中国科学技术大学
University of Science and Technology of China

GAMES 301：第8讲

无翻转光滑映射

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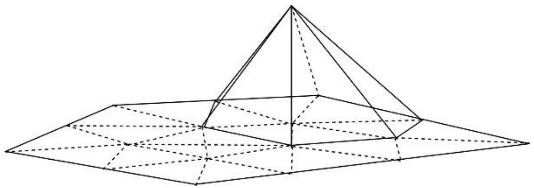
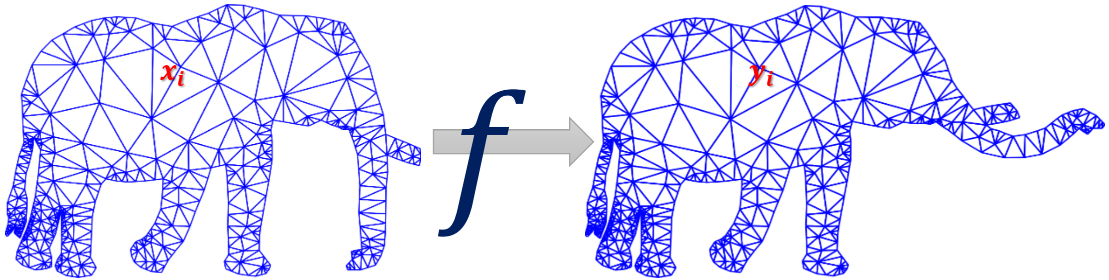
三角网格-分片线性映射

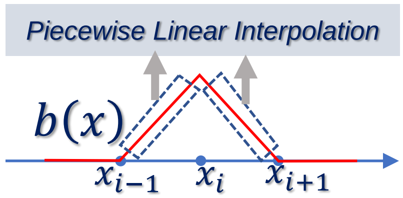
分片线性映射不是光滑映射

- 基函数不光滑
- 相邻片映射导数不连续

$$f(x_i) = \sum_j y_j b_j(x_i) \qquad b_j(x_i) = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$





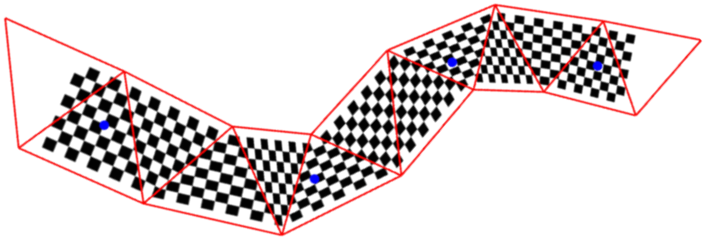


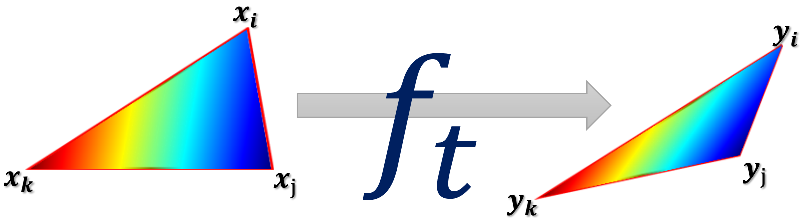
三角网格-分片线性映射

分片线性映射不是光滑映射

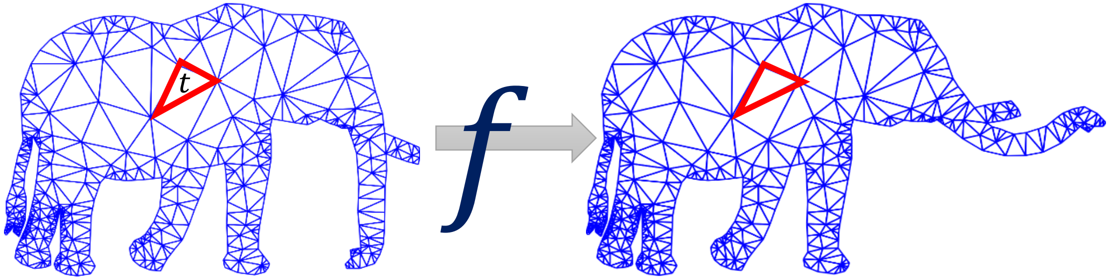
- 基函数不光滑
- 相邻片映射导数不连续

How to improve smoothness of the mapping?





$$f_t(x) = Ax + b$$



光滑映射构造

基于光滑基函数的光滑映射

1. RBF
2. 广义重心坐标
3. 调和映射
4. 样条 (B-Spline)

Radial Basis Function (RBF)

$$f(x) = \sum_i a_i b_i(x)$$

$$b_i(x) = g(|x - p_i|)$$

- $g(r) = \frac{1}{r+\epsilon}, g(r) = e^r, g(r) = \frac{1}{r^2}$

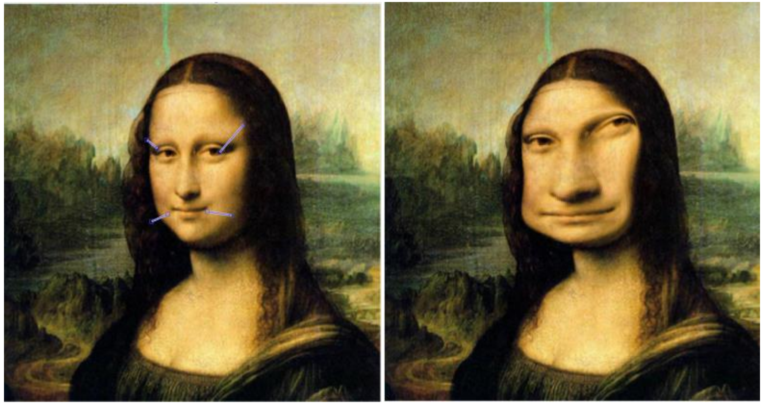
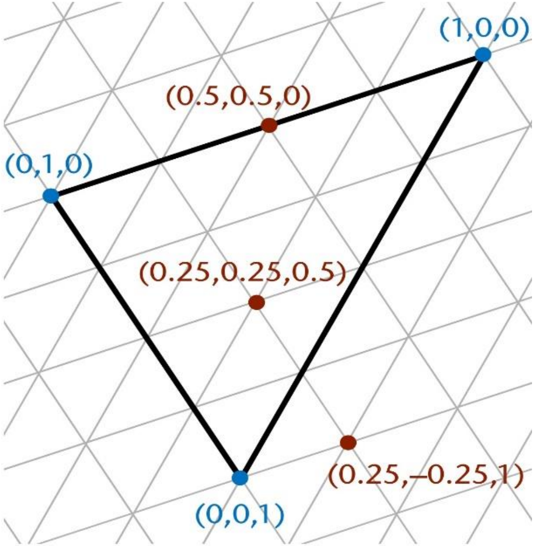


Image Warping based on RBF

- Pro: smooth
- Con: does not span polynomial (linear) functions

广义重心坐标



[Möbius 1827]

Point (a, b, c) with 3 coordinates w.r.t. base points A, B, C

Mathematically:

$$(a, b, c) = a \cdot A + b \cdot B + c \cdot C$$

where $A = (1,0,0)$, $B = (0,1,0)$, $C = (0,0,1)$ and $a + b + c = 1$

Barycenter:

$$v = \frac{a v_A + b v_B + c v_C}{a + b + c}$$

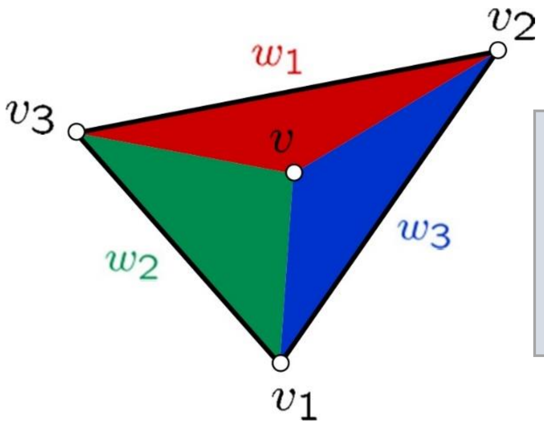
$v = (x, y)$

重心坐标

广义重心坐标

Theorem [Möbius 1827]

The barycentric coordinates $w_1,\ldots,w_{d+1}$ of $v \in \mathbb{R}^d$ w.r.t $v_1,\ldots,v_{d+1}$ are unique up to a common factor.



$$v = \frac{w_1 v_1 + w_2 v_2 + w_3 v_3}{w_1 + w_2 + w_3} \Rightarrow w_i = A(v, v_{i+1}, v_{i+2})$$

$$b_i = \frac{w_i}{A(v_1, v_2, v_3)}$$

- Properties
  - Partition of unity $\sum_i b_i(v) = 1$
  - Reproduction $\sum_i b_i(v) v_i = v$
  - Positivity $b_i(v) \geq 0, \forall v \in \Delta$

重心坐标

Ideal for interpolation

广义重心坐标 (GBC)

- For arbitrary n-polygon

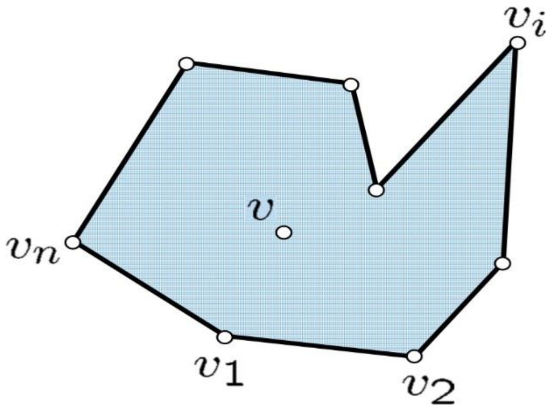
- Barycentric coordinates $w_1(v), \dots, w_n(v)$

$$v = \frac{\sum_{i=1}^{n} w_i(v) v_i}{\sum_{i=1}^{n} w_i(v)}$$

- Normalized coordinates

$$b_i(v) = \frac{w_i(v)}{\sum_j w_j(v)}$$

- Properties
  - Partition of unity $\sum_i b_i(v) = 1$
  - Reproduction $\sum_i b_i(v) v_i = v$
  - ~~Non-negative~~

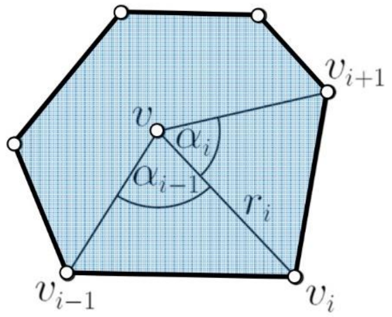


$$\sum_i b_i(v) f(v_i) = f(v), \forall \text{ linear function } f$$

Examples of GBC

Mean value coordinates (Floater '97)

$$w_i = \frac{\tan\frac{\alpha_{i-1}}{2} + \tan\frac{\alpha_i}{2}}{r_i}$$



- Barycenter $v = \frac{\sum_{i=1}^{n} w_i(v) v_i}{\sum_{i=1}^{n} w_i(v)}$
- Non-negative for star-shape polygon

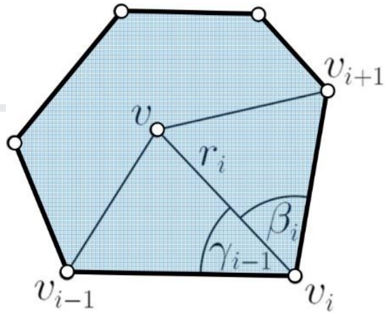
Mean value interpolation

$$f(v) = \frac{\oint w(x,v) f(x) dx}{\oint w(x,v) dx}$$

Examples of GBC

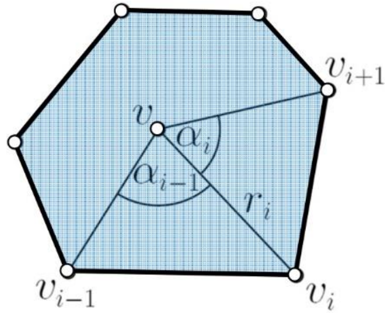
- **Mean value coordinates**

$$w_i = \frac{\tan\frac{\alpha_{i-1}}{2} + \tan\frac{\alpha_i}{2}}{r_i}$$



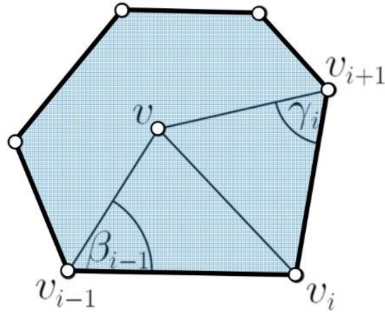
- **Wachspress coordinates**

$$w_i = \frac{\cot\gamma_{i-1} + \cot\beta_i}{r_i^2}$$



- **Discrete harmonic coordinates (cot weights)**

$$w_i = \cot\beta_{i-1} + \cot\gamma_i$$



Closed-form GBC

1. Wachspress [Wachspress 1975]
2. Discrete Harmonic [Pinkall & Polthier 1993]
3. Mean value [Floater 2003]
4. Positive mean value [Lipman et al 2007]
5. Gordon-Wixom [Belyaev 2006]
6. Positive Gordon-Wixom [Manson et al. 2011]
7. Poisson [Li & Hu 2013]
8. Power [Budninsky et al 2016]
9. Blended [Anisimov et al 2017]

GBC can be negative in general.

Computational GBC

- **Harmonic coordinates [Joshi et al 2007]**

$$\Delta b_i = 0, \qquad s.t.\ b_i(v_j) = \delta_{ij}$$

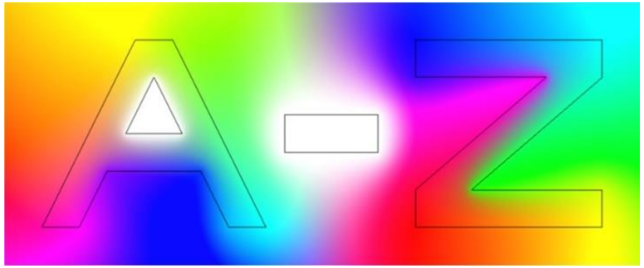
C^∞ smooth & Non-negative!

- **Maximum entropy coordinates [Hormann & Sukumar 2008]**

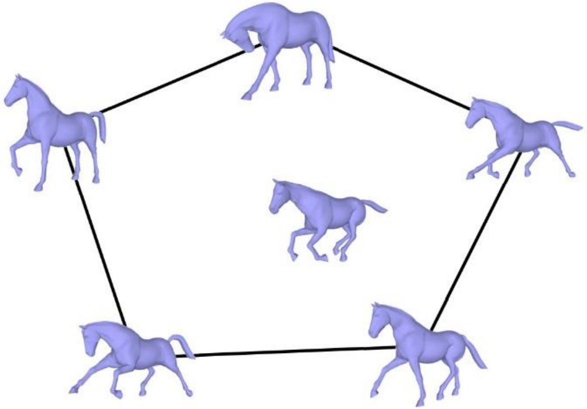
- **Moving least square coordinates [Manson & Schaefer 2010]**

- **Local barycentric coordinates [Zhang et al 2014]**

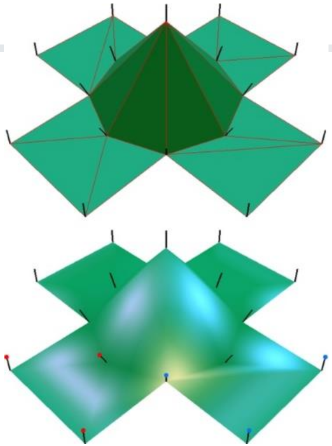
Applications of GBC



Interpolation



Mesh Animation



Shading

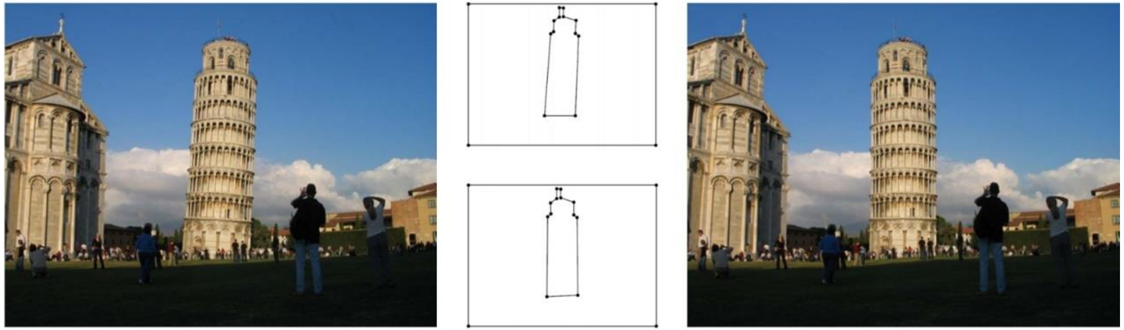
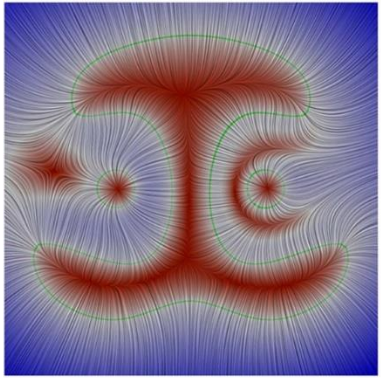
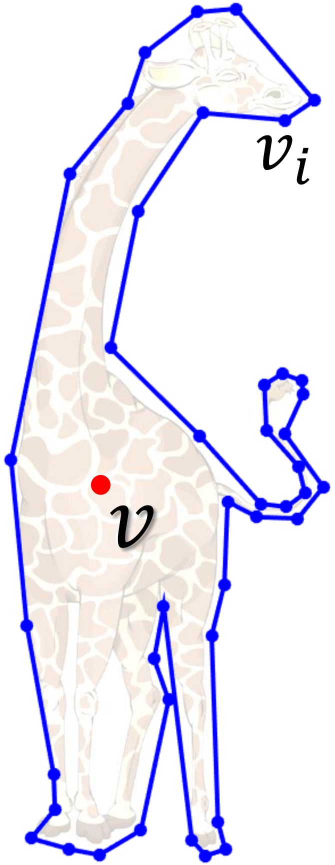


Image Editing



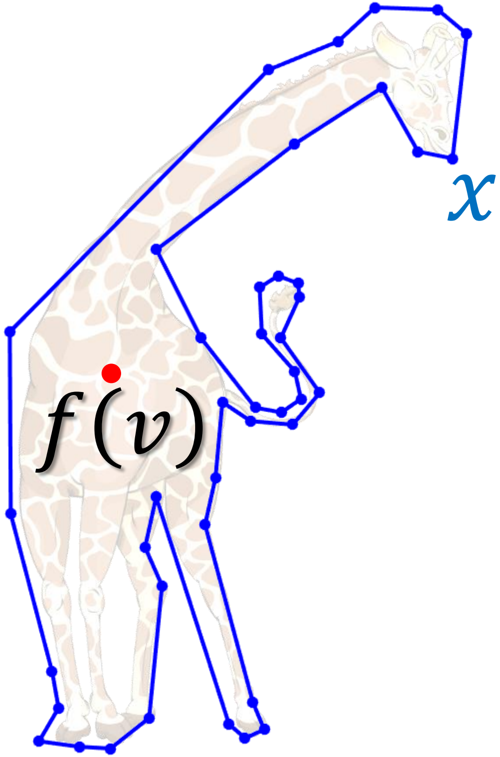
Vector Fields

GBC based Smooth Mapping

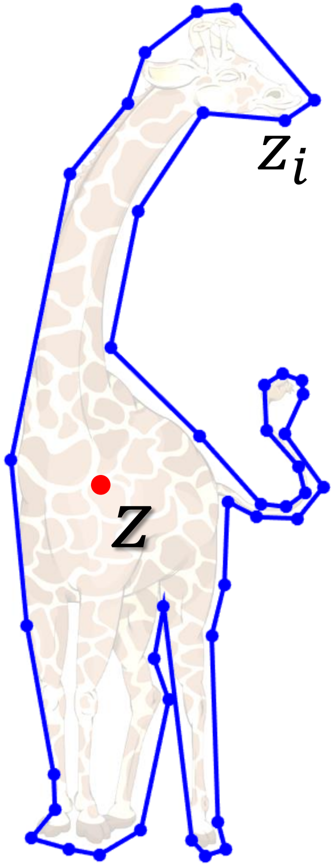


$$v = \sum_i b_i(v) v_i$$

$$f(v) = \sum_i b_i(v) x_i$$



Cauchy Complex Barycentric Coordinate



$$z = \sum_i c_i(z) z_i$$

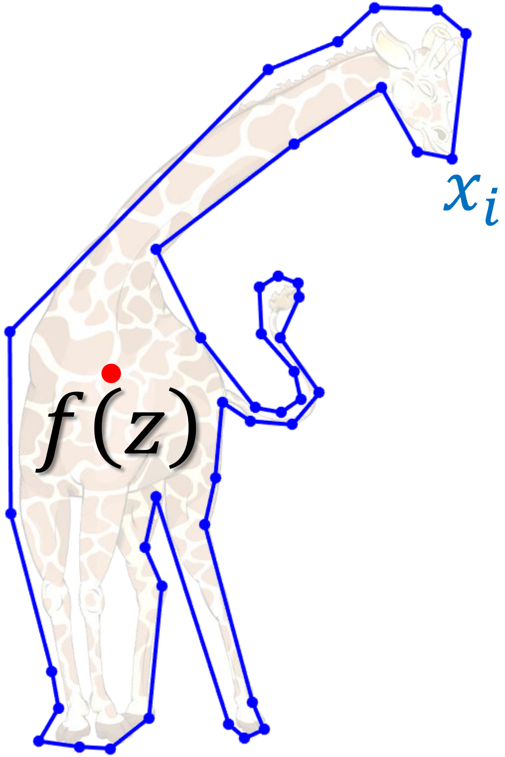
$$c_i(z) \in \mathbb{Z}$$

$$f(z) = \sum_i c_i(z) x_i$$

$\frac{\partial f}{\partial \bar{z}} = 0 \Rightarrow$ Holomorphic f

⇓

⇒ Harmonic mapping



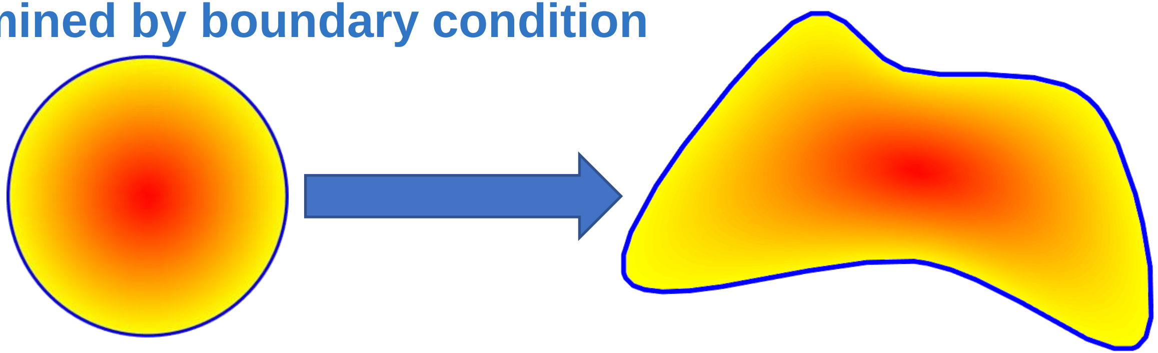
[Weber et al ‘09]

Harmonic Mapping

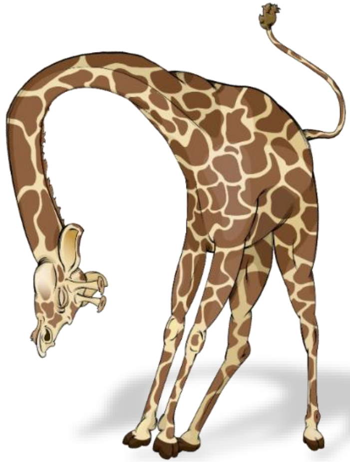
$$f(x, y) = \big(u(x, y), v(x, y)\big) \qquad f: \Omega \to \mathbb{R}^2$$

$$\Delta u = 0, \qquad \Delta v = 0 \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

1. C^∞ smooth
2. Maximum/minimum principle
3. Uniquely determined by boundary condition



Harmonic Mapping Space



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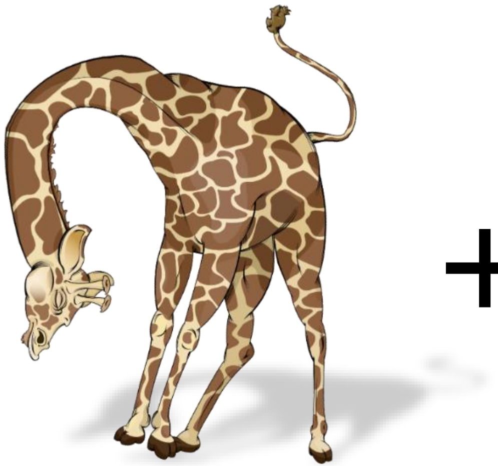

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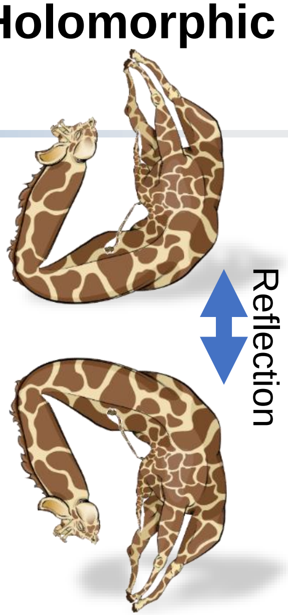


$$f = \Phi + \overline{\Psi}$$

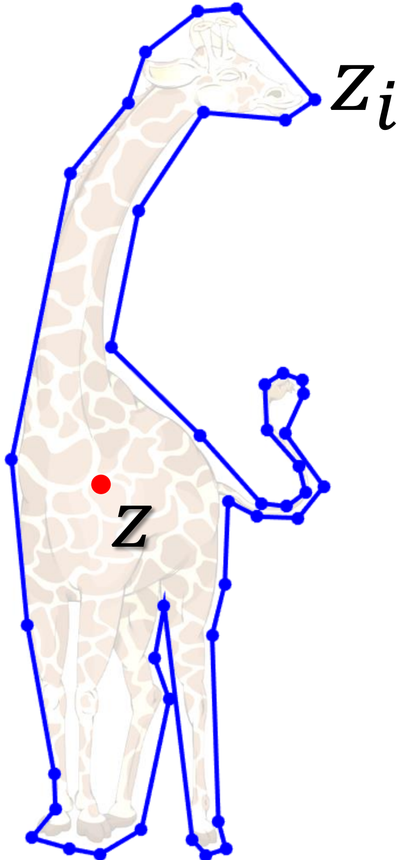
Harmonic

Holomorphic
(complex analytic)

Anti-Holomorphic

Cauchy complex barycentric coordinates

Harmonic Mapping with Cauchy Coordinate



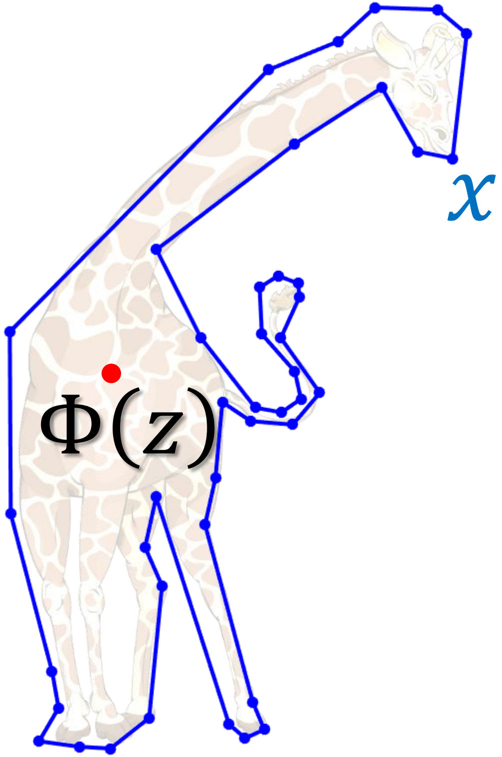
$$f = \Phi + \overline{\Psi} \leftrightarrow (x, y)$$

$$\Phi(z) = \sum_i C_i(z) x_i$$

$$\Psi(z) = \sum_i C_i(z) y_i$$

$$f_z = \Phi'(z)$$

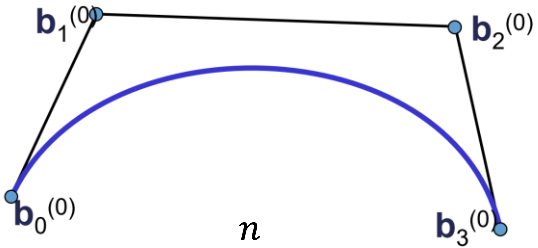
$$f_{\bar{z}} = \overline{\Psi'(z)}$$



样条基 - Bézier曲线

Bézier curves for curve design:

- Rough form specified by the control polygon
- Smooth curve approximating the control points



$$x(t) = \sum_{i=0}^{n} B_i^n(t) b_i$$

$$B_i^n(t) = C_i^n t^i (1-t)^{n-i}$$

- Problems:
  I. High polynomial degree
  II. Non-local support
  III. Interpolation of points

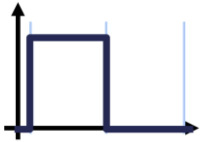
Properties

- Smoothness
- Pseudo-local support
- Convex hull
  - Partition of unity
  - Non-negative

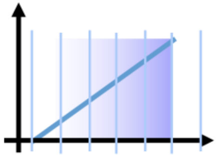
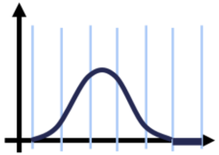
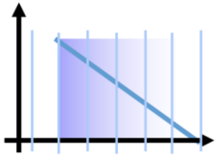
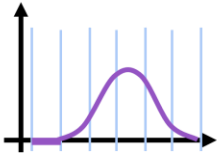
B样条基

The **uniform** B-spline basis of order k (degree $k-1$) is given as

$$N_i^1(t) = \begin{cases} 1, \text{if } i \le t < i+1 \\ 0, \quad \text{otherwise} \end{cases}$$

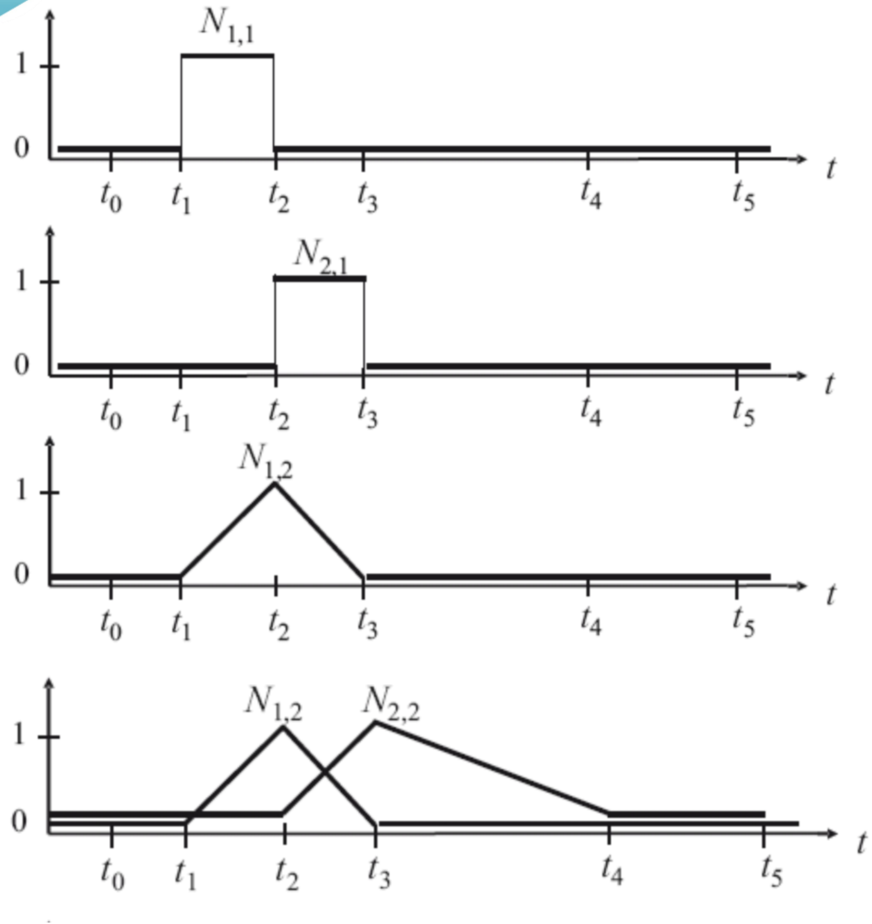


$$N_i^k(t) = \frac{t-i}{(i+k-1)-i} N_i^{k-1}(t) \qquad + \qquad \frac{(i+k)-t}{(i+k)-(i+1)} N_{i+1}^{k-1}(t)$$

$$= \frac{t-i}{k-1} N_i^{k-1}(t) \qquad + \qquad \frac{i+k-t}{k-1} N_{i+1}^{k-1}(t)$$

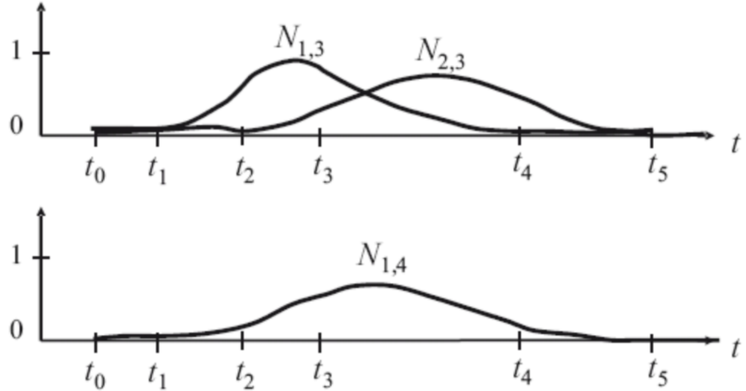
B样条基 - general case



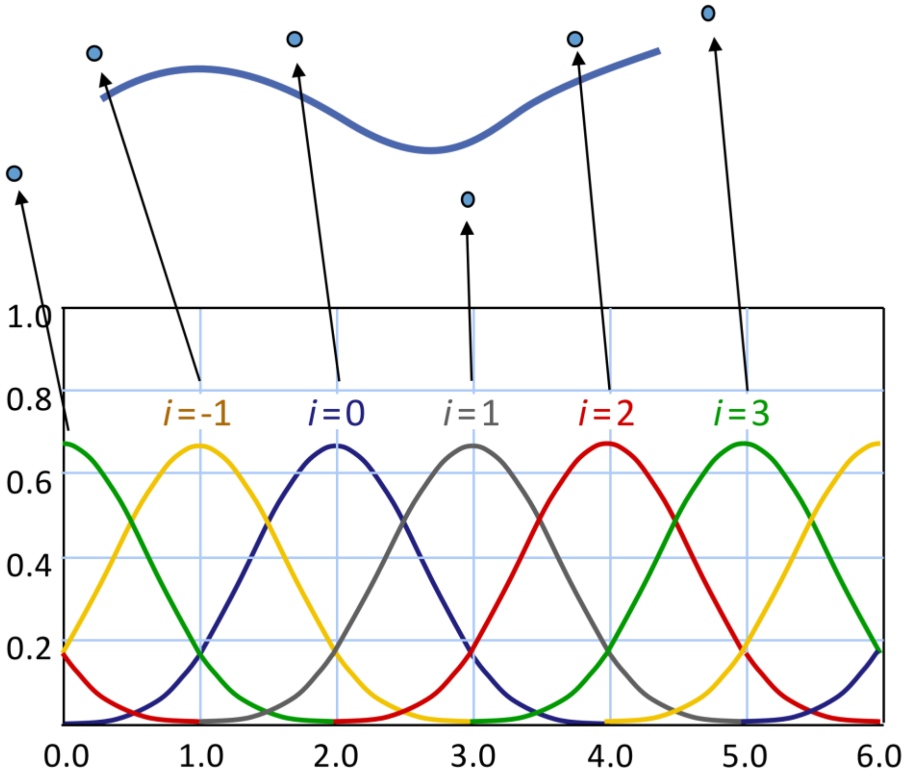
$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

for $k > 1$ and $i = 0, \ldots, n$

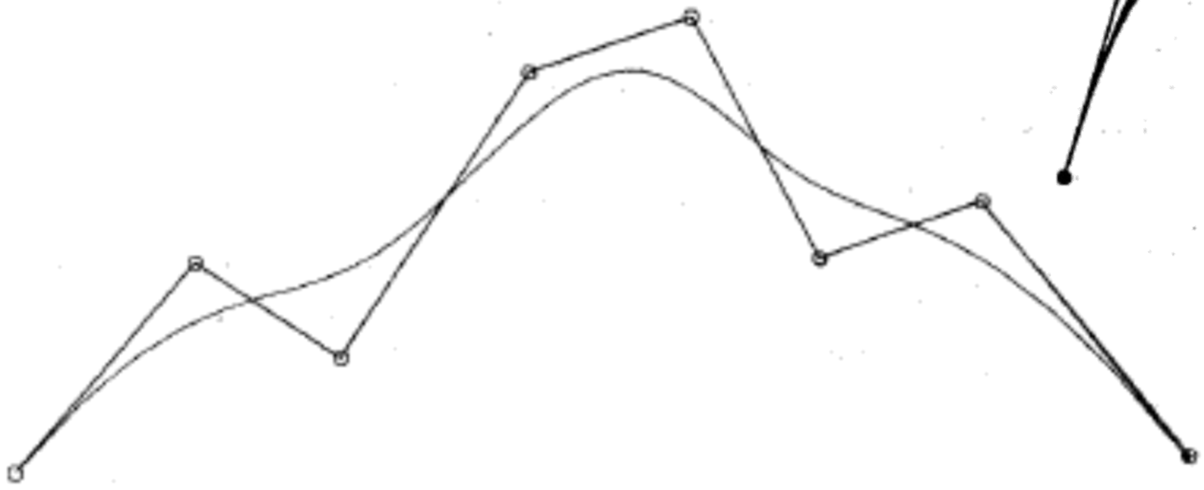


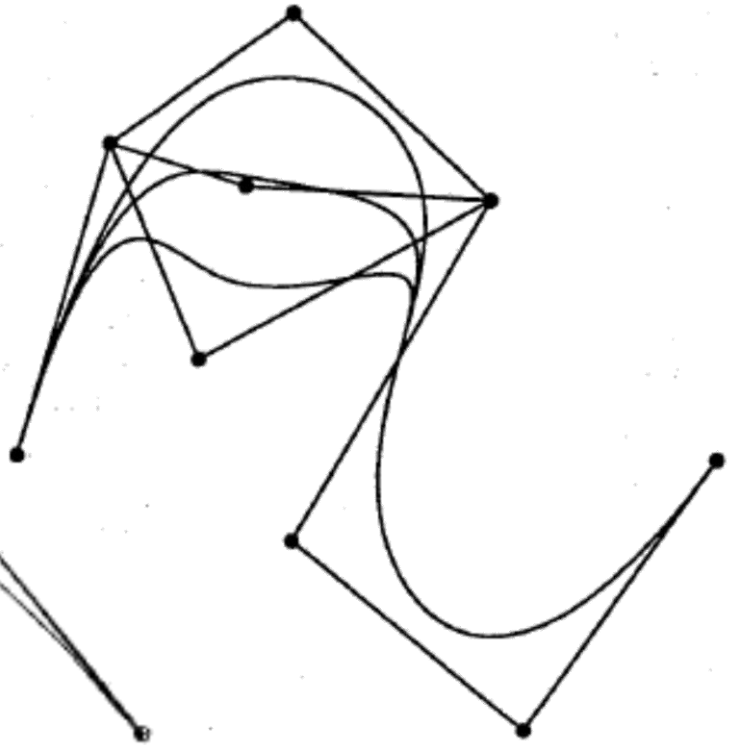
B-Spline curves

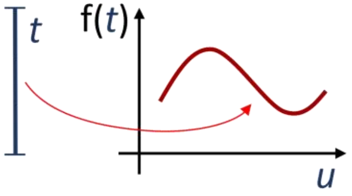
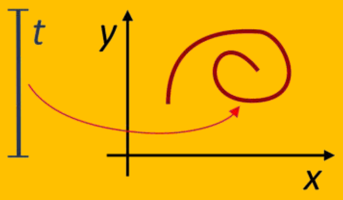
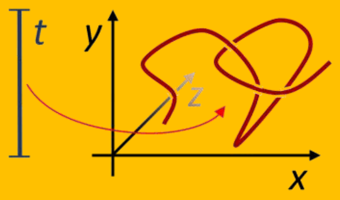
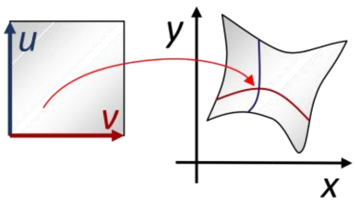
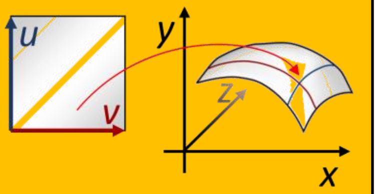
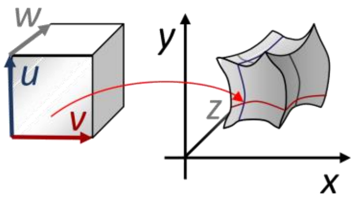


Shifted basis function $b(t)$

B-Spline curves



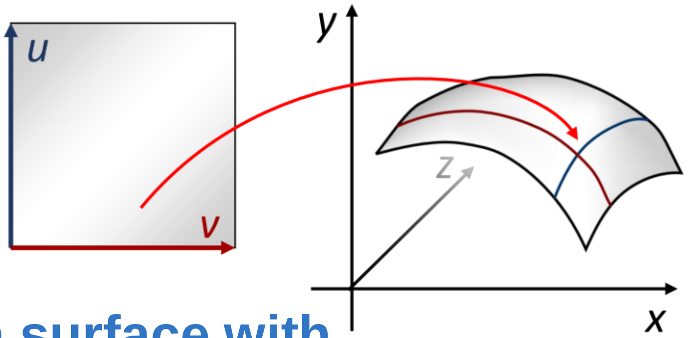


	Output: 1D	Output: 2D	Output: 3D
Input: 1D	 Function graph	 Plane curve	 Space curve
Input: 2D		 Plane warp	 Surface
Input: 3D			 Space warp

Spline Surfaces

Parametric spline surfaces:

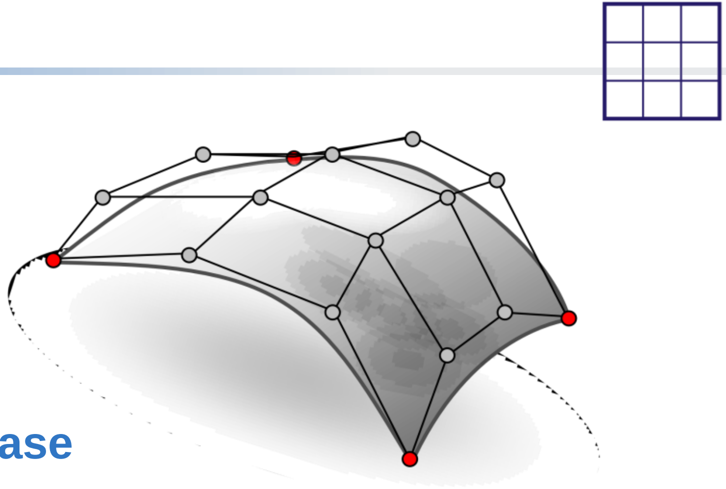
- Two parameter coordinates (u, v)
- Piecewise bivariate polynomials
- Assemble multiple pieces to form a surface with continuity
- Each piece is called *spline patch*



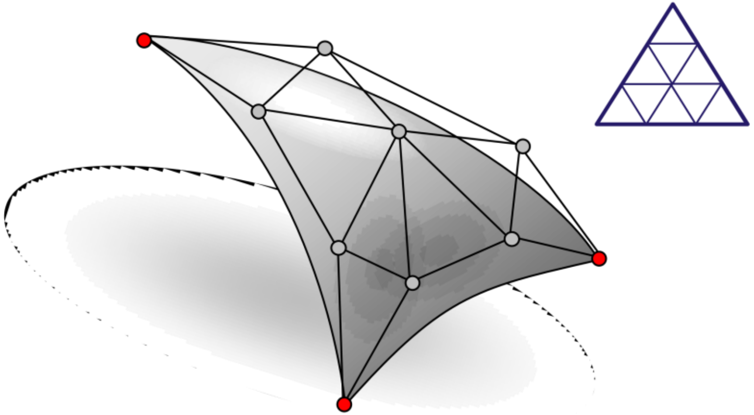
Spline Surfaces

Two different approaches

- Tensor product surfaces
  I. Simple construction
  II. Everything carries over from curve case
  III. Quad patches



- Total degree surfaces
  I. Not as straightforward
  II. Isotropic degree
  III. Triangle patches

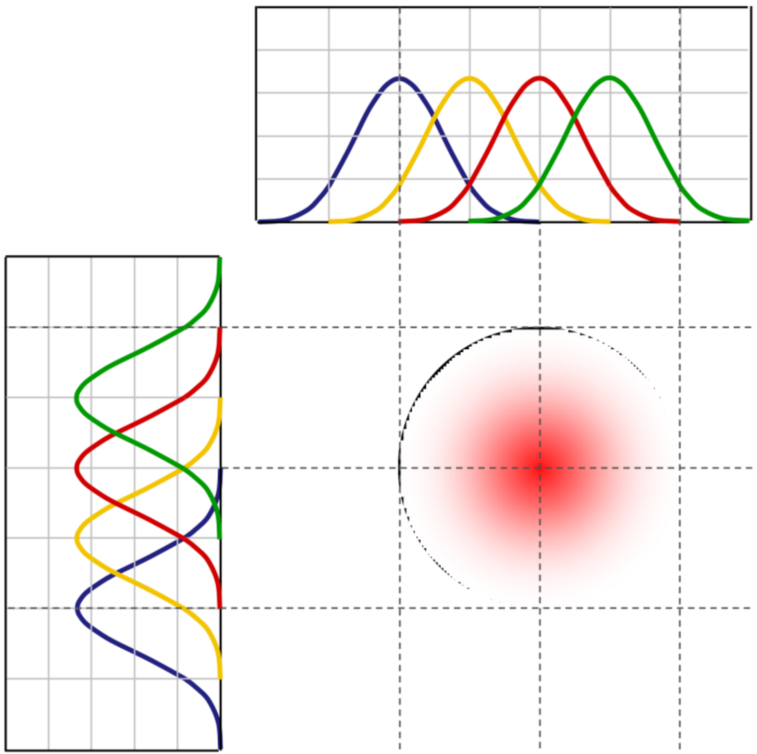


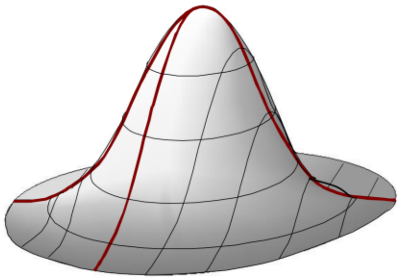
Tensor Product Surfaces

Tensor product basis

	$b_1(u)$	$b_2(u)$	$b_3(u)$	$b_4(u)$
$b_1(v)$	$b_1(v)b_1(u)$	$b_1(v)b_2(u)$	$b_1(v)b_3(u)$	$b_1(v)b_4(u)$
$b_2(v)$	$b_2(v)b_1(u)$	$b_2(v)b_2(u)$	$b_2(v)b_3(u)$	$b_2(v)b_4(u)$
$b_3(v)$	$b_3(v)b_1(u)$	$b_3(v)b_2(u)$	$b_3(v)b_3(u)$	$b_3(v)b_4(u)$
$b_4(v)$	$b_4(v)b_1(u)$	$b_4(v)b_2(u)$	$b_4(v)b_3(u)$	$b_4(v)b_4(u)$

Tensor Product Surfaces





Tensor Product Surfaces

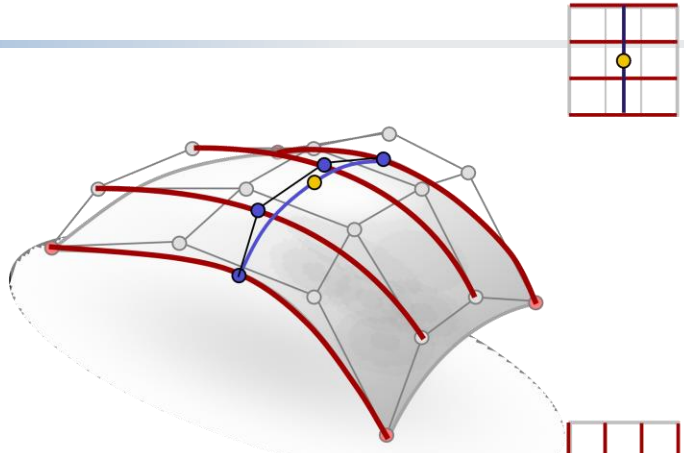
Tensor Product Surfaces

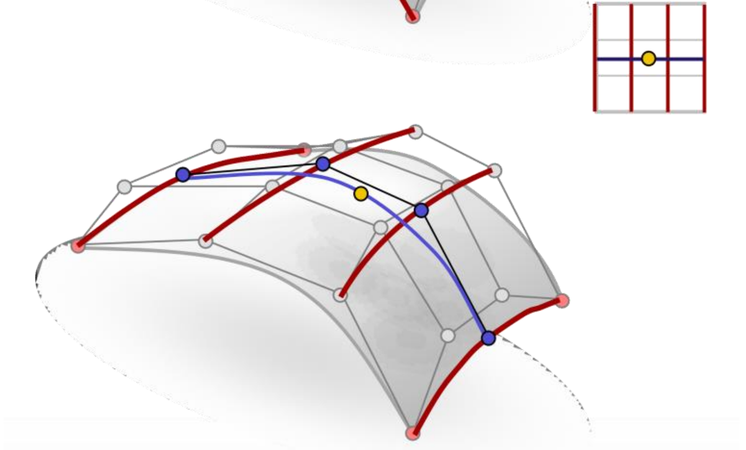
$$\boldsymbol{f}(u,v) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_i(u) b_j(v) \boldsymbol{p}_{i,j}$$

$$= \sum_{i=1}^{n} b_i(u) \sum_{j=1}^{n} b_j(v) \boldsymbol{p}_{i,j}$$

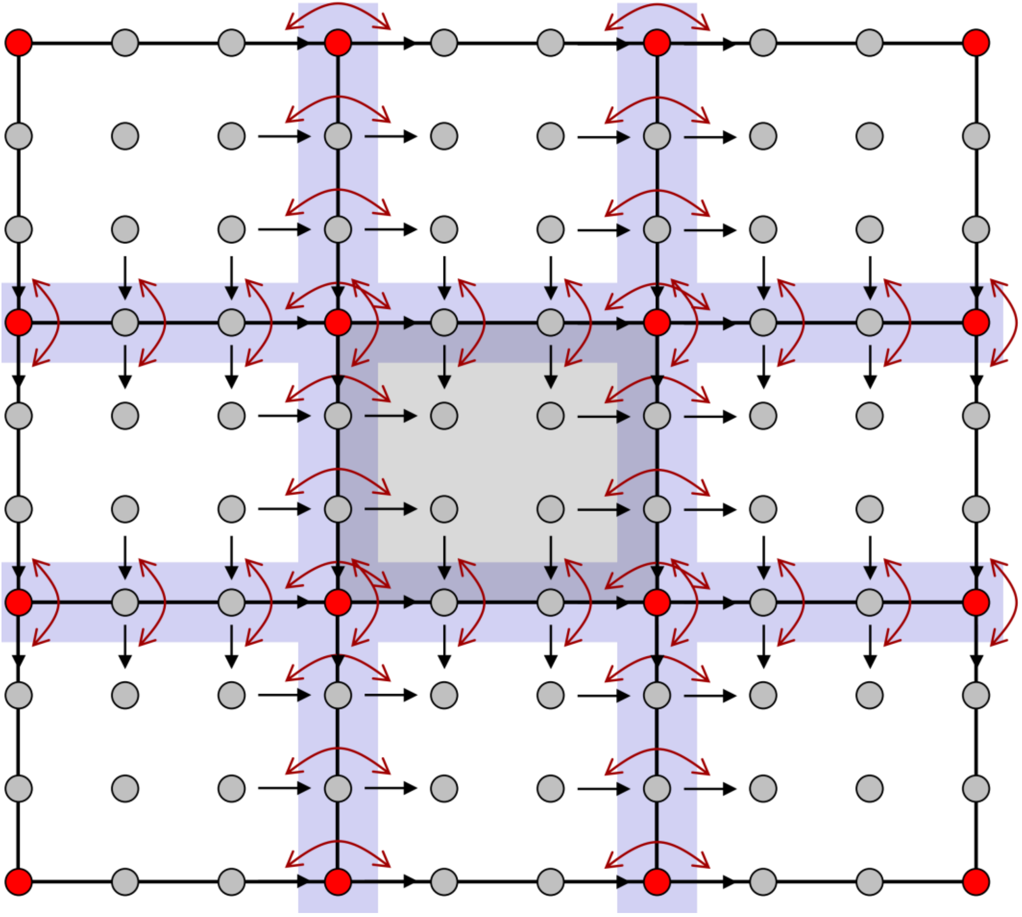
$$= \sum_{j=1}^{n} b_j(v) \sum_{i=1}^{n} b_i(u) \boldsymbol{p}_{i,j}$$

- “Curves of Curves”





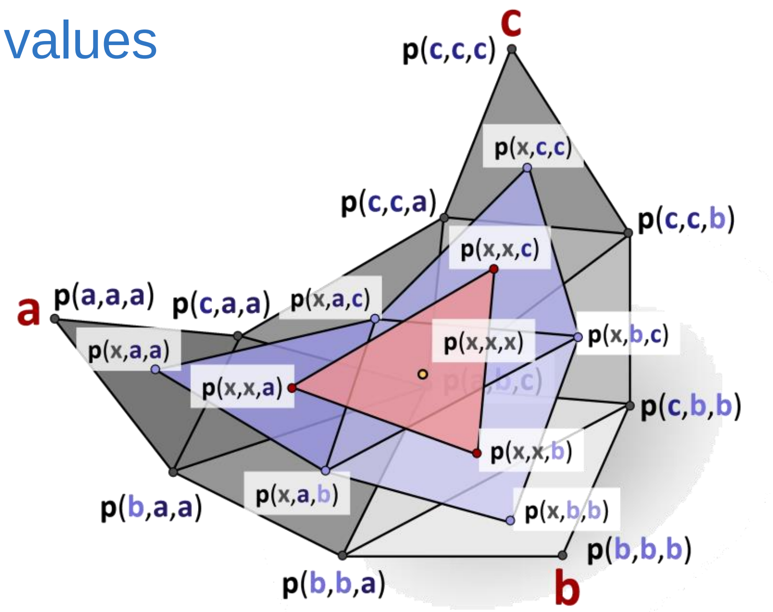
Tensor Product Surfaces C^1 Continuity



Bézier Triangles

Derived using a triangular de Casteljau algorithm

- Blossoming formalism for defining Bézier Triangles
- Barycentric interpolation of blossom values



$$\boldsymbol{x} = \alpha\boldsymbol{a} + \beta\boldsymbol{b} + \gamma\boldsymbol{c},$$
$$\alpha + \beta + \gamma = 1$$

无翻转光滑映射

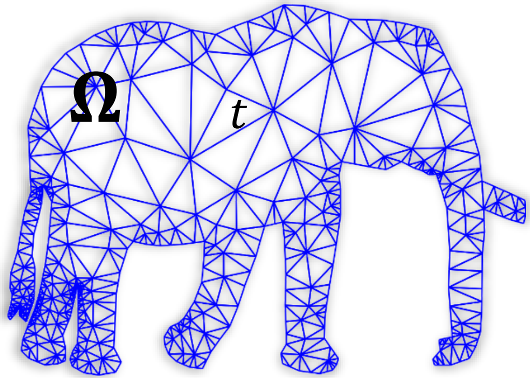
无翻转映射优化

e.g. $D(p) = |J(p)|_F^2 + |J(p)|_F^{-2}$

分片线性映射

Minimize $E = \int_\Omega D(s)\,ds = \sum_{t\in T} A_t D(J_t)$

$s.t.\ |J_t| > 0, \forall t \in T$

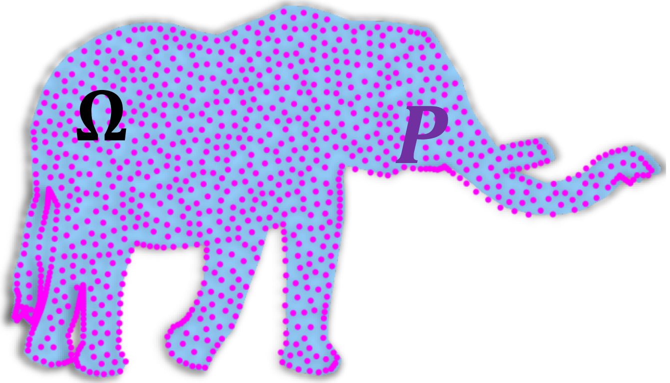


光滑映射

Minimize $E = \int_\Omega D(s)\,ds \approx \sum_{p\in P} D(p)$

$s.t.\ |J(s)| > 0, \forall s \in P$

Infeasible!



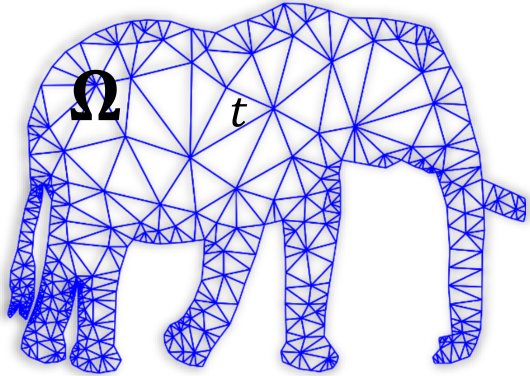
无翻转映射优化

e.g. $D(p) = |J(p)|_F^2 + |J(p)|_F^{-2}$

分片线性映射

Minimize $\sum_{t \in T} A_t D(J_t)$

$s.t.\ |J_t| > 0, \forall t \in T$

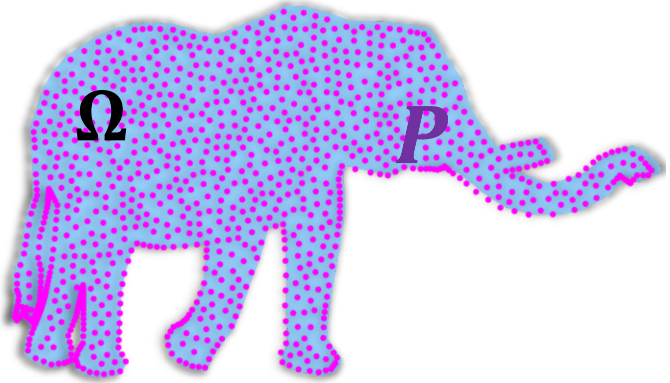


光滑映射

Minimize $\sum_{p \in P} D(p)$

$s.t.\ |J(s)| > 0, \forall s \in \boldsymbol{P}$

$|J(s)| > 0, \forall s \notin P$

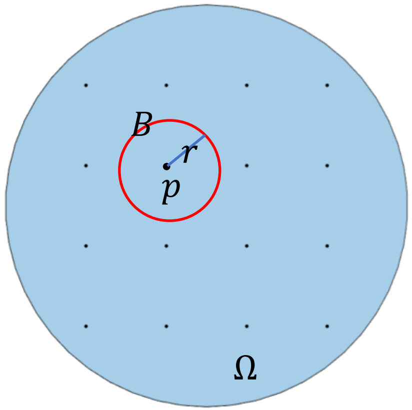


无翻转光滑映射

基于Lipschitz连续性的无翻转光滑映射

$$J(p) > Lr \xRightarrow{\text{Lipschitz连续}} J(q) \geq J(p) - Lr > 0, \forall q \in B$$

$$L = \sup_{q \in B}(|\nabla J(q)|_F)$$

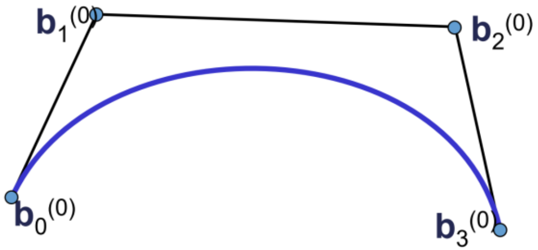


全局无翻转条件: $J(p) > Lr, \forall p$

无翻转光滑映射

基于样条基的凸包性质的无翻转映射

$$f(u,v) = \left(f^1(u,v), f^2(u,v)\right) = \sum_i B_i(u,v)\, P_i$$



$$f(t) = \sum_i B_i(t)\, P_i$$

$$J = \begin{pmatrix} f_x^1 & f_x^2 \\ f_y^1 & f_y^2 \end{pmatrix}$$

$$B_i^n(t) = C_i^n t^i (1-t)^{n-i}$$

$$|J(u,v)| = f_x^1 f_y^2 - f_x^2 f_y^1 = \cdots = \sum_i |J_i| B_i(u',v')$$

Convex hull Property
- Partition of unity
- Non-negative

$$\forall i, |J_i| \geq 0 \Rightarrow |J(u,v)| \geq 0, \forall u,v \in [0,1]$$

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谢 谢 ！

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