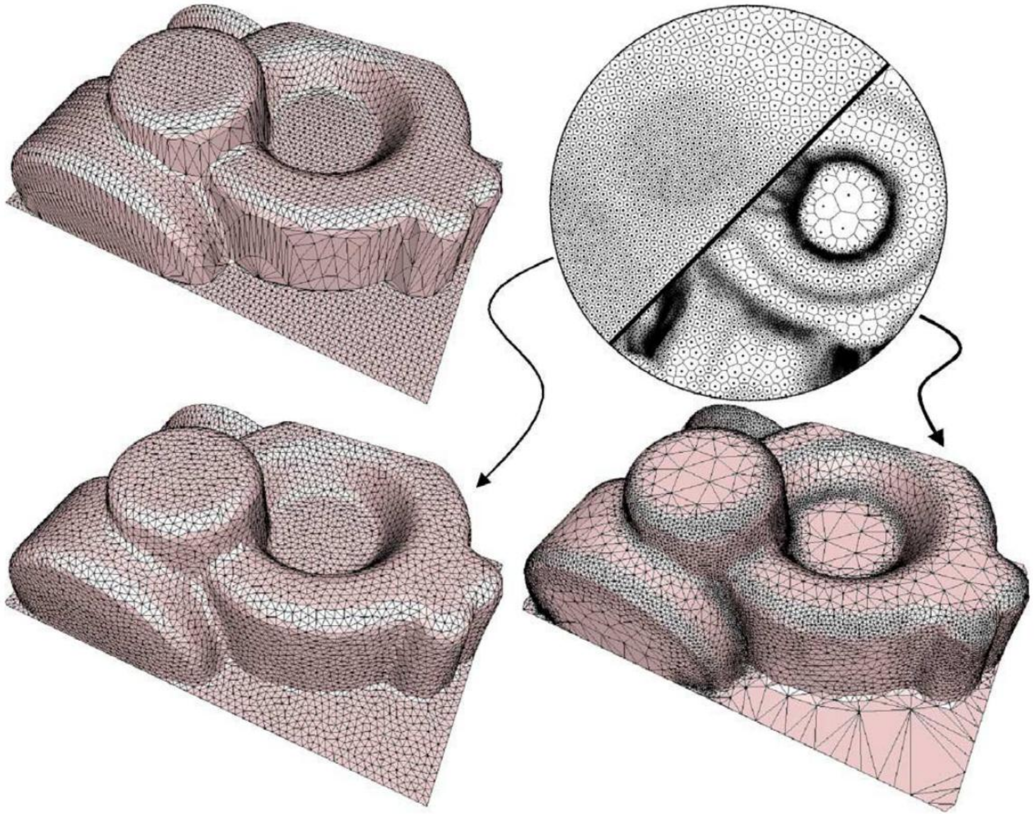
Irregular Semiregular Regular

# Method overview

- Delaunay triangulation / Voronoi diagram
- Advancing front
- Local operators
- Parameterization-based methods
- Topology structure optimization
- .....

# Parameterization-based methods

- It is easy to perform meshing/remeshing in the parameter domain.

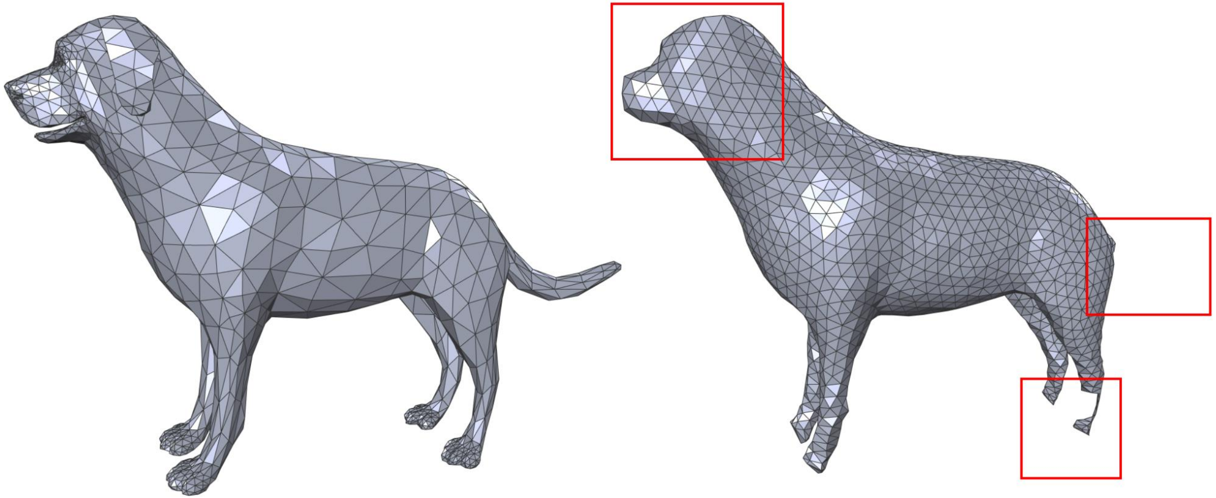


# Requirements of parameterizations

- Low distortion
  - keeping shapes from the parameter domains
- Cuts
  - parameterization-based method requires cut paths
  - visit at least twice

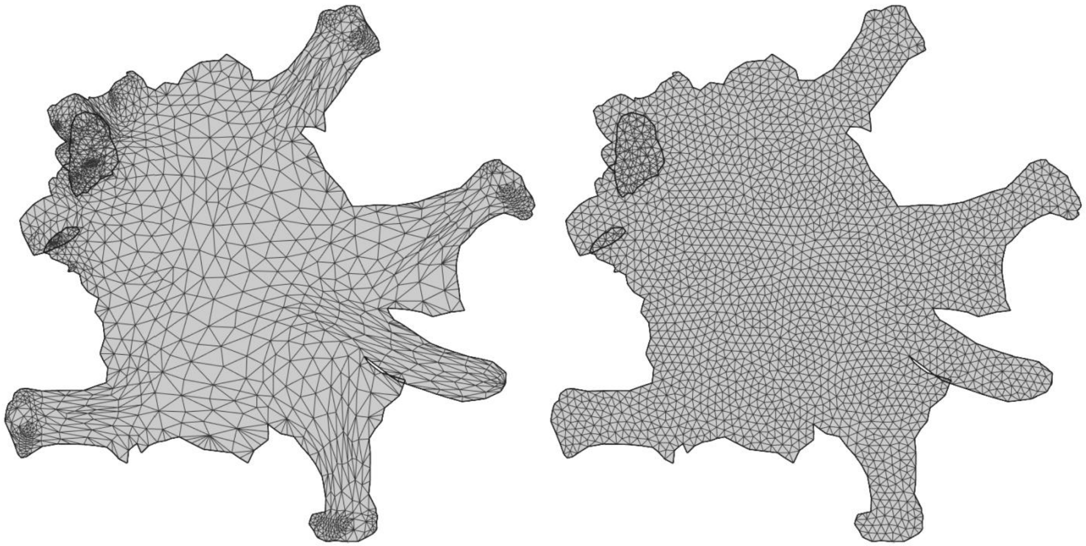
# Isotropic triangular meshing

- Target: regular triangles
- Keeping closeness
  - projection onto the input surfaces
    - time-consuming
    - may be incorrect for small-scale features



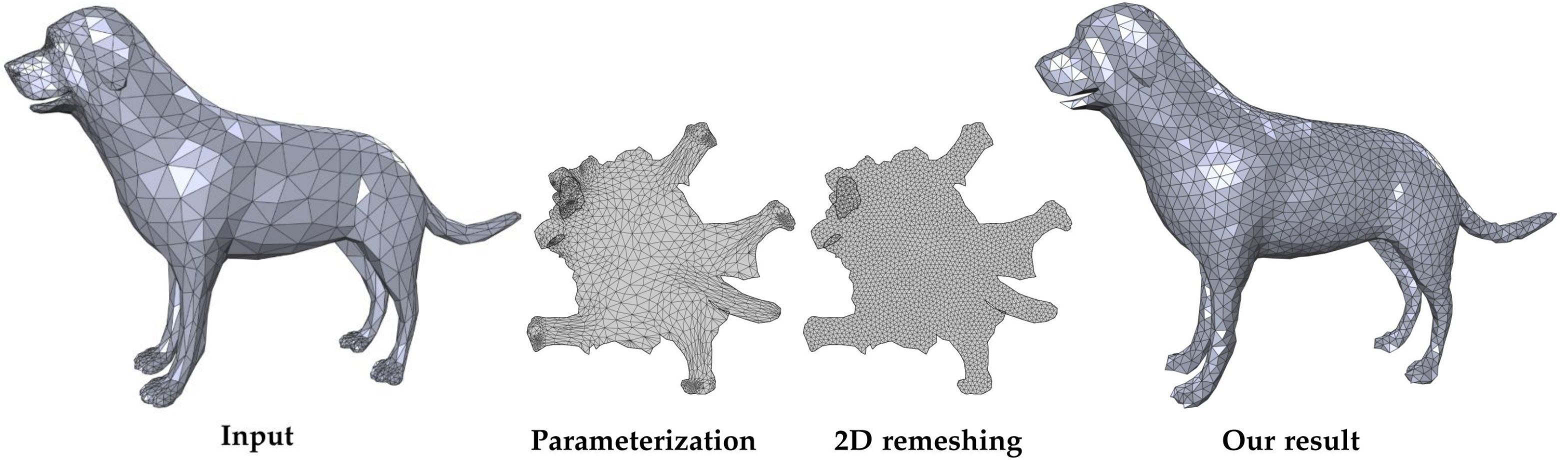
# Key observation

- Remeshing on the plane
  - no projection
  - the Euclidean distance approximates the geodesic distance when the parameterizations is nearly isometric



# Using planar parameterizations

- Cut the input surface to be disk topology
- Compute parameterizations
- Remesh the parameterized domain
- Project back to the input surface



# Anisotropic remeshing

- Input:
  - Domain: $\Omega \in \mathbb{R}^d$
  - Metric field: $\boldsymbol{M}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega$
    - $d \times d$ positive-definite matrix
- Isotropic remeshing
  - All edge lengths are as equal as possible.
- Anisotropic remeshing
  - All edge lengths with metric are as equal as possible.

# Metric

- A metric on a set $X$ is a function (called the distance function or simply distance)

$$d: X \times X \rightarrow [0, \infty)$$

where $[0, \infty)$ is the set of non-negative real numbers.

- For all $x, y, z \in X$, the following conditions are satisfied:
  - Non-negativity or separation axiom
    - $d(x, y) \geq 0$
  - Identity of indiscernibles
    - $d(x, y) = 0 \Leftrightarrow x = y$
  - Symmetry
    - $d(x, y) = d(y, x)$
  - Subadditivity or triangle inequality
    - $d(x, z) \leq d(x, y) + d(y, z)$

# Metric

- Conditions 1 and 2 together define a positive-definite function.
- The first condition is implied by the others.

- In practice, the metric can be represented by a positive-definite symmetric $m \times m$ matrix $M(x)$.
  - $M(x) = Q(x)^T Q(x)$.

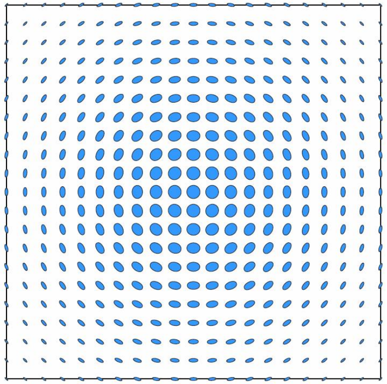
- Given a $M(x)$, its decomposition to $Q(x)$ is non-unique.

# Length

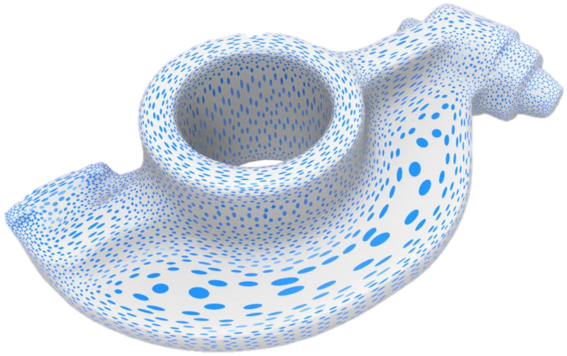
- Given the metric field $M(x)$ and an open curve $C \subset \Omega$, the length of $C$ is defined as the integration of the length of tangent vector along the curve $C$ with metric $M(x)$
- The anisotropic distance $d_M(x, y)$ between two points $x$ and $y$ can be defined as the length of the (possibly non-unique) shortest curve (assuming line segment) that connects $x$ and $y$.

$$\int_0^1 \sqrt{(x-y)^T M(tx+(1-t)y)(x-y)}dt$$

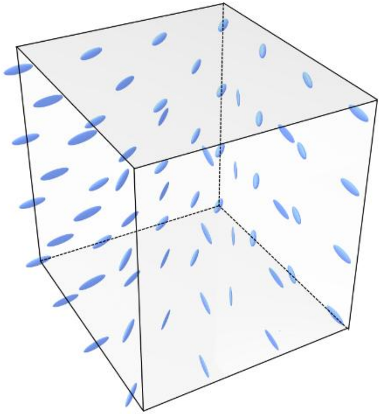
# Input examples



$\Omega$ = 2D square,
$M(p)$ = Hessian of given $u$



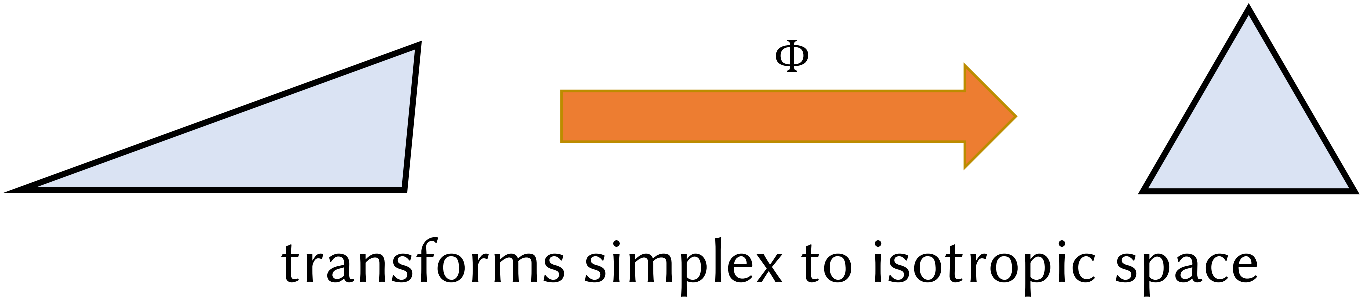
$\Omega =$ 3D surface,
$M(p)$ = mesh curvature



$\Omega$ = 3D cube,
$M(p)$ = given tensor field

# Anisotropic remeshing

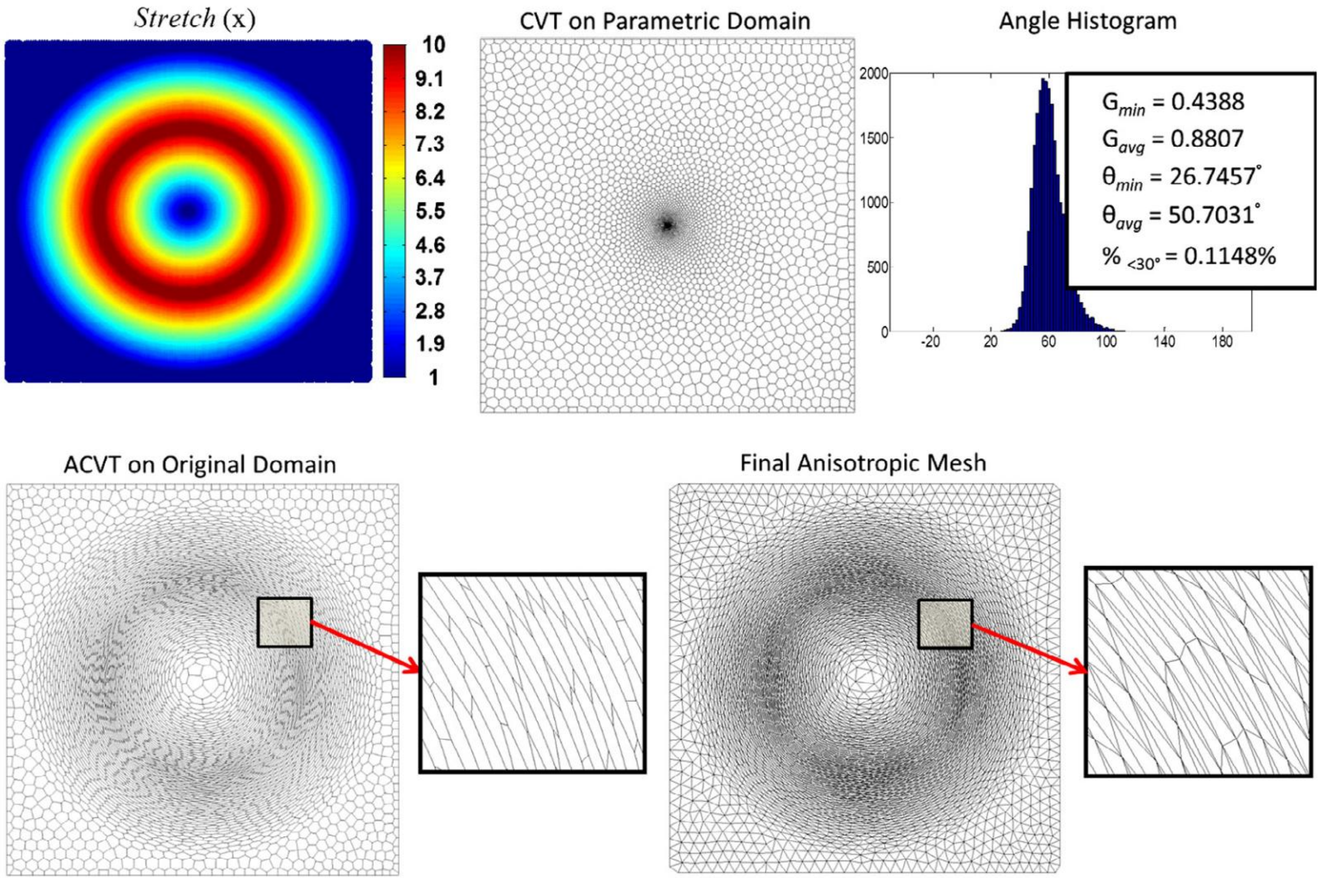
- Eigen-decomposition: $M(x) = U(x)\Lambda U(x)^T$
- Transformation $\Phi = \Lambda^{1/2}U(x)^T$
- The quality metrics are measured in the transformed space.



# Key observation

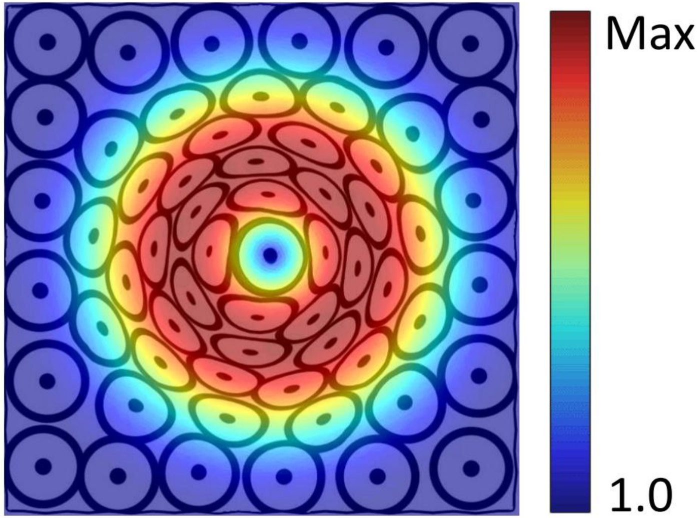
- Key idea: convert anisotropic meshing to isotropic meshing through parameterization/mapping
- Two common mappings:
  - Conformal mapping
    - Uniformization Theorem
    - any surface admits a Riemannian metric of a constant Gaussian curvature, which is conformal to the original one.
  - High-dim isometric embedding
    - Nash embedding theorem
    - every Riemannian manifold can be isometrically embedded into some high-dimensional (high-d) Euclidean space
    - In such high-d embedding space, the metric is uniform and isotropic.

# Using conformal parameterizations

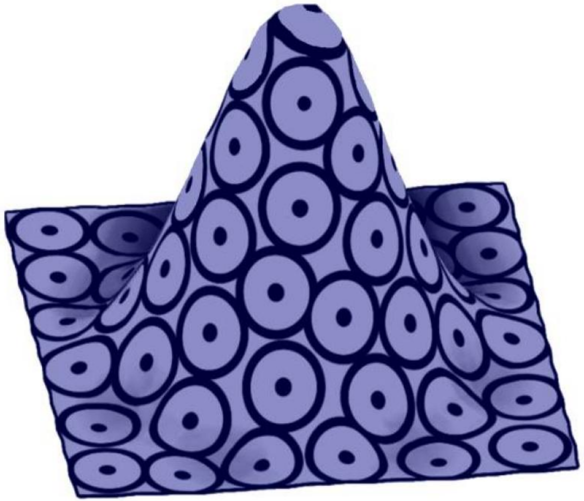


# High-dim isometric embedding

- For an arbitrary metric field $M(x)$ defined on the surface or volume $\Omega \subset R^m$, there exists a high-d space $R^l$ $(m < l)$, in which $\Omega$ can be embedded with Euclidean metric as $\overline{\Omega} \subset R^l$.




$\Omega$



$\overline{\Omega}$

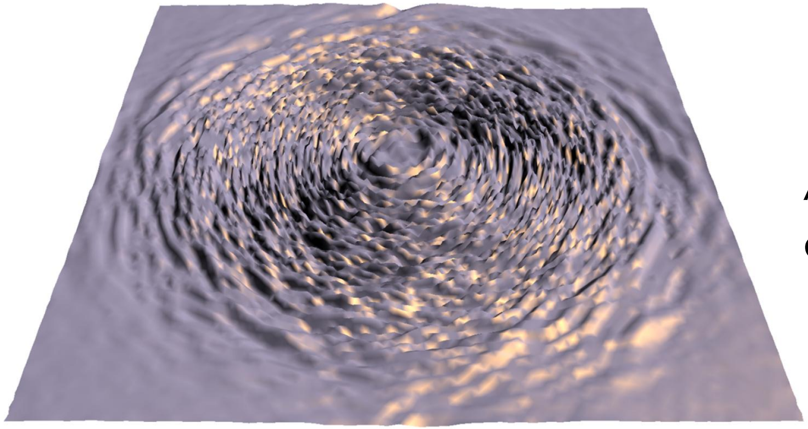
# Computing high-dim embedding

$$E_{\text{embedding}} + \mu E_{\text{smooth}}$$

$E_{\text{embedding}}$: measure the rigidity, like ARAP

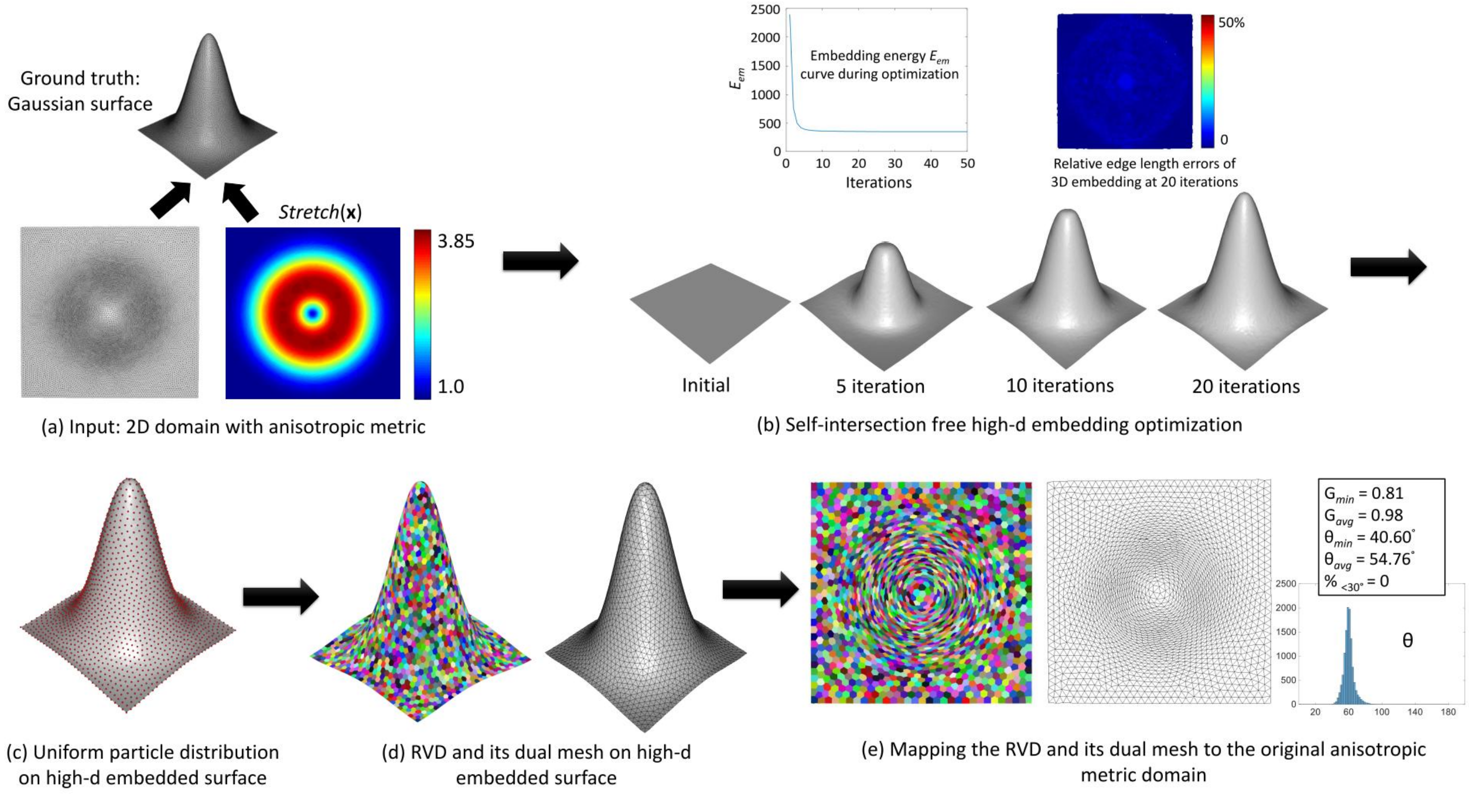
$E_{\text{smooth}}$: measure the smoothness of the embedding

Solver: local-global solver



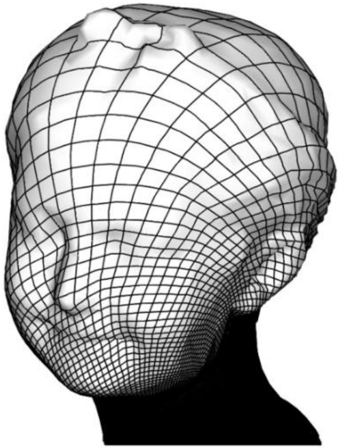
A 3D embedding from a 2D domain with an anisotropic metric

# Pipeline and results

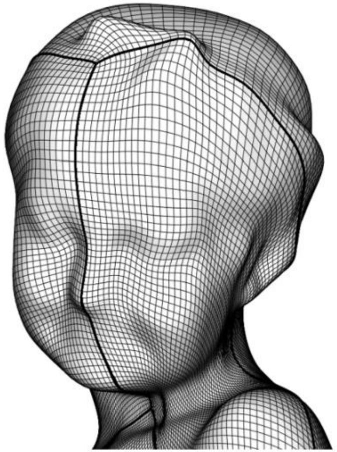


# Quad meshing

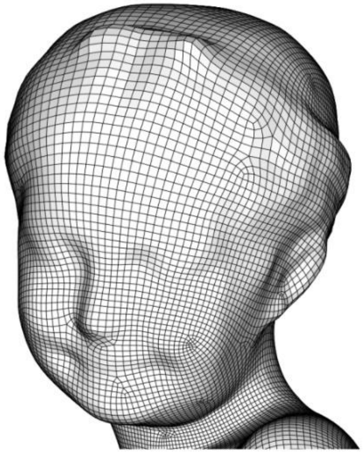
- Higher accuracy



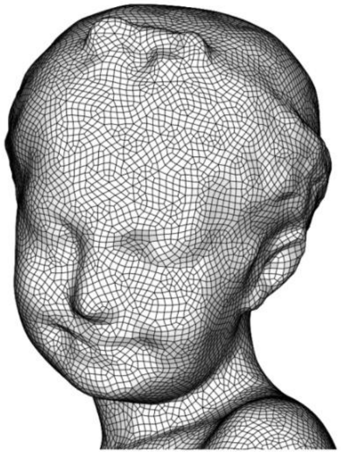
Regular



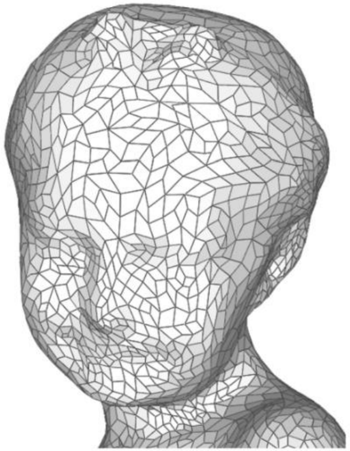
Semi-regular



Valence semi-regular

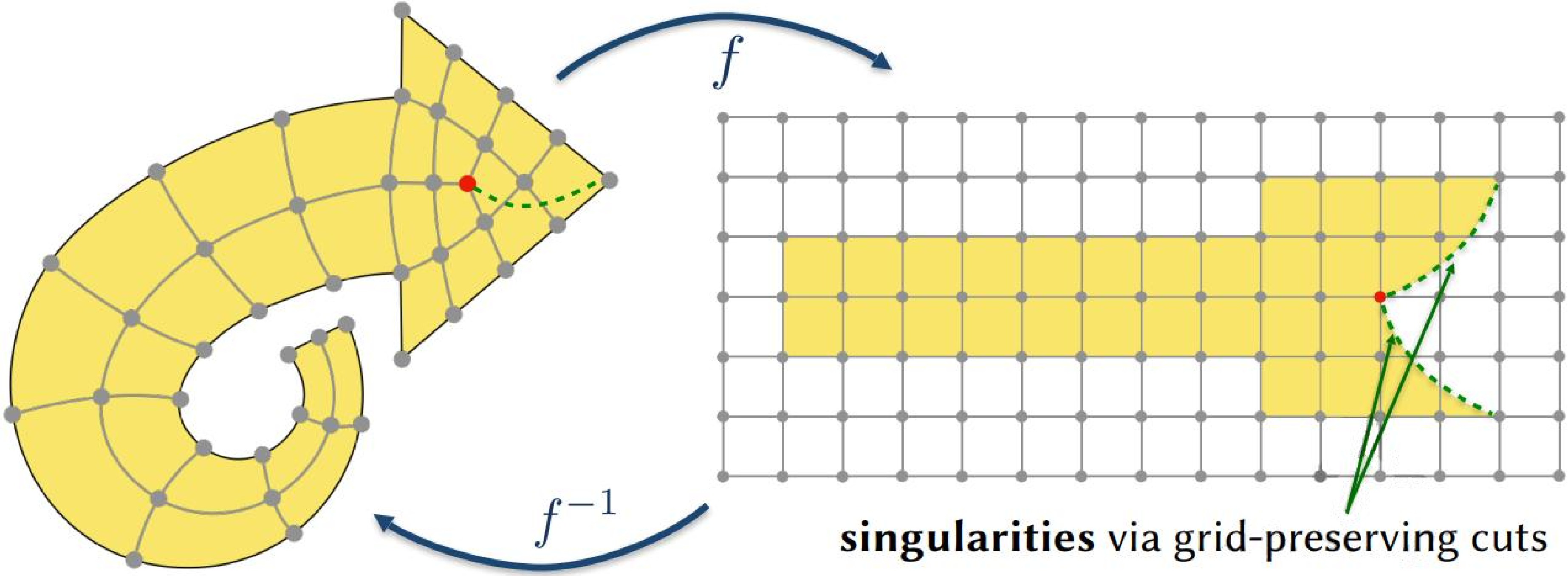


Unstructured

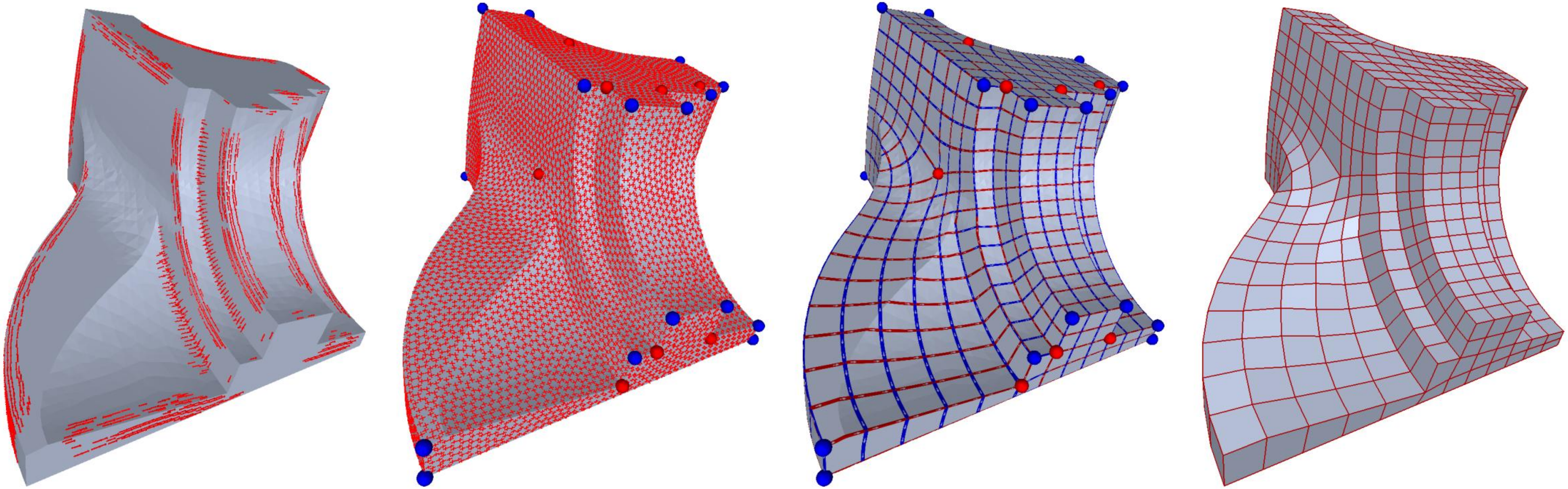


Unstructured

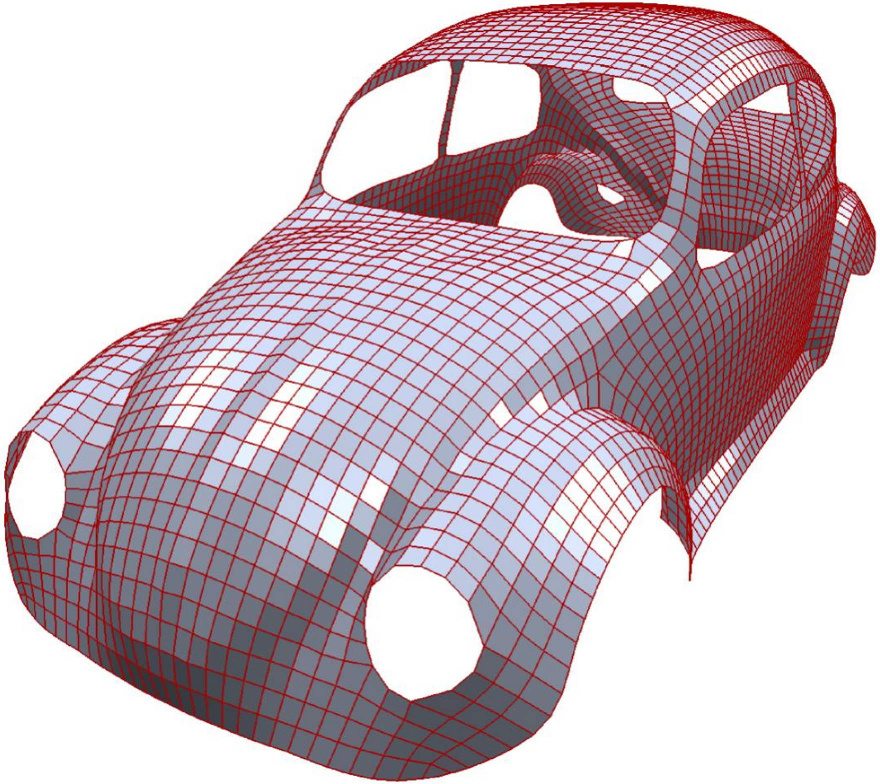
# Key observation

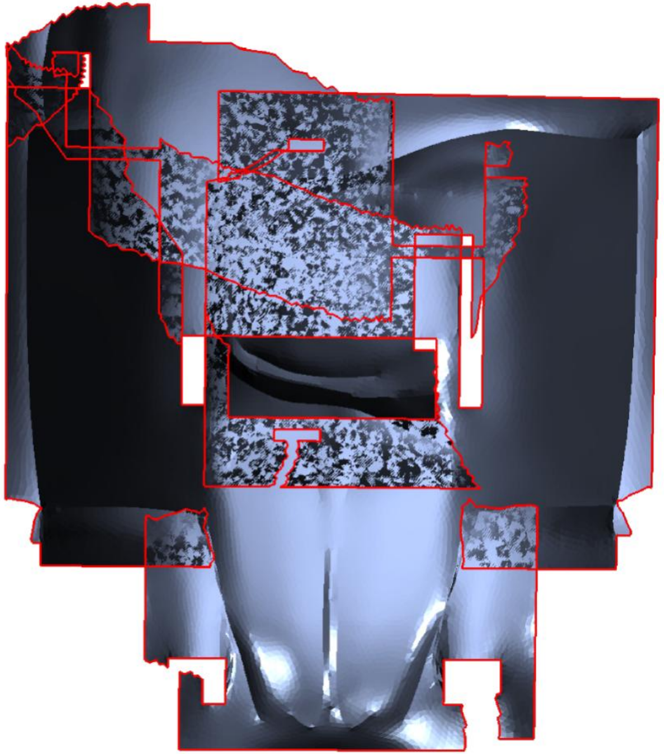


# Pipeline



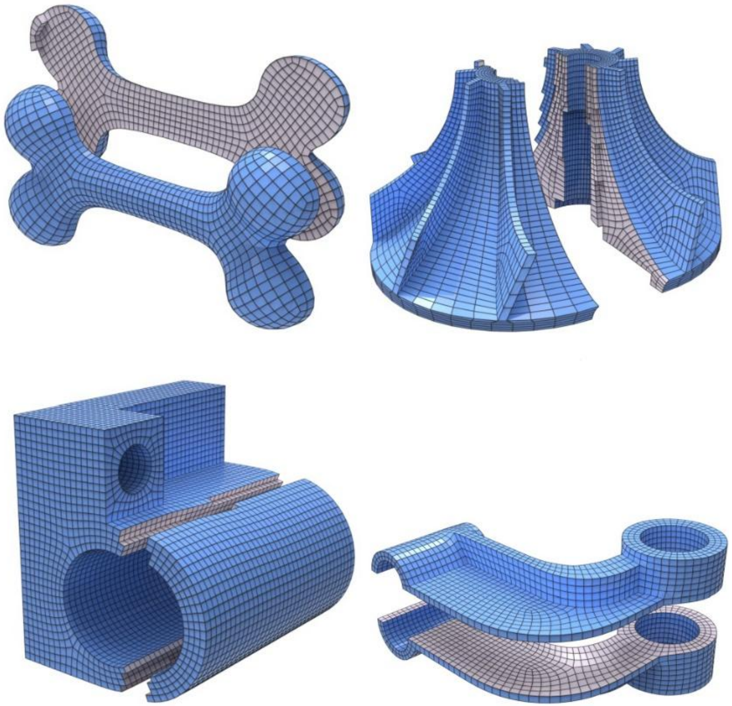
# Results



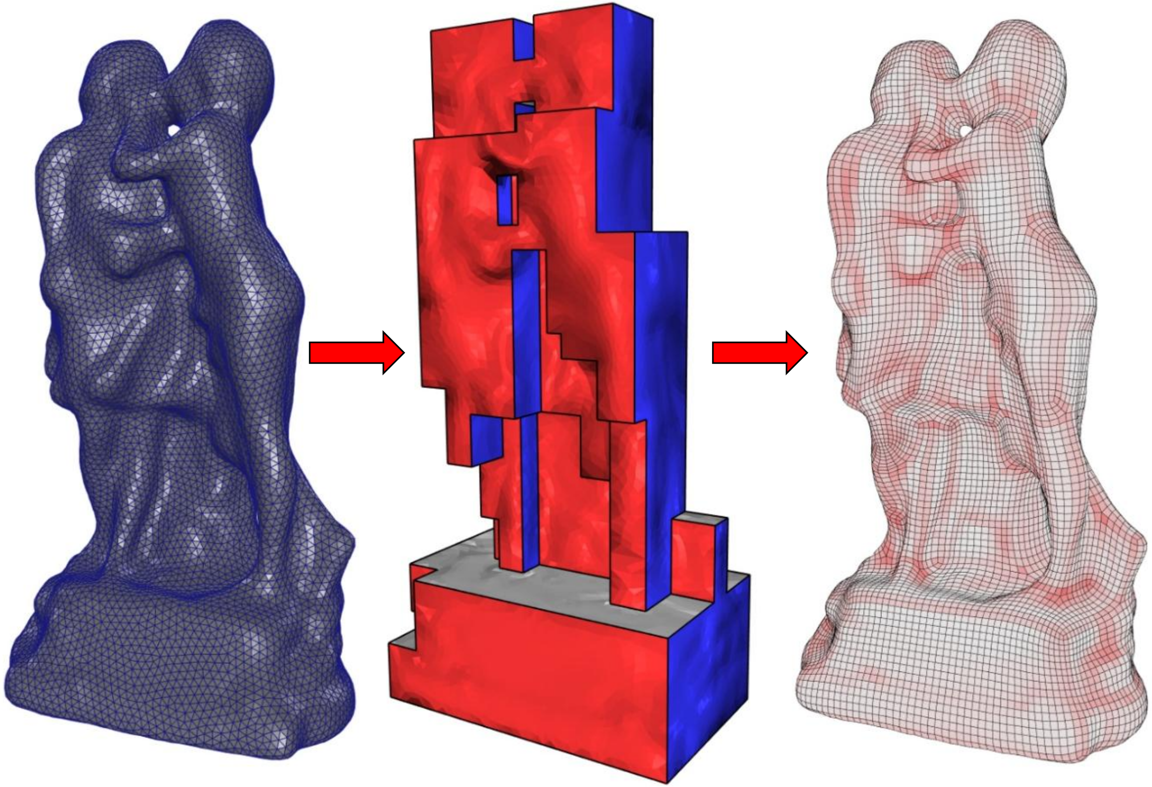


# All-hex meshing

- Fewer elements and higher accuracy

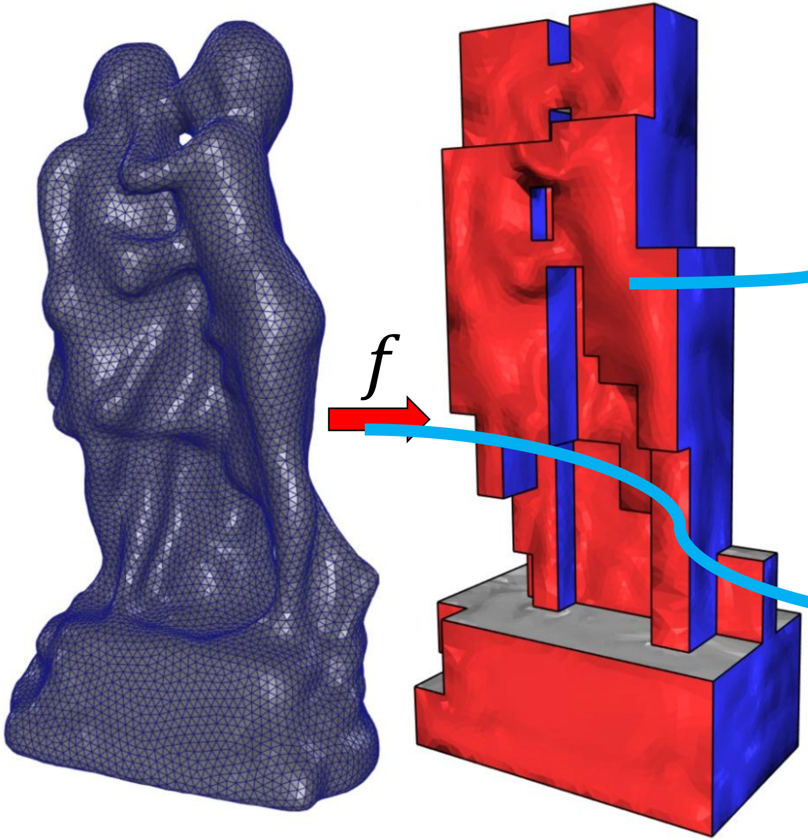


# Key observation



Tetrahedral Mesh Grid domain All-Hex Mesh

# PolyCube-maps




Tetrahedral Mesh

PolyCube

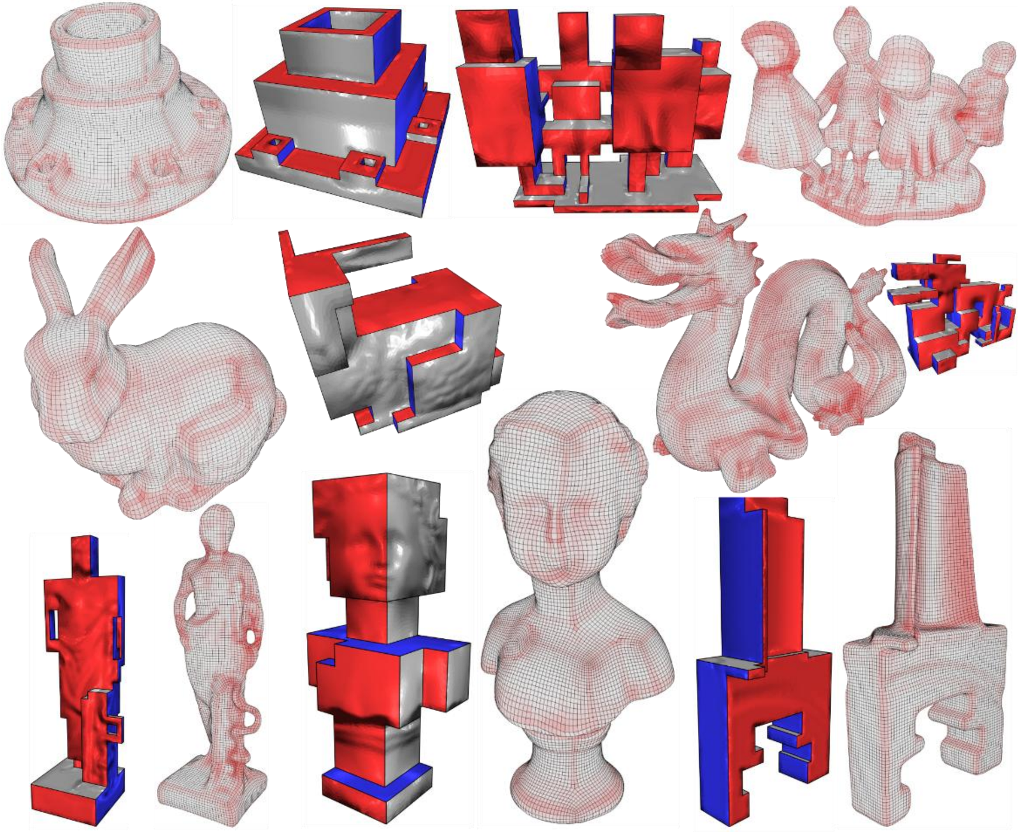
PolyCube:

1. Compact representations for closed complex shapes
2. Boundary normal aligns to the axes.
3. Axes: $(\pm 1,0,0)^T$, $(0,\pm 1,0)^T$, $(0,0,\pm 1)^T$.

PolyCube-Map $f$:

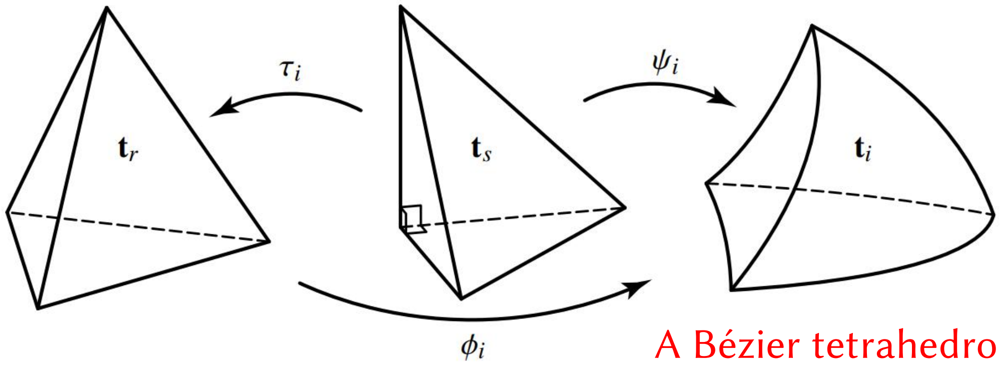
1. A mesh-based map.
2. Inversion-free and low distortion.

# Results

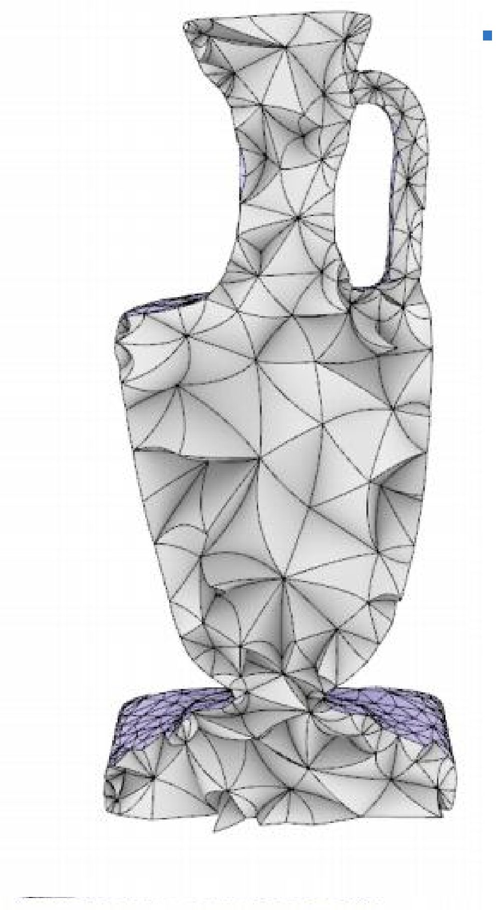


# High-order meshes

- Meshes with high-order elements

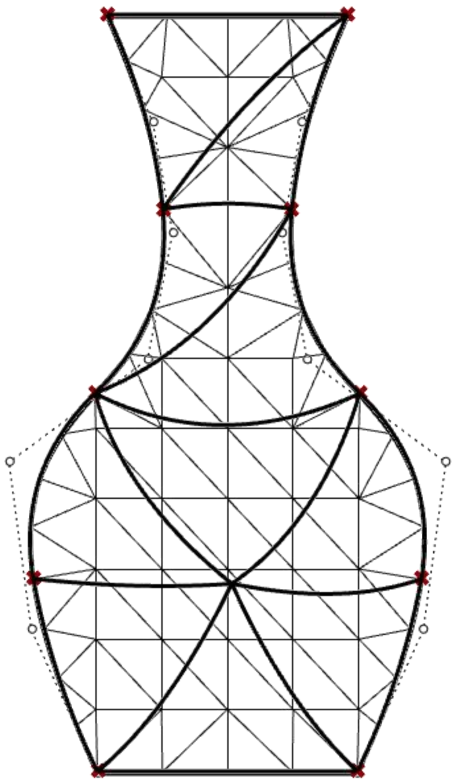



A Bézier tetrahedron



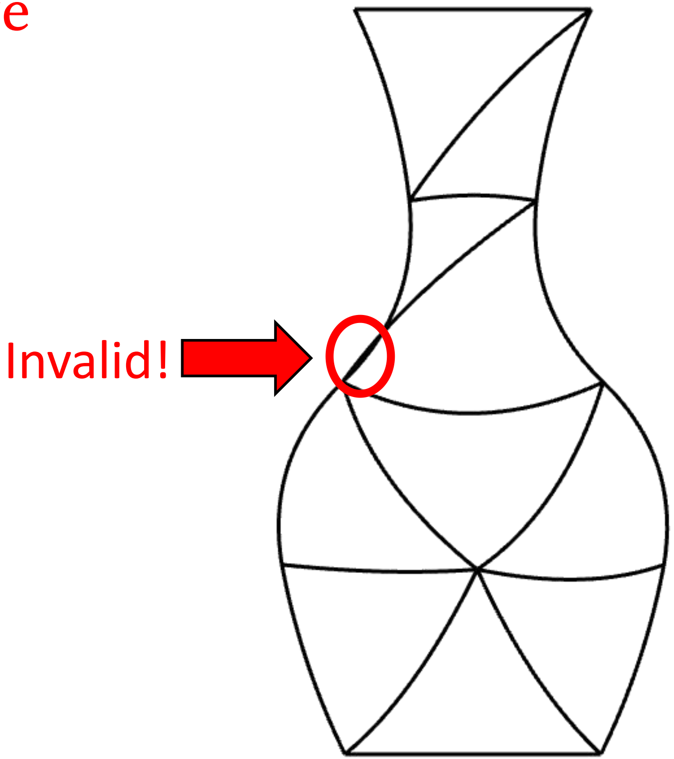
# VS. linear meshes

- Capture the boundary using much fewer elements
- Higher solution accuracy for the FEM simulation



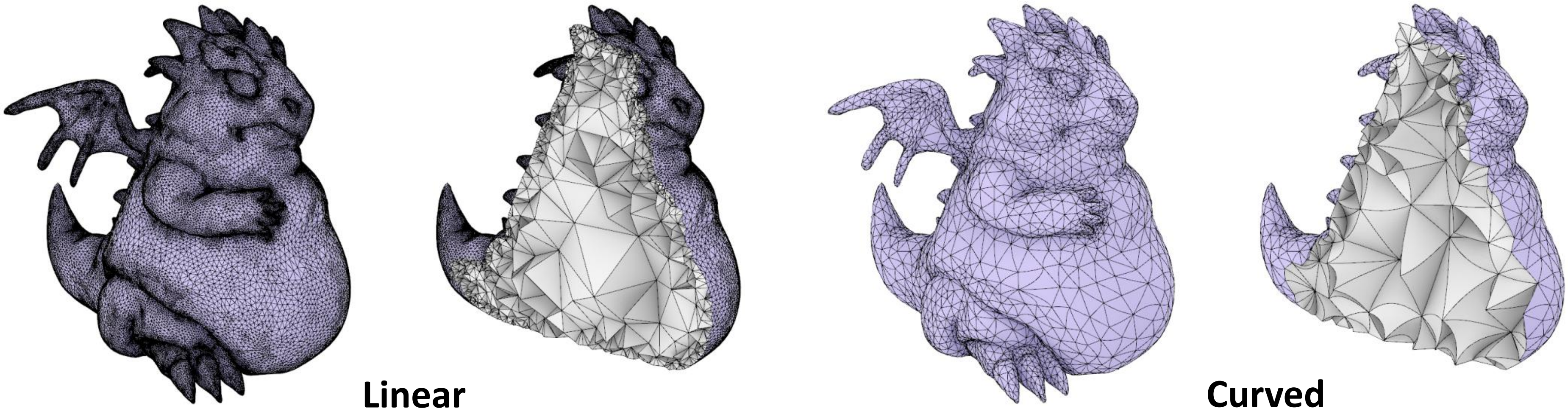
# Validity constraint

- The Bézier map is bijective



# Deforming linear meshes

- Construct a coarse linear mesh
- Deform the coarse linear mesh to be curved using different energies while keeping bijection.



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