中国科学技术大学
University of Science and Technology of China

# GAMES 301：第10讲

# 共形参数化1

## Spin变换、Circle填充 & 共轭调和函数

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# Content

1. Introduction to conformal mapping
2. Differential of conformal mapping
3. Spin transformations
4. Circle packing and circle patterns
5. Conjugate harmonic functions

# 1

# Introduction to conformal mapping

# Angle-based flattening (ABF)

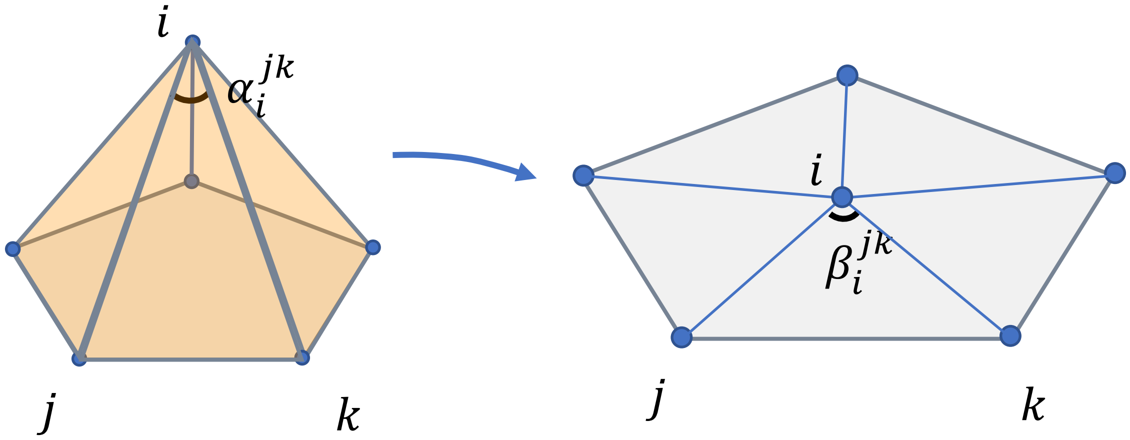
- Key observation: the parameterized triangles are uniquely defined by all the angles at the corners of the triangles.
  - Find angles instead of uv coordinates.
  - Use angles to reconstruct uv coordinates.

- Angel preservation:
  - Interior vertex:

$$\beta_i^{jk} = \frac{\alpha_i^{jk} \cdot 2\pi}{\sum_i \alpha_i^{jk}}.$$

  - Boundary vertex:

$$\beta_i^{jk} = \alpha_i^{jk}.$$




Sheffer, Alla, and Eric de Sturler. "Parameterization of faceted surfaces for meshing using angle-based flattening." Engineering with computers 17.3 (2001): 326-337.

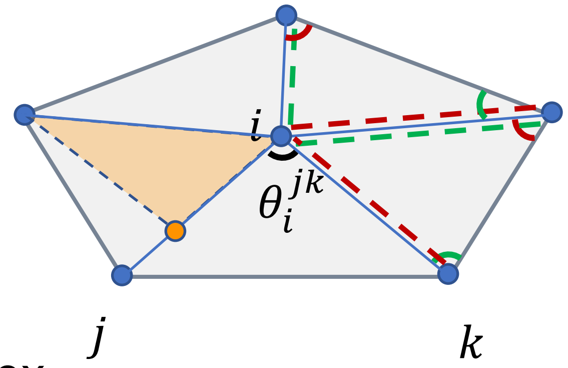
# Angle-based flattening (ABF)

- Least square optimization:

$$\min_{\theta>0} \sum_{ijk} \left(\beta_i^{jk} - \theta_i^{jk}\right)^2 + \left(\beta_j^{ki} - \theta_j^{ki}\right)^2 + \left(\beta_k^{ij} - \theta_k^{kj}\right)^2$$

*s.t.*

$$\begin{cases} \sum_{t_{ijk} \in St(i)} \theta_i^{jk} = 2\pi, \ \forall\, i \text{ interior vertex} \\ \theta_i^{jk} + \theta_j^{ki} + \theta_k^{kj} = \pi, \quad \forall\, t_{ijk} \\ \prod_{t_{ijk} \in St(i)} \frac{\sin\theta_j^{ki}}{\sin\theta_k^{ij}} = 1, \forall\, i \text{ interior vertex} \end{cases}$$

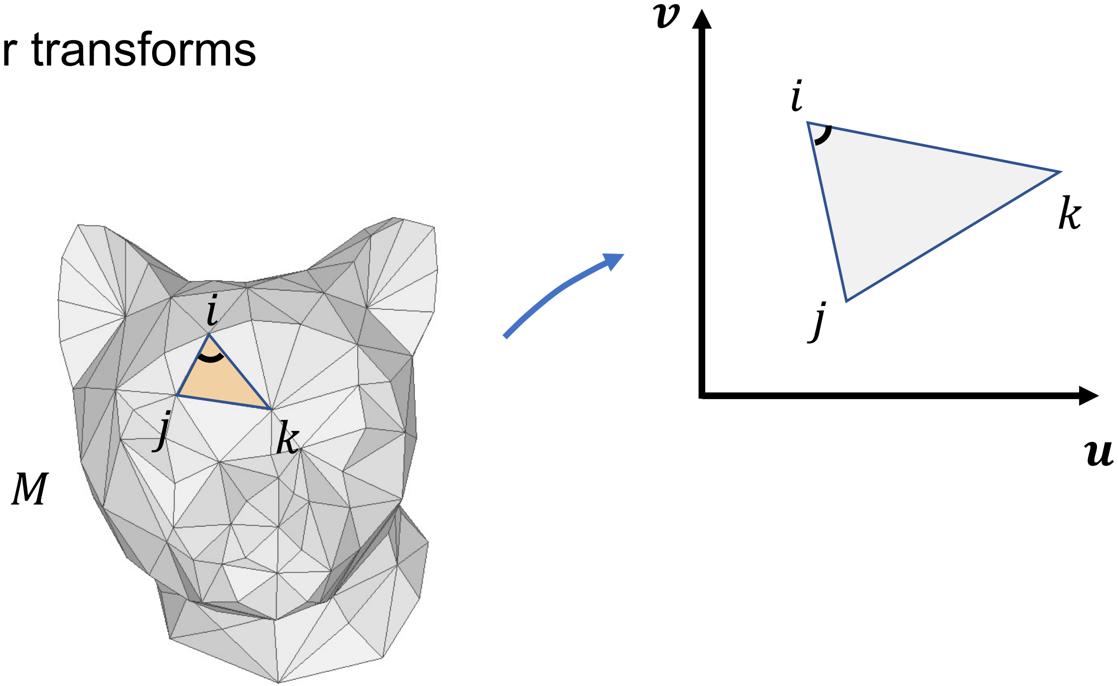



$$\frac{\sin\theta_j^{ki}}{\sin\theta_k^{ij}} = \frac{l_{ki}}{l_{ij}}$$

Sheffer, Alla, and Eric de Sturler. "Parameterization of faceted surfaces for meshing using angle-based flattening." Engineering with computers 17.3 (2001): 326-337.

# Least-square conformal mapping (LSCM)

- Angle preservation → similar transforms




Lévy, Bruno, et al. "Least squares conformal maps for automatic texture atlas generation." ACM transactions on graphics (TOG) 21.3 (2002): 362-371.

# Least-square conformal mapping (LSCM)

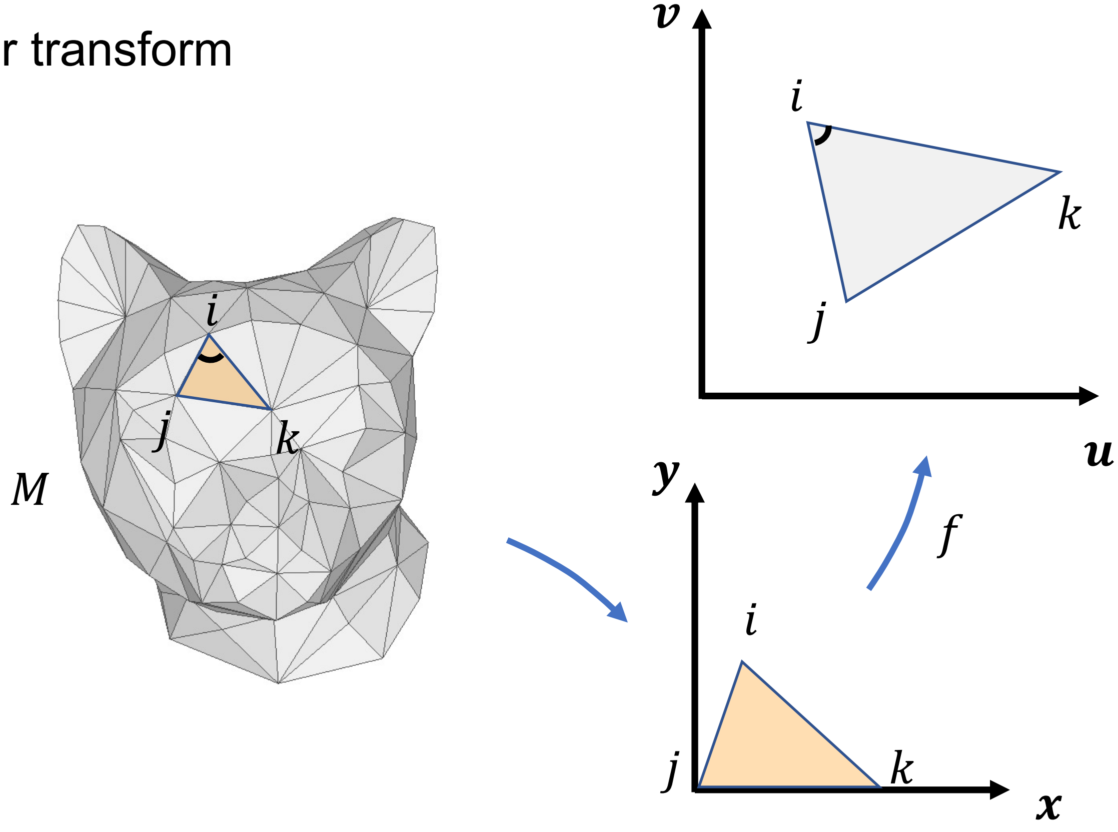
- Angle preservation → similar transform
- For triangle $t_{ijk}$:

$$\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$\downarrow$$

$$s \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\Longrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



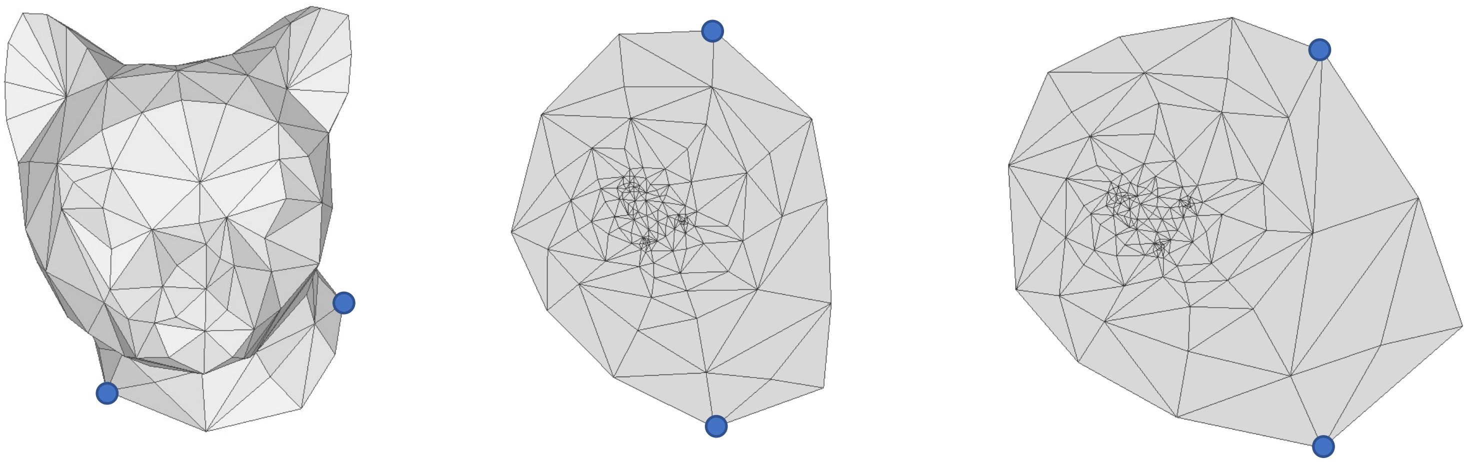

Lévy, Bruno, et al. "Least squares conformal maps for automatic texture atlas generation." ACM transactions on graphics (TOG) 21.3 (2002): 362-371.

# Least-square conformal mapping (LSCM)

- Least square optimization:

$$E_{LSCM} = \sum_{ijk} A_{ijk} \left( \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right)$$

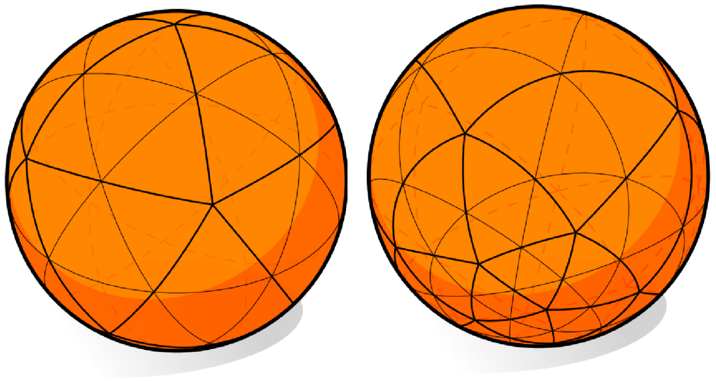
- Not unique minimizer → fixing at least two vertices.

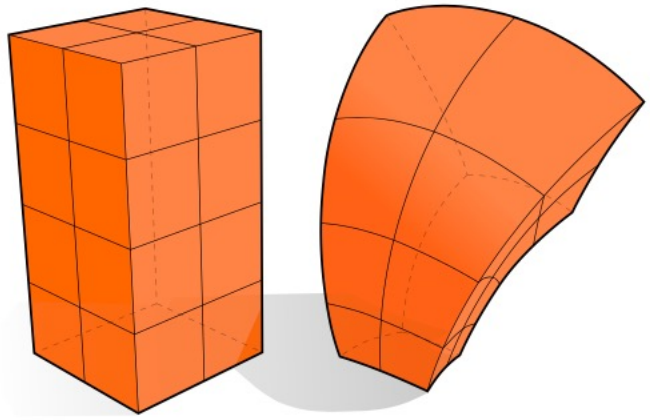


Lévy, Bruno, et al. "Least squares conformal maps for automatic texture atlas generation." ACM transactions on graphics (TOG) 21.3 (2002): 362-371.

# Conformal mapping

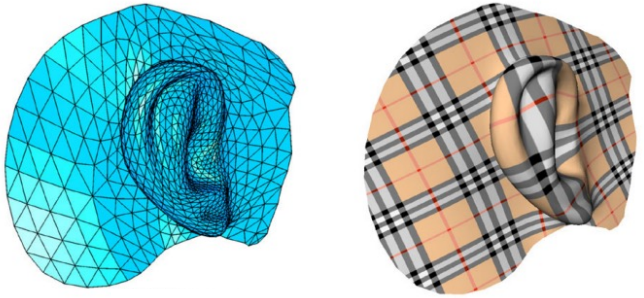
- Angle preservation

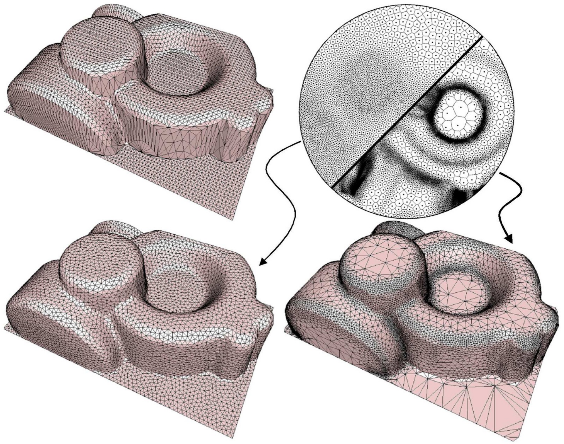


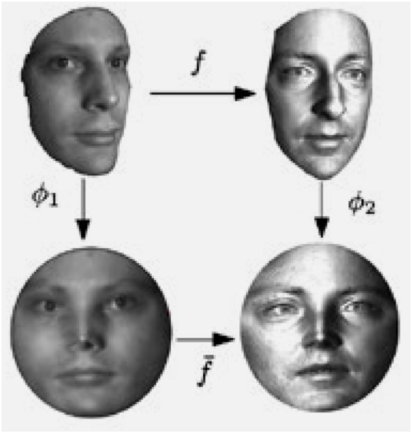


# Applications

- Texturing
- Morphing
- Remeshing
- Shape analysis
- …

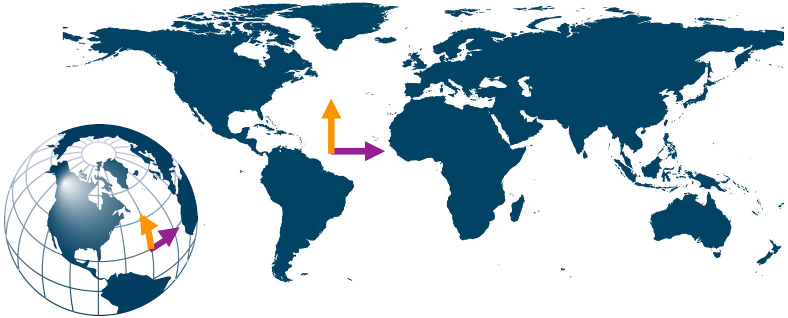


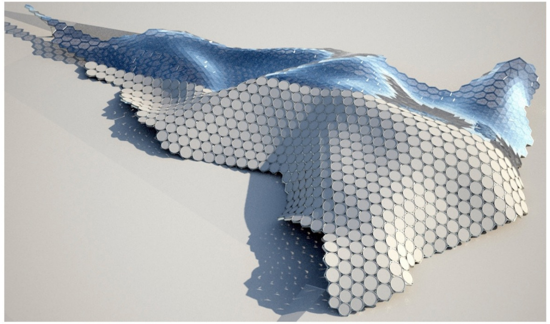


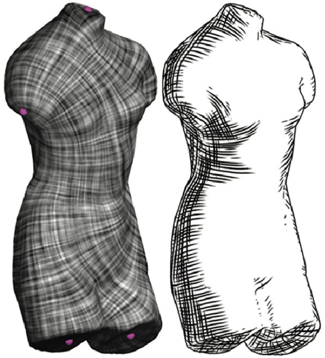


# Applications

- Cartography
- Architecture
- Art design
- Fabrication
- …

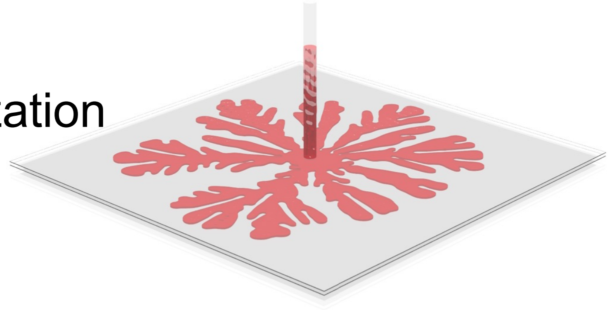


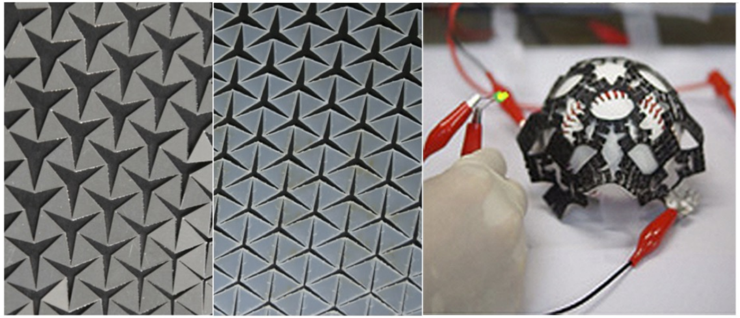


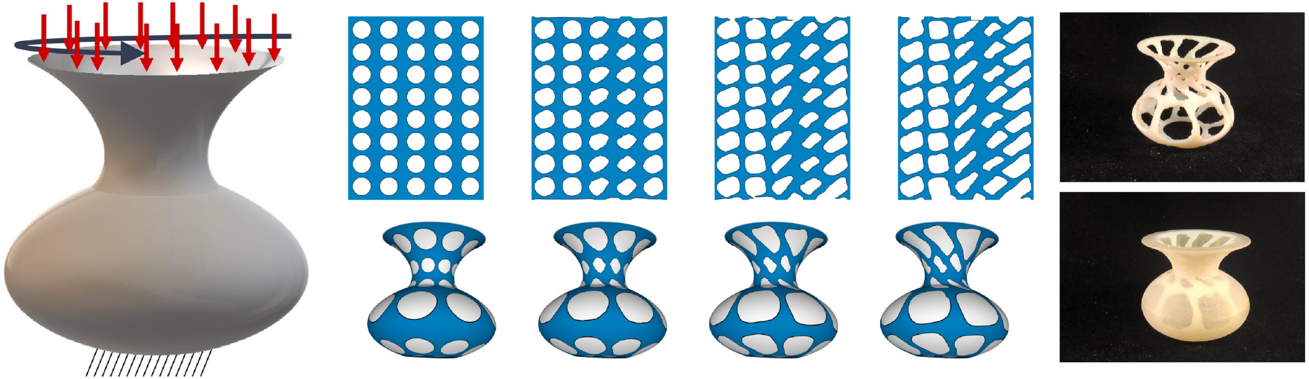


# Applications

- Fluids
- Microstructures
- Topology optimization
- …







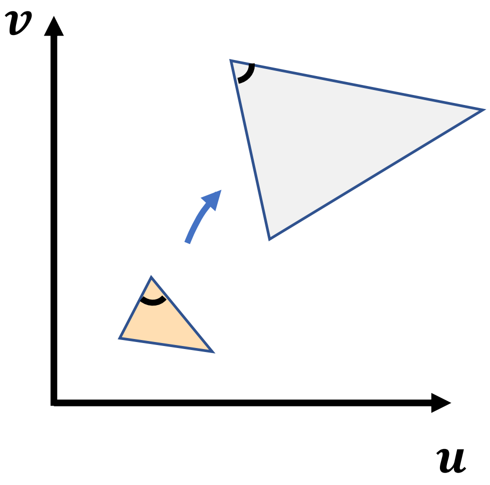
# 2

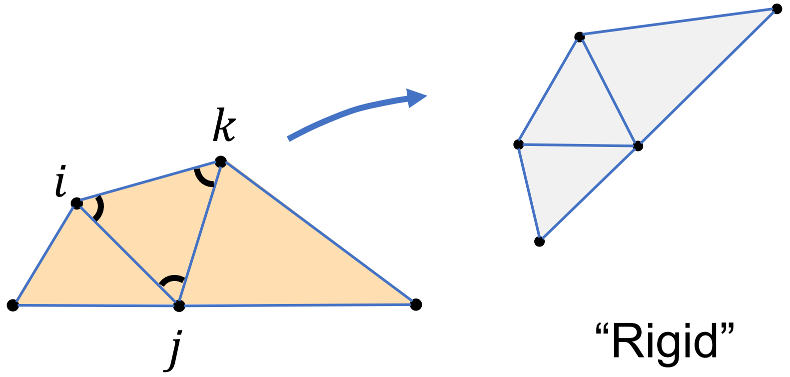
# Differential of conformal mapping

# Conformal mapping

- Discrete conformal mapping.
  - Angle of triangles.
  - Piecewise linear function.

$$\frac{l_{jk}}{\sin\theta_i^{jk}} = \frac{l_{ki}}{\sin\theta_j^{ki}} = \frac{l_{ij}}{\sin\theta_k^{ij}} = 2R$$

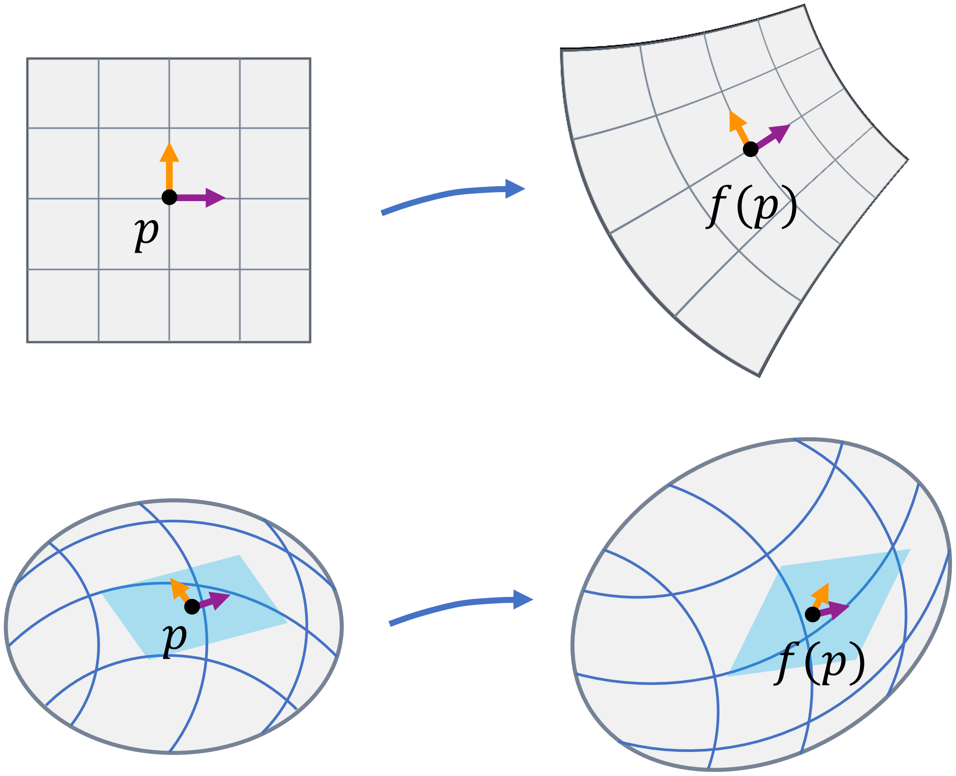


# Conformal mapping

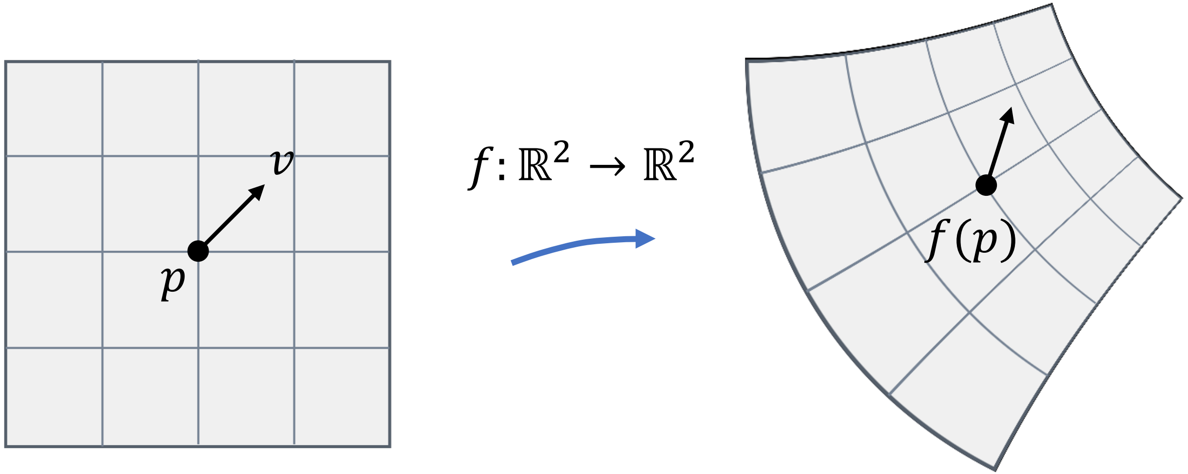
- Discrete conformal mapping.
  - Angle of triangles.
  - Piecewise linear function.

- Continuous conformal mapping.
  - Angle between vectors.
  - Smooth function.



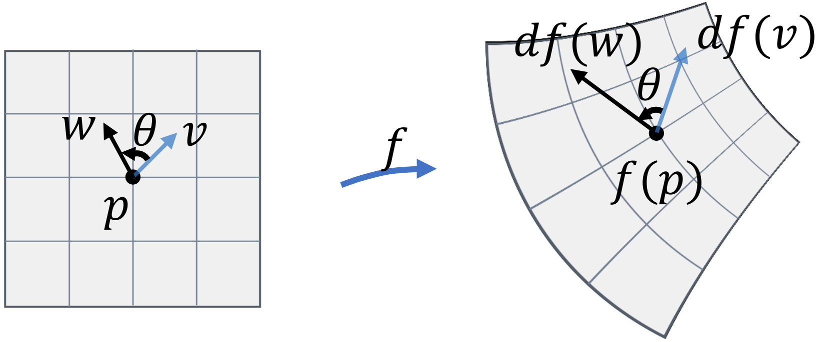
# Differential

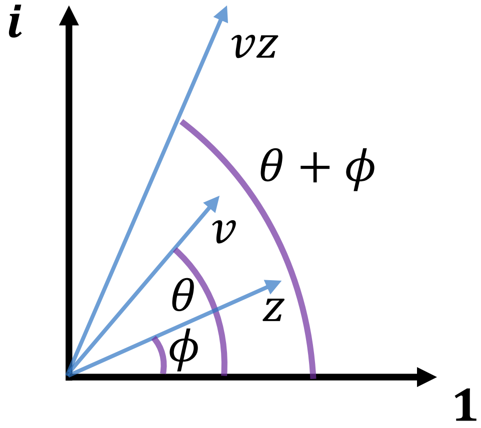
- Push forward of vector: $df\,(v) = \lim_{t \to 0} \frac{f(p+tv)-f(p)}{t}$
- Linear operator: $df(v) = J_f v,\ J_f$ Jacobian matrix.



# Differential

- Angle preservation:
  - $\theta[v,w] \Leftrightarrow \theta[df(v), df(w)]$
- Complex number
  - Rotate & scale : $z = |z|e^{i\phi}$
  - $v \to w \;\Leftrightarrow\; w = vz = |v||z|e^{i(\theta+\phi)}$



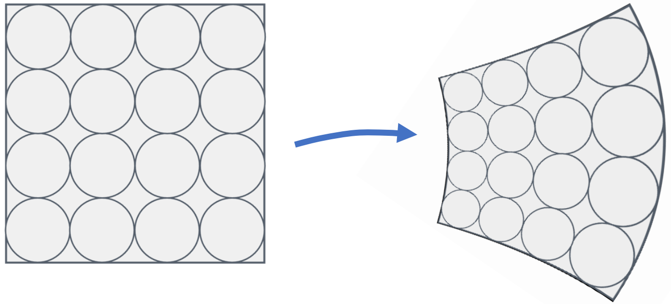

$$Arg(v^{-1}w) = Arg(df^{-1}(v)df(w))$$

$$v^{-1}w = df^{-1}(v)df(w)$$

Complex linear: $df(zv) = zdf(v)$

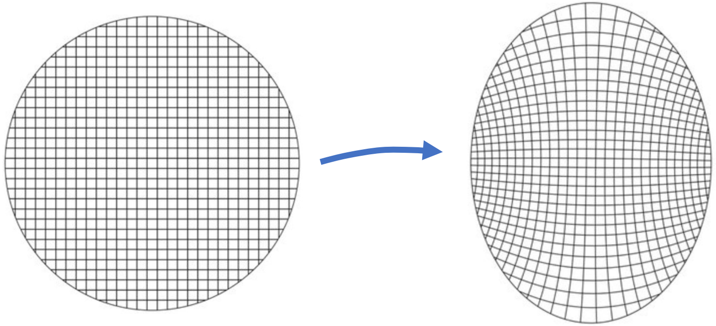
# Differential

- Let $f = f_x + if_y$.
  - $df(i) = \lim_{t\to 0} \frac{f(x+yi+ti)-f(x+yi)}{t} = \frac{\partial f_x}{\partial y} + \frac{\partial f_y}{\partial y} i$
  - $df(1) = \lim_{t\to 0} \frac{f(x+yi+t)-f(x+yi)}{t} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial x} i$



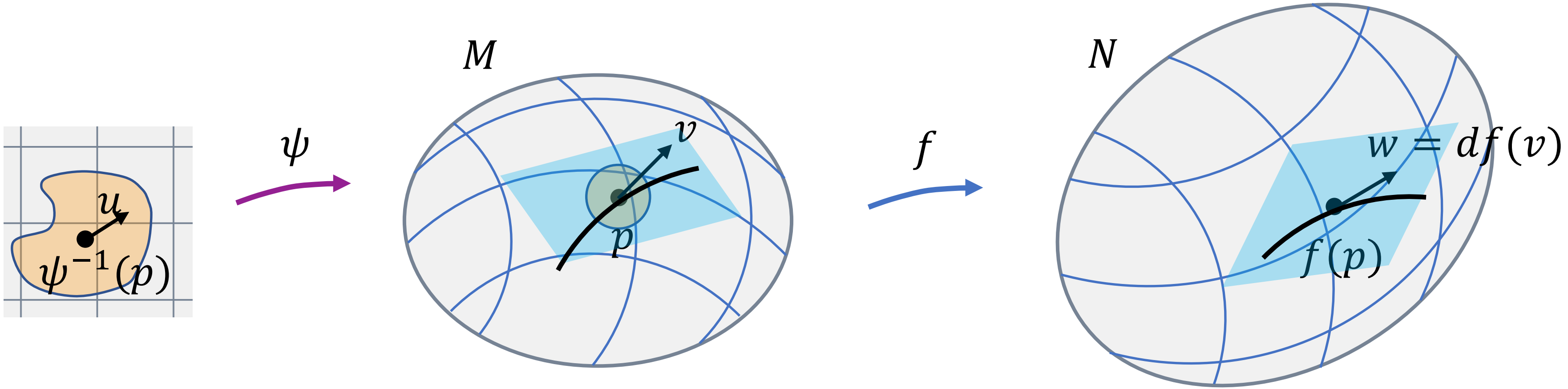
- Cauchy-Riemann equation: $df(i) = i df(1)$
  - $\begin{cases} \frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} \\ \frac{\partial f_x}{\partial y} = -\frac{\partial f_y}{\partial x} \end{cases} \Leftrightarrow J_f = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$
  - $\forall\, v \in \mathbb{C}, df(v) = df(1)v$

Similar transform!



# Differential

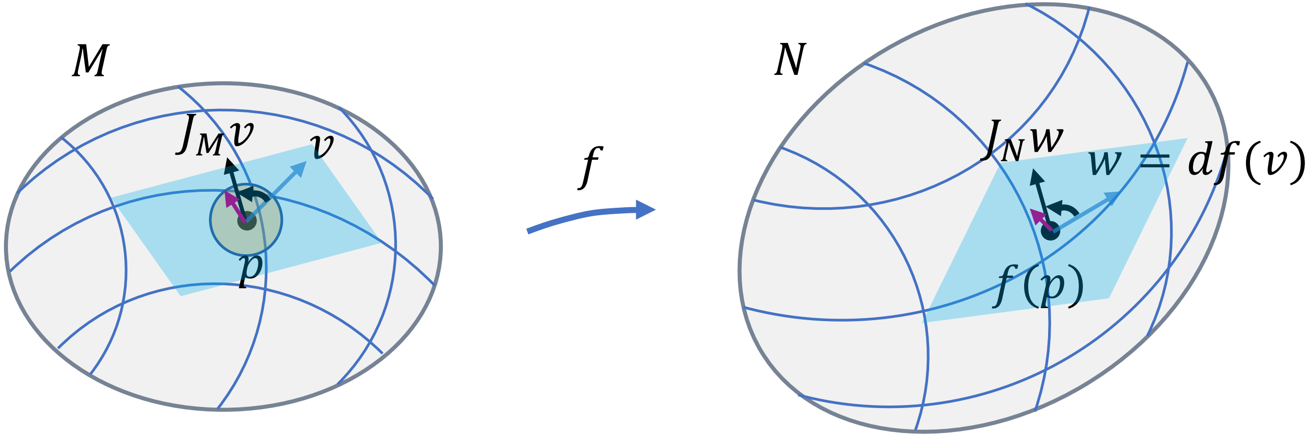
- Plane to plane : $df\,(v) = \lim_{t \to 0} \frac{f(p+tv) - f(p)}{t}$
- Manifold to manifold
  - Curve : $\Gamma(0) = \mathrm{p}, \Gamma'(0) = v$
  - $df(v) = \lim_{t \to 0} \frac{f(\Gamma(t)) - f(\Gamma(0))}{t}$ ⇒ $df(v) : v \in T_pM \;\to w \in T_{f(p)}N$




Independent of $\Gamma$ !

# Differential

- Cauchy-Riemann equation
  - Plane : $df(i) = idf(1)$
  - Manifold : $df\,(J_M v) = J_N df(v), \forall v \in T_p M$

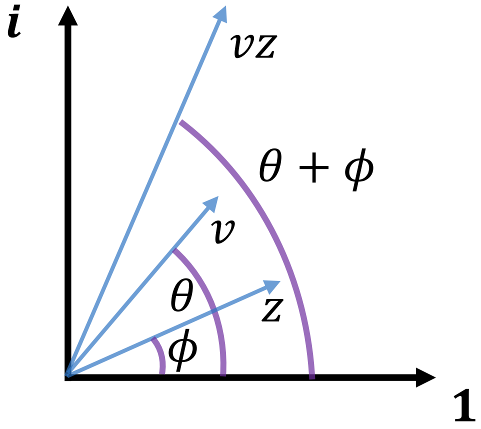


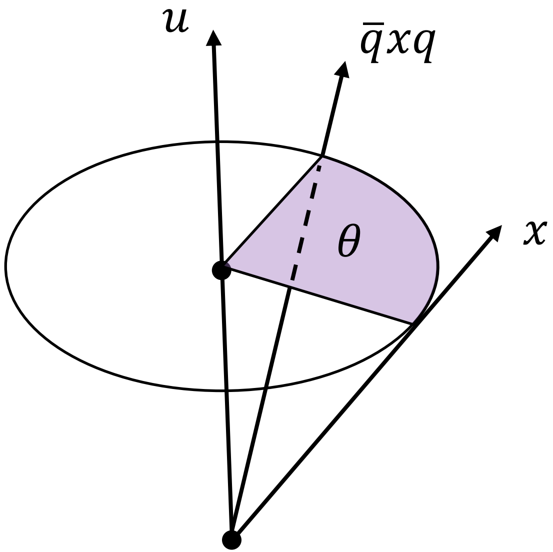
# 3

# Spin transformation

# Quaternions

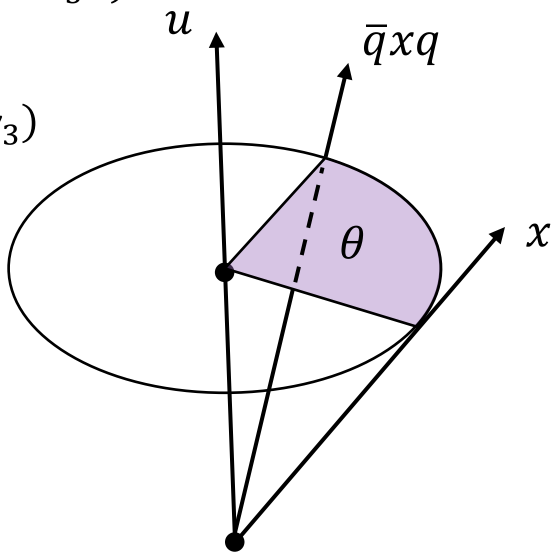
- Conformal mapping : rotate and scale
  - 2D plane : complex number
  - Surface embedding in $\mathbb{R}^3$ : quaternions



# Quaternions

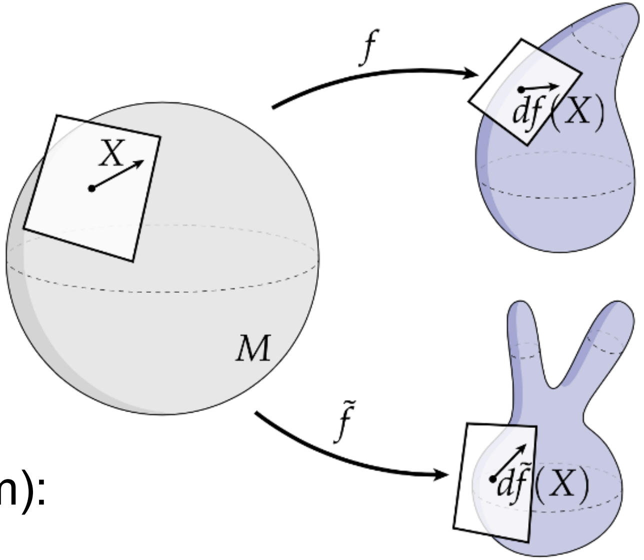
- From $\mathbb{R}^3$ to $\mathbb{H}$ : $\vec{x} = (x_1, x_2, x_3) \rightarrow x = (0, \vec{x}) = 0 + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k}$
- Rotation around the axis $\vec{u} = (u_1, u_2, u_3), \|\vec{u}\| = 1$
  - $q = \left(\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\vec{u}\right) = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}(u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k})$
  - $\bar{q} = \left(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}\vec{u}\right),\ q\bar{q} = 1 \rightarrow \bar{q} = q^{-1}$
  - $y = \bar{q}xq = 0 + y_1 i + y_2 j + y_3 k \rightarrow \vec{y} = (y_1, y_2, y_3)$
- Scale
  - $q' = cq, c \in \mathbb{R} \Longrightarrow y' = \overline{q'}xq = c^2 y$




$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$$

# Spin transformation

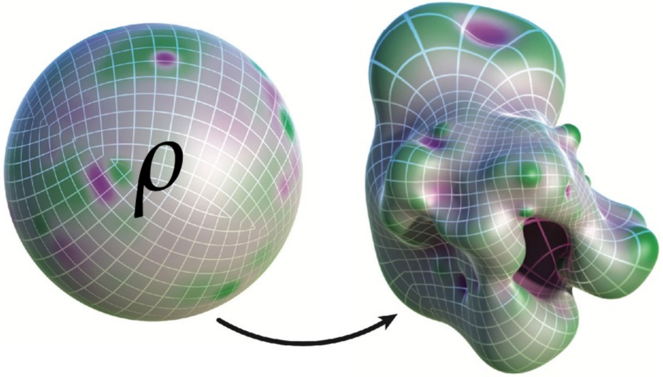
- Spin equivalence:
  - $f: M \to \mathbb{R}^3$
  - $\tilde{f}: M \to \mathbb{R}^3$
  - $d\tilde{f}(X) = \bar{\lambda} df(X)\lambda,\ \exists\, \lambda: M \to \mathbb{H}$
- Dirac equation (integrable condition):
  - $D\lambda = -\frac{df \wedge d\lambda}{|df|^2}$
  - $(D - \rho)\lambda = 0, \exists\, \rho: M \to \mathbb{R}$
- Given initial $\rho$, solve $\lambda$ (eigenvalue problem):
  - $(D - \rho)\lambda = \gamma\lambda$
  - $\rho \to \rho + \lambda$



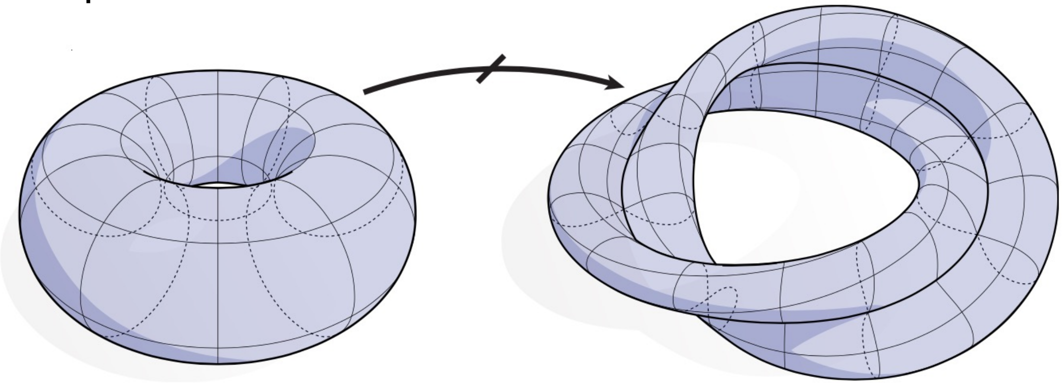

Crane, Keenan, Ulrich Pinkall, and Peter Schröder. "Spin transformations of discrete surfaces." ACM SIGGRAPH 2011 papers. 2011. 1-10.

# Spin transformation

- Mean curvature half-density:
  - $(D-\rho)\lambda = 0, \exists\, \rho, \lambda: M \to \mathbb{R}$
  - $d\tilde{f}(X) = \bar{\lambda} df(X) \lambda \to \tilde{H}|d\tilde{f}| = H|df| + \rho|df|$




- Relation to conformal equivalence
  - Spin $\rightarrow$ conformal
  - Conformal $\nrightarrow$ spin



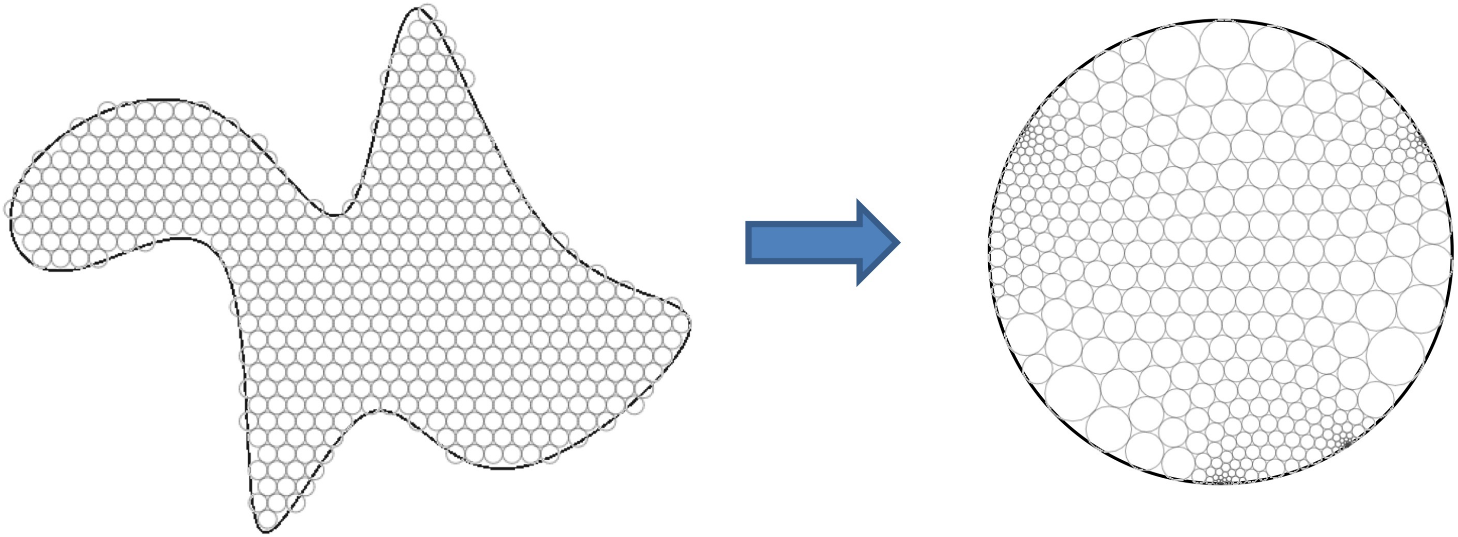
Crane, Keenan, Ulrich Pinkall, and Peter Schröder. "Spin transformations of discrete surfaces." ACM SIGGRAPH 2011 papers. 2011. 1-10.

# 4

# Circle packing and circle patterns

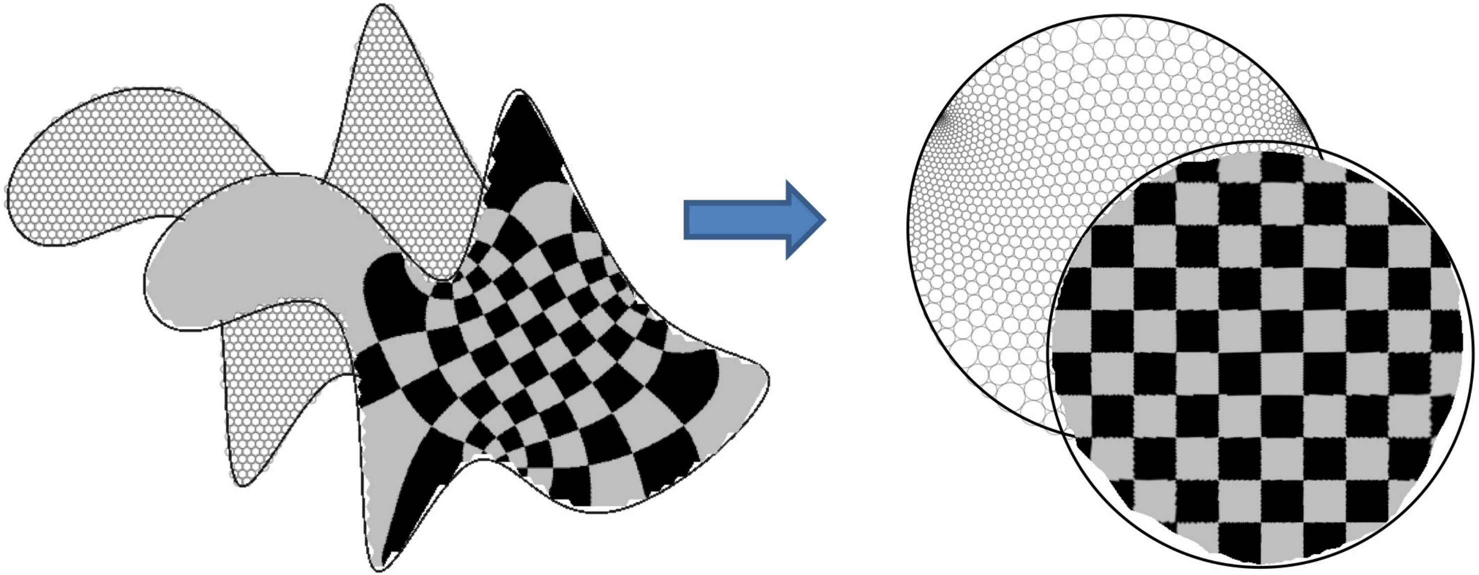
# Circle packing

- Smooth: infinitesimal circles preservation
- Discrete: preserve circles associated with mesh elements



# Circle packing

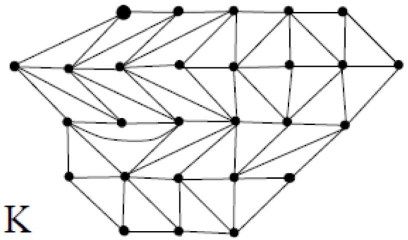
- Limit → smooth conformal map

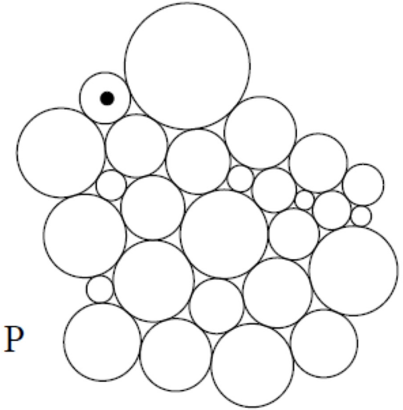


Rodin, B., & Sullivan, D. (1987). The convergence of circle packings to the Riemann mapping. *Journal of Differential Geometry*.

# Circle packing

- For a triangulation $K$, $P = \{c_v\}$ is the circle packing of $K$ if:
  - The center of $c_v \Leftrightarrow v \in V \subset K$
  - $\forall e_{ij} = v_i v_j,\ c_{v_i}, c_{v_j}$ are tangent
  - $\forall f_{ijk} = v_i v_j v_k,\ c_{v_i}, c_{v_j}, c_{v_k}$ form a positively oriented triple


K


P

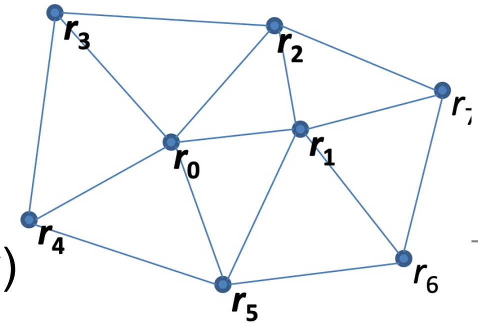
Collins, C. R., & Stephenson, K. (2003). A circle packing algorithm. *Computational Geometry*.

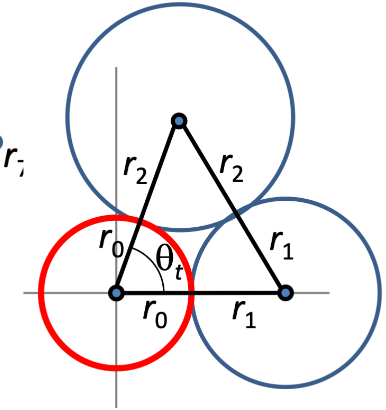
# Circle packing

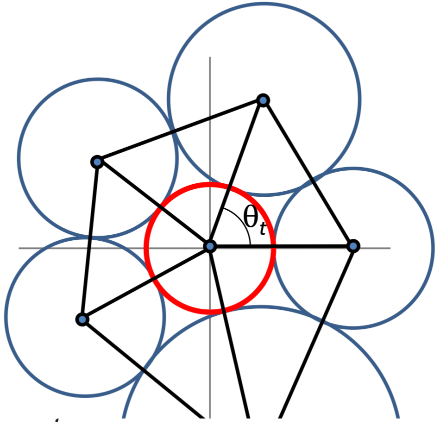
- Necessary and Sufficient Condition:

  Given a triangulation K of a topological disk and a constraint radius at each boundary vertex, there is an (essentially) unique circle packing realizing the boundary constraints, with interior angles summing to 2π.

$$\cos\theta_t = \frac{(r_0 + r_1)^2 + (r_0 + r_2)^2 - (r_1 + r_2)^2}{2(r_0 + r_1)(r_0 + r_2)}$$



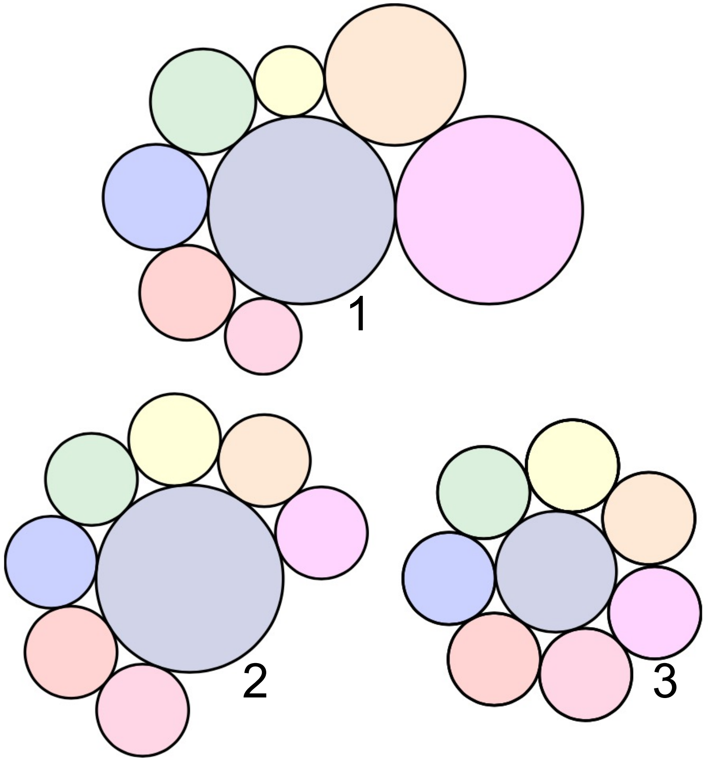

Collins, C. R., & Stephenson, K. (2003). A circle packing algorithm. *Computational Geometry*.

# Circle packing

- Algorithm: repeat

For each $v_i \in V^\circ$:

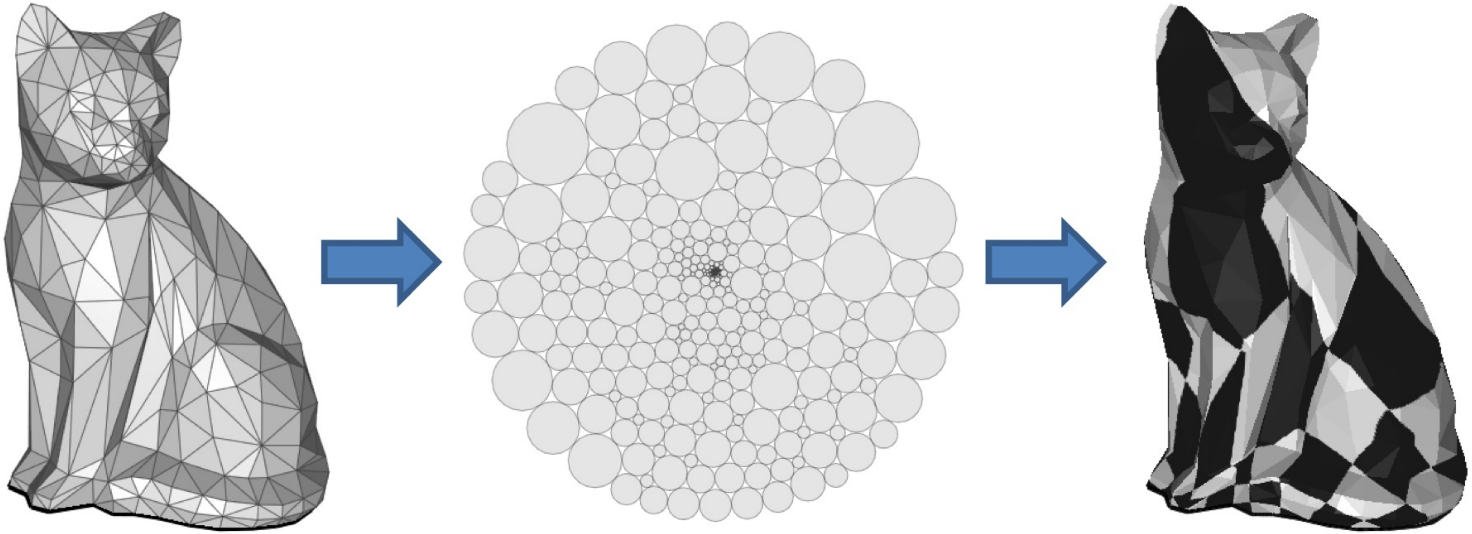
1. Let $\theta$ be total angle currently covered by k neighbors
2. Let r be radius such that k neighbors of radius r also cover $\theta$
3. Set new radius of $c_{v_i}$ such that $k$ neighbors of radius $r$ cover $2\pi$




Collins, C. R., & Stephenson, K. (2003). A circle packing algorithm. Computational Geometry.

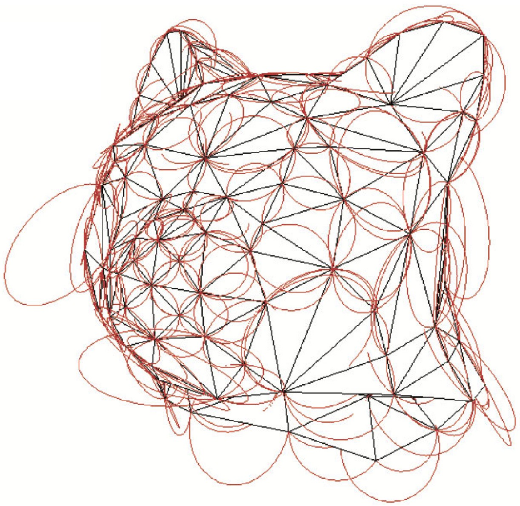
# Circle packing

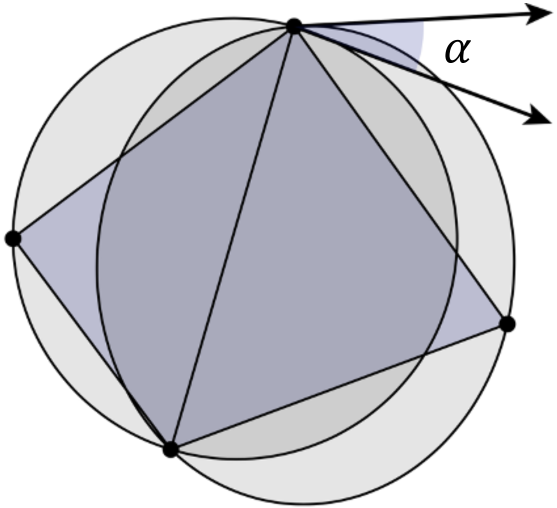
- Lack of geometry information



# Circle patterns

- Associate each face with its circumcircle
- Preserve circle intersection angles



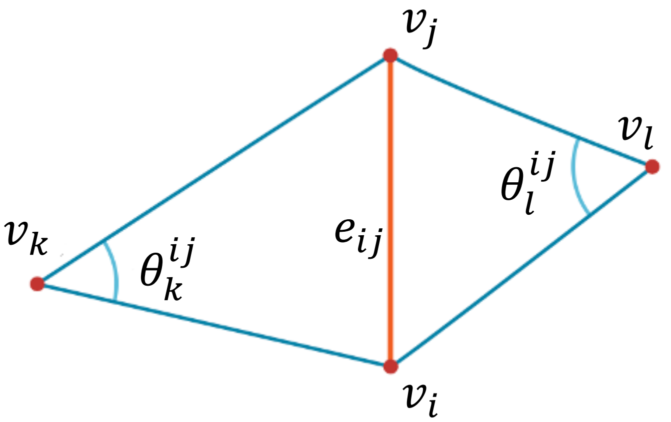


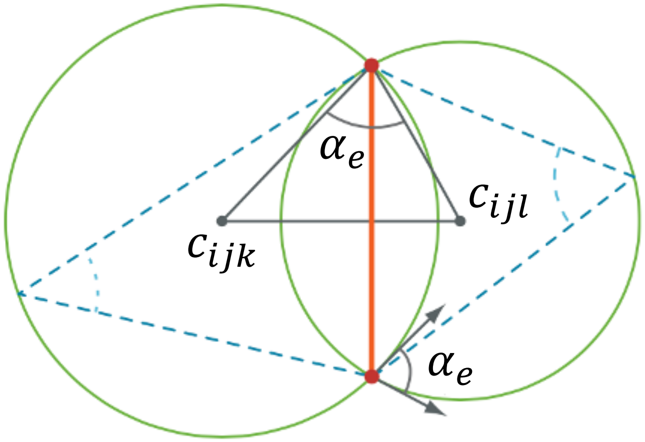

Kharevych, L. et al. (2006). Discrete conformal mappings via circle patterns. *ACM Transactions on Graphics*.

# Circle patterns

- For planar Delaunay triangulation

$$\forall\, e_{ij} \in E : \alpha_e = \begin{cases} \pi - \theta_k^{ij} - \theta_l^{ij}, & \text{for interior edges} \\ \pi - \theta_k^{ij}, & \text{for boundary edges} \end{cases}$$

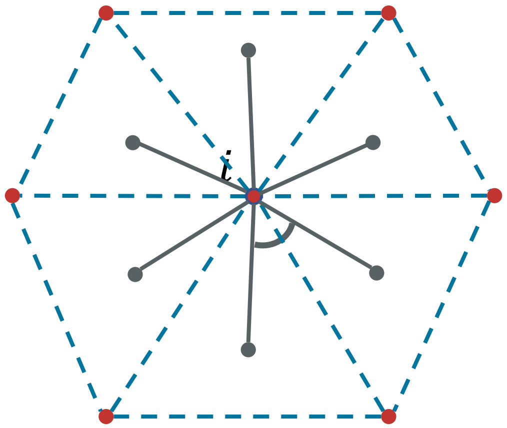


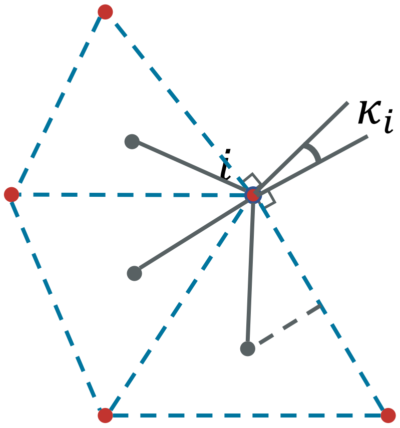




Kharevych, L. et al. (2006). Discrete conformal mappings via circle patterns. *ACM Transactions on Graphics*.

# Circle patterns

- For planar Delaunay triangulation
  - $\forall\ e_{ij} \in E : 0 < \alpha_e < \pi$
  - $\forall\ v_i$ interior vertices : $\sum_{e \ni v_i} \alpha_e = 2\pi$
  - $\forall\ v_i$ boundary vertices : $\sum_{e \ni v_i} \alpha_e = 2\pi - \kappa_i$




Kharevych, L. et al. (2006). Discrete conformal mappings via circle patterns. *ACM Transactions on Graphics*.

# Circle patterns

- For planar Delaunay triangulation
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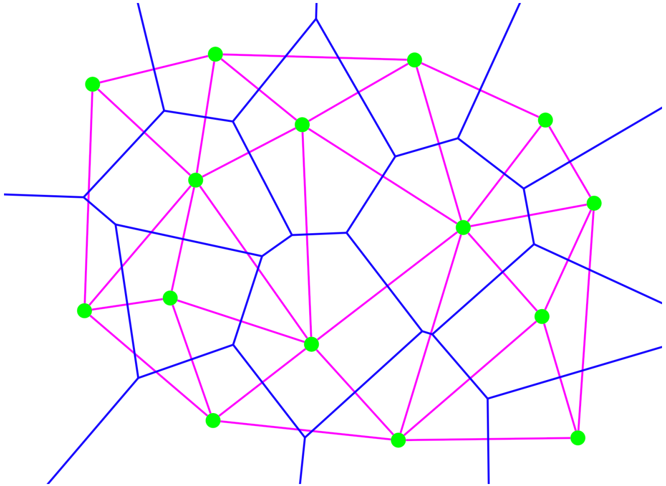
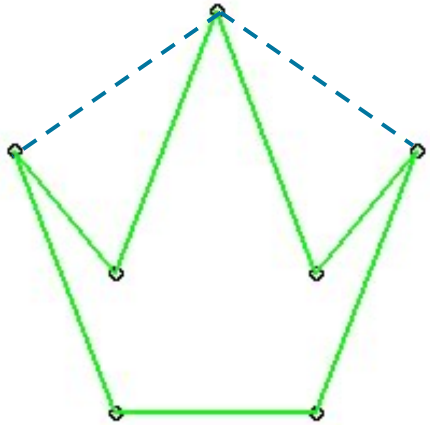
Local Delaunay!





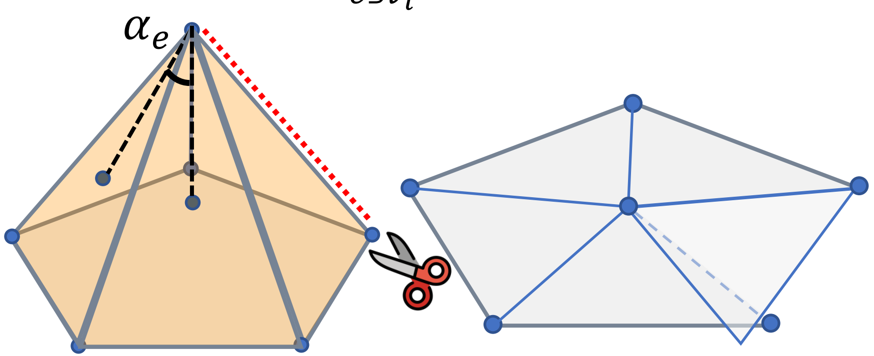
# Circle patterns

- Parameterization
  - Input Delaunay triangulation :

$$\sum_{e \ni v_i} \alpha_e \; != 2\pi$$

  - UV mesh :

$$\hat{\alpha}_e > 0 \text{ and } \sum_{e \ni v_i} \hat{\alpha}_e = 2\pi$$




- Coherent angle system for $\hat{\alpha}_e$ :

$\exists\ \hat{\theta}_k^{ij}$, s.t.

- $\hat{\theta}_k^{ij} > 0$
- $\forall\ t_{ijk} \in T,\ \hat{\theta}_k^{ij} + \hat{\theta}_i^{jk} + \hat{\theta}_j^{ki} = \pi$
- $\forall\ e_{ij} \in E$ :

$$\hat{\alpha}_e = \begin{cases} \pi - \hat{\theta}_k^{ij} - \hat{\theta}_l^{ij}, & \text{for interior edges} \\ \pi - \hat{\theta}_k^{ij}, & \text{for boundary edges} \end{cases}$$

Optimize $\hat{\theta}_k^{ij}$!

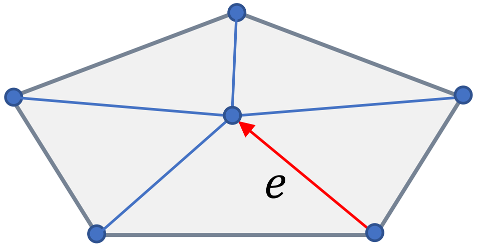
$$\min_{\hat{\theta}_k^{ij}} \sum \left( \hat{\theta}_k^{ij} - \theta_k^{ij} \right)^2$$

Kharevych, L. et al. (2006). Discrete conformal mappings via circle patterns. *ACM Transactions on Graphics*.

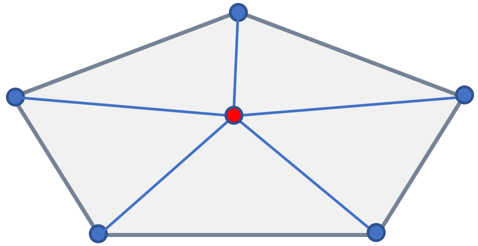
# 5 Conjugate harmonic functions

# Conjugate harmonic functions

- Cauchy-Riemann equation on complex plane
  - $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Leftrightarrow \nabla v = \boldsymbol{i}\nabla u$
- Harmonic functions
  - $\begin{cases} \Delta u = \nabla \cdot (\nabla u) = \nabla \cdot (-\boldsymbol{i}\nabla v) = 0 \\ \Delta v = \nabla \cdot (\nabla v) = \nabla \cdot (\boldsymbol{i}\nabla u) = 0 \end{cases}$
- Discretization on triangular meshes
  - Edges (conjugate harmonic 1-forms)
  - Vertices (conjugate harmonic coordinates)




1-form $\omega$



uv coordinates

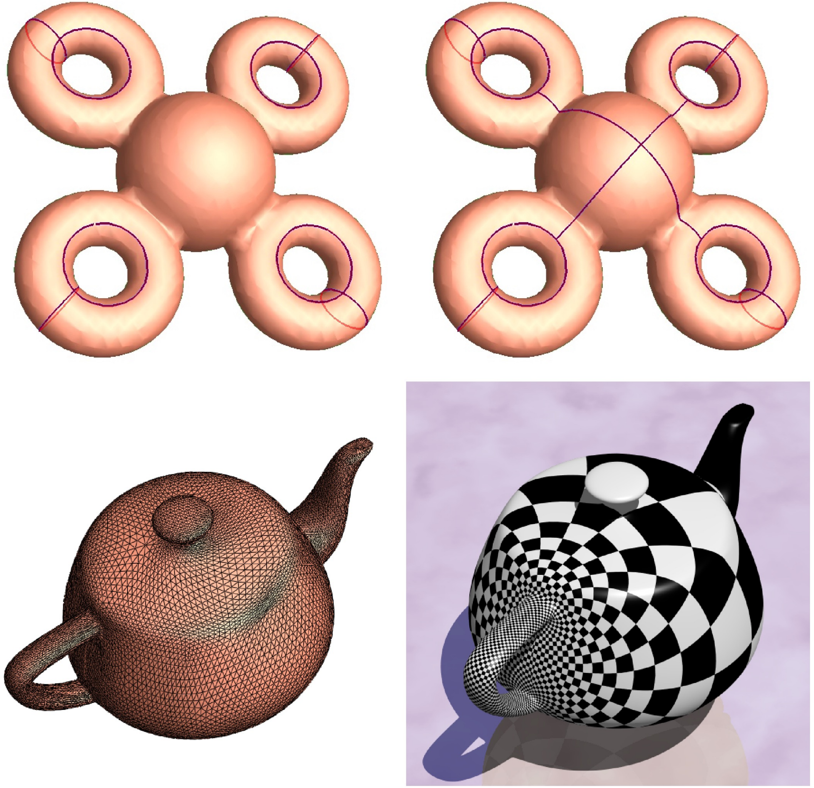
# Conjugate harmonic 1-forms

- Harmonic 1-forms:
  - $\forall\, t_{ijk},\ \omega_{ij} + \omega_{jk} + \omega_{ki} = 0$
  - $\forall v_i \text{interior vertex}, \sum_{e_{ij} \ni v_i} \alpha_{ij}\omega_{ij} = 0$

- Dimension of harmonic 1-forms:
  - Genus $g \implies \{\omega^{(1)}, \dots, \omega^{(2g)}\}$
  - Homology basis $P_1, \dots, P_{2g}$:

$$\sum_{e_{ij} \in \mathrm{P}_k} \omega_{ij} = c_k, k = 1, \dots, 2g$$

- Conjugate gradients
  - $\{{}^*\omega^{(1)}, \dots, {}^*\omega^{(2g)}\}$
  - Integrate $\omega^{(k)} + \sqrt{-1}\,{}^*\omega^{(k)}$



Gu, X., & Yau, S. T. (2003). Global conformal surface parameterization. In Proceedings of the 2003 Eurographics/ACM SIGGRAPH symposium on Geometry processing.

# Conjugate harmonic coordinates

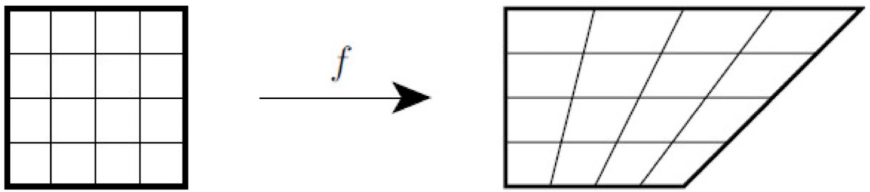
- Solving Laplacian equations:
  - For interior vertices $\begin{cases} \Delta u = 0 \\ \Delta v = 0 \end{cases}$
  - Boundary control

- Dirichlet boundary condition:
  - Boundary curve $\gamma: \partial M \to \mathbb{R}^2$

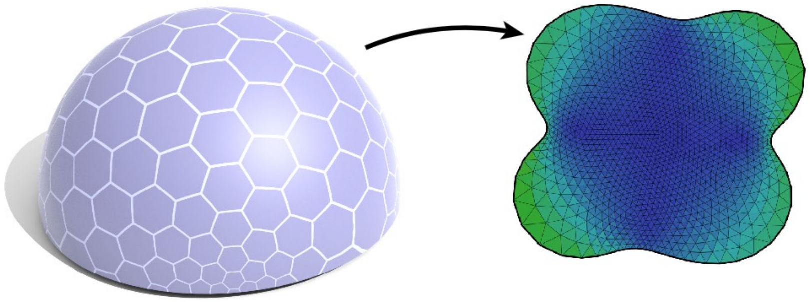
$$u\Big|_{\partial M} = \gamma_u, v\Big|_{\partial M} = \gamma_v$$

- Neumann boundary condition:
  - Boundary gradients $\mathrm{h}: \partial M \to \mathbb{R}^2$

$$\partial_M u = h_u, \partial_M v = h_v$$




Hamonic, not conformal



Conformal, conjugate gradients

Sawhney, R., & Crane, K. (2017). Boundary first flattening. ACM Transactions on Graphics.

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# 谢 谢 ！