中国科学技术大学
University of Science and Technology of China

GAMES 301：第11讲

共形参数化2

离散共形等价类、Möbius变换 & 曲率流

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Content

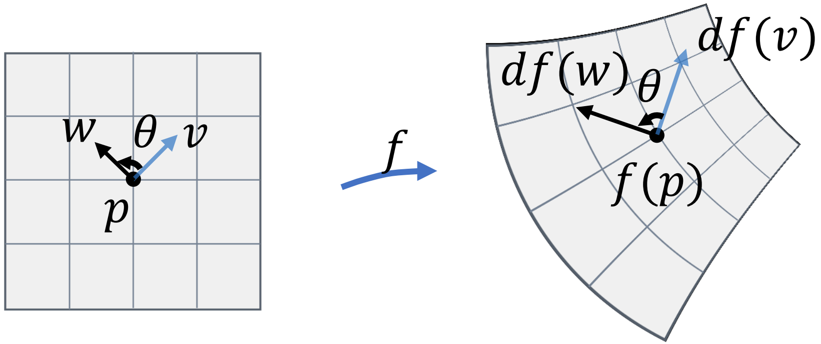
1. Conformal mapping on Riemann metric
2. Conformal equivalence of triangle meshes
3. Piecewise Möbius transformation
4. Ricci flow and Calabi flow

1

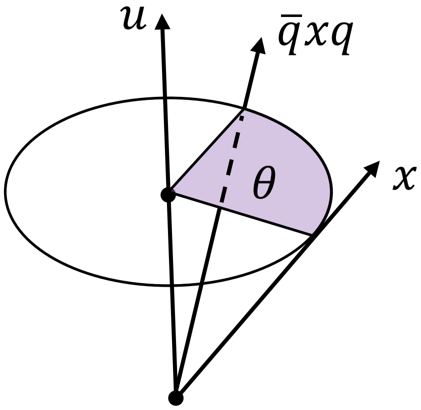
Conformal mapping on Riemann metric

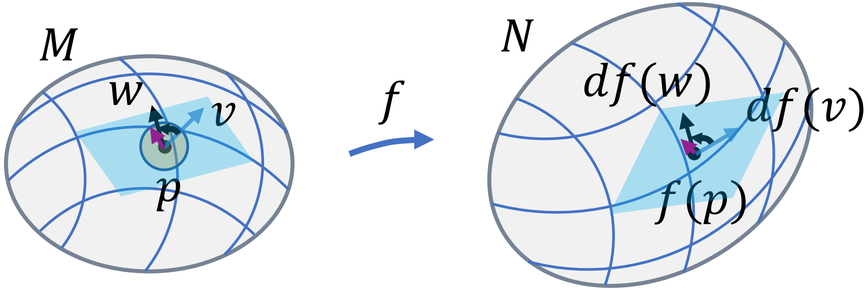
Differential

- Cauchy-Riemann equation
  - Plane : $df(i) = idf(1)$
  - Manifold : $df\,(J_M v) = J_N df(v), \forall v \in T_p M$



- Spin transformation:
  - Quaternions



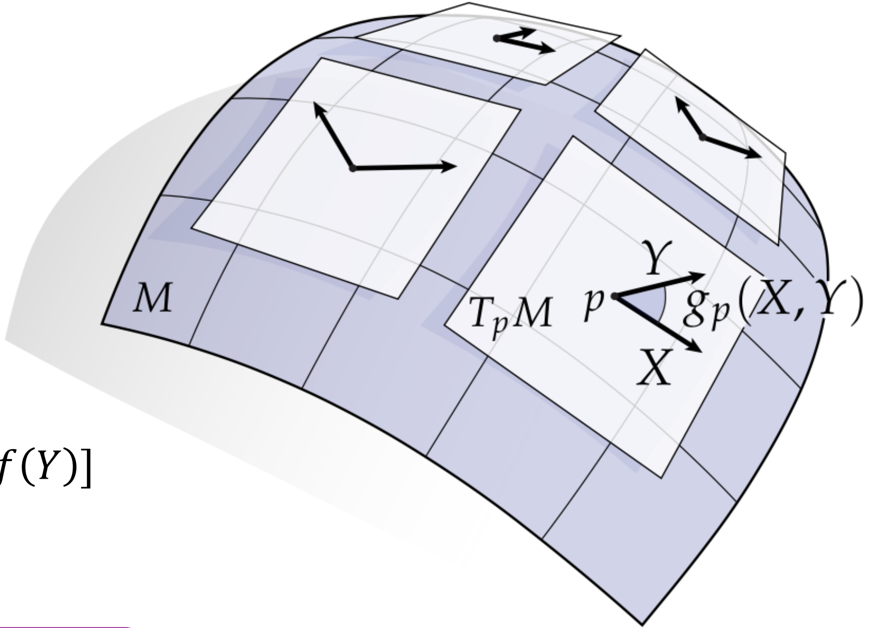


Riemann metric

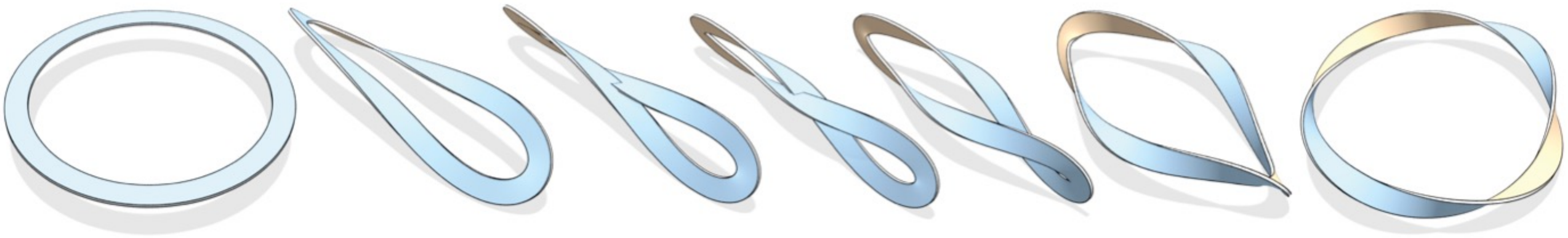
- Riemann metric
  - $g_p : T_pM \times T_pM \to \mathbb{R}$ bilnear
  - $|X| = \sqrt{g_p(X,X)}, \forall X \in T_pM$
  - $\theta[X,Y] = \arccos(g_p(X,Y)/|X||Y|)$

- Change with conformal mapping
  - $g_p(df \circ X, df \circ Y) \Rightarrow g'_p : T_pM \times T_pM \to \mathbb{R}$
  - $g'_p(X,Y) = |df(X)||df(Y)| \cos\theta[df(X), df(Y)]$
  - $g'_p(X,Y) = s^2 g_p(X,Y), \ \forall X,Y \in T_pM$

$g'_p = e^{2\lambda} g_p, \quad \lambda :$ log conformal factor

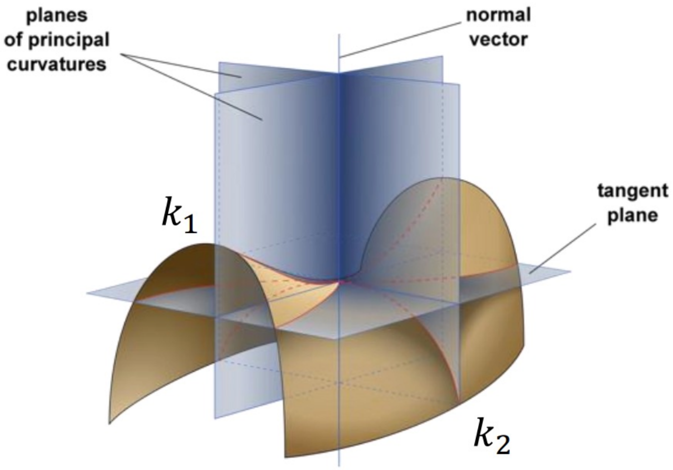


Isometric deformation



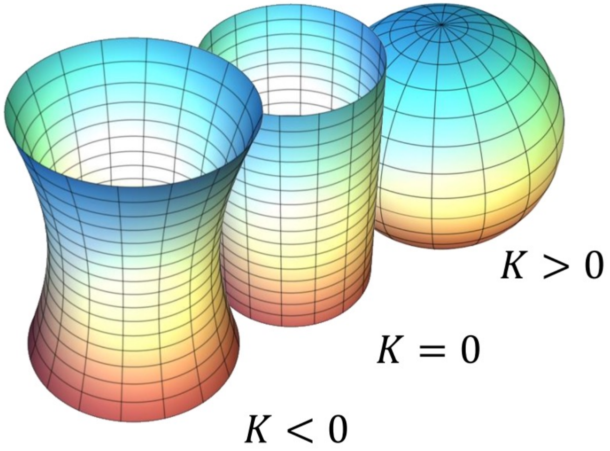
Curvature

- Normal curvature
- Principle curvature



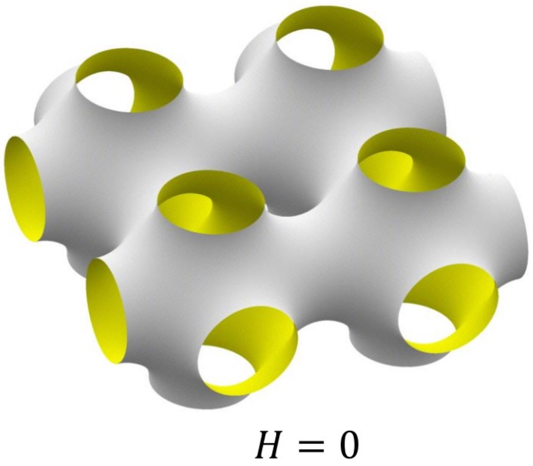
Gaussian curvature:

$$K = k_1 \times k_2$$



Mean curvature:

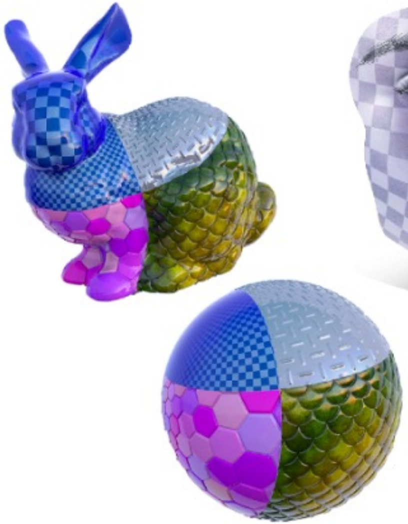
$$H = \frac{k_1 + k_2}{2}$$

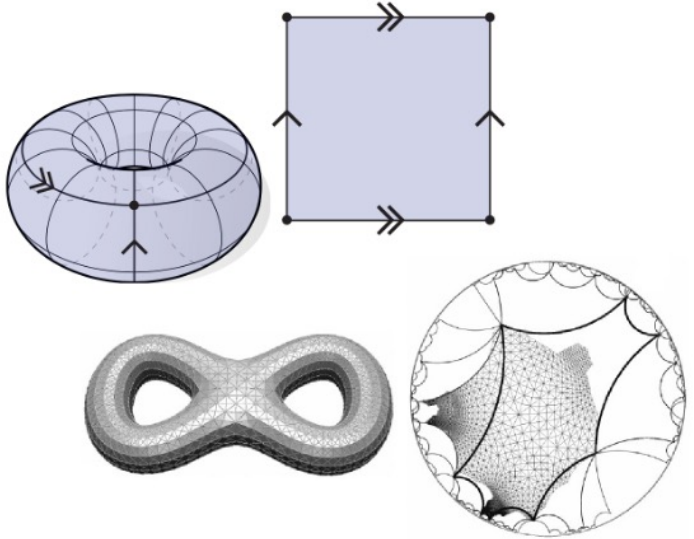


Uniformization Theorem

- Riemannian metric on any surface is conformally equivalent to one with constant Gaussian curvature (flat, spherical, hyperbolic).

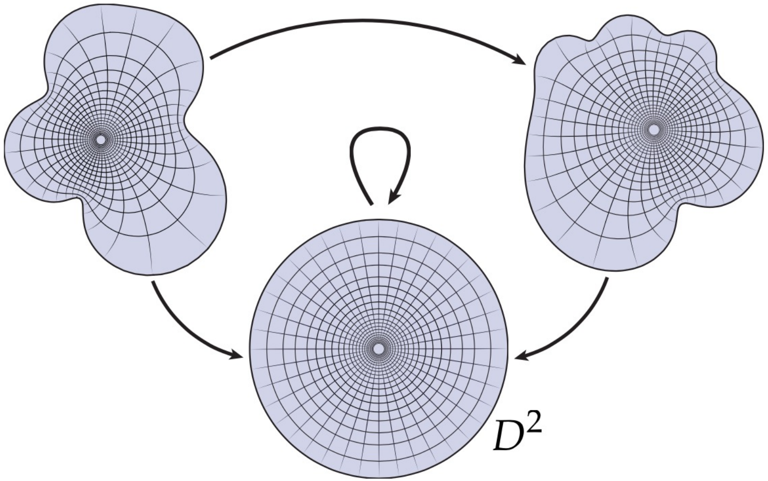
$$\boxed{g' = e^{2\lambda} g}$$

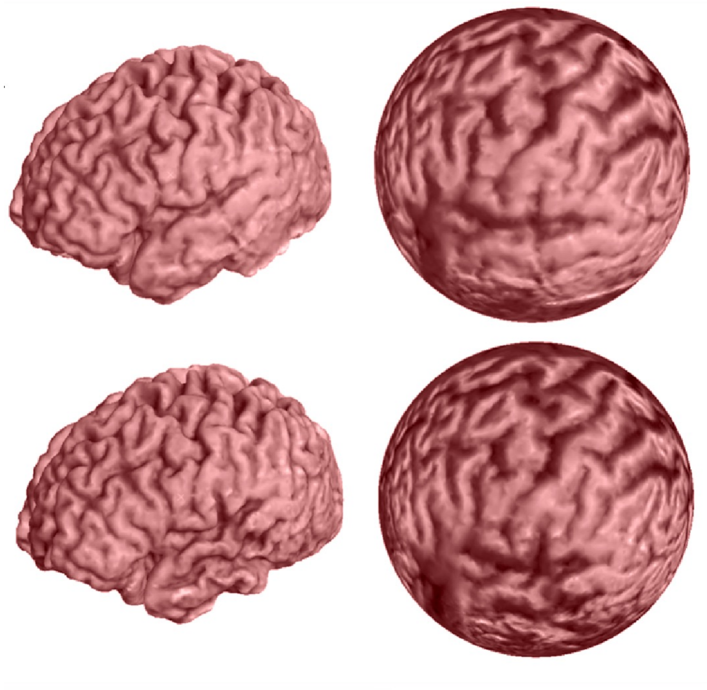




Uniformization Theorem

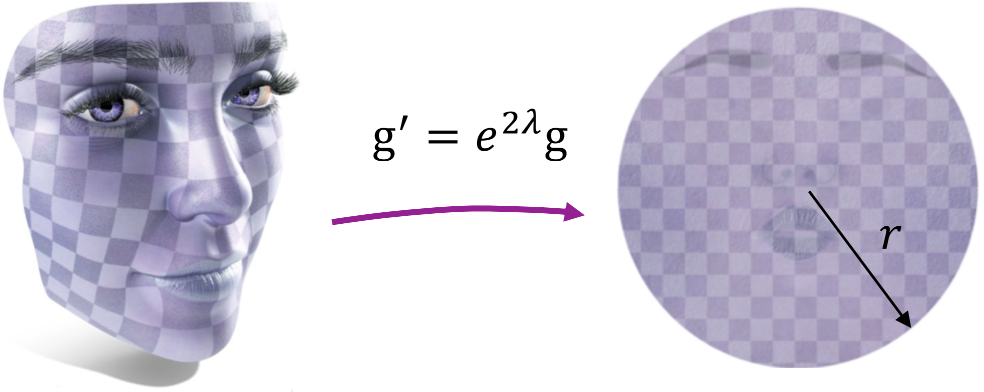
- Parameterization to canonical domain
- Cross-parameterization



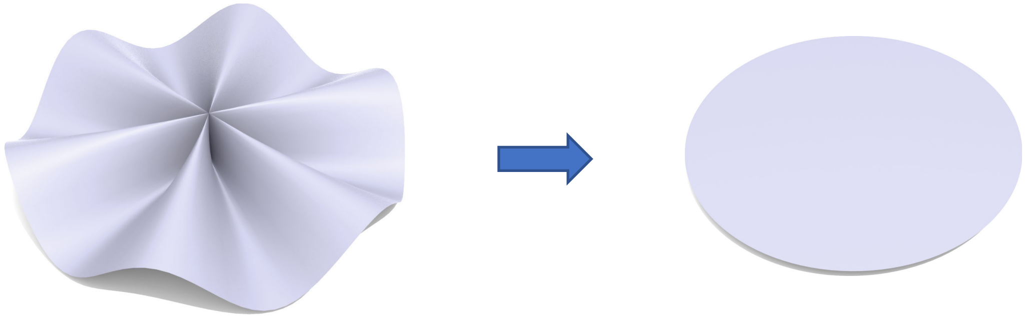


Uniformization Theorem

- From curvature to metric
  - Target curvature
    $$K' = 0, \qquad \kappa' = \frac{1}{r}$$
  - Log conformal factor
    $$\lambda : M \to \mathbb{R}$$



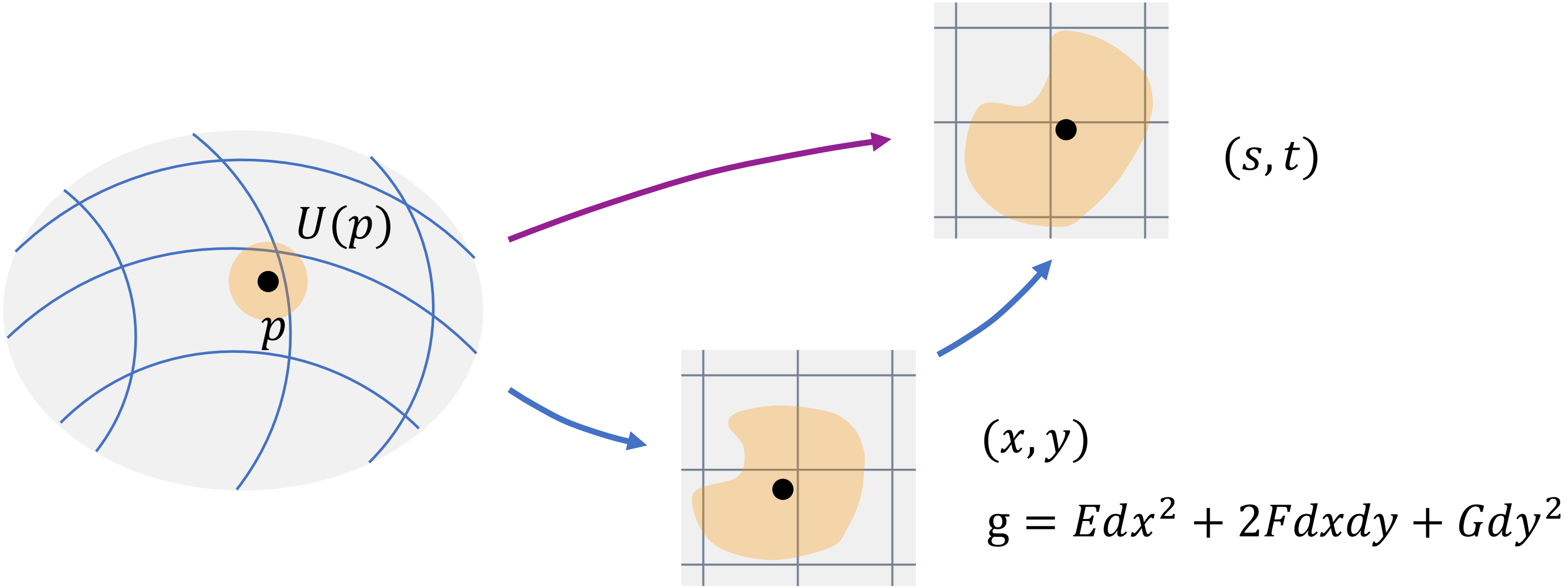
- Flattening to plane
  - $g' = e^{2\lambda} g$
  - No distortion



Isothermal coordinate

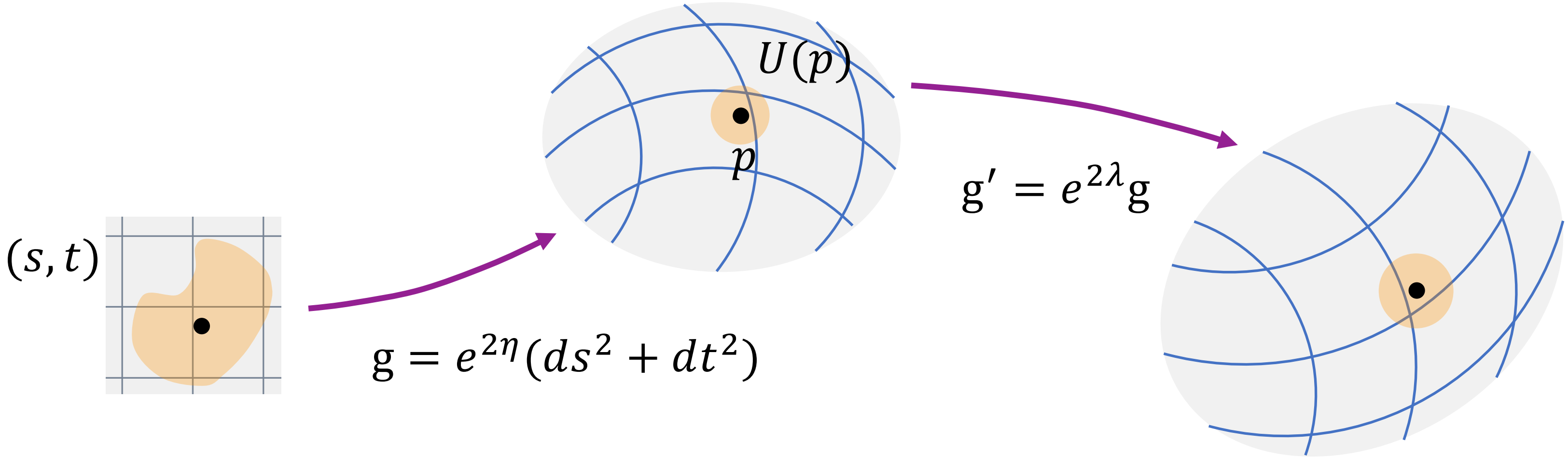
- For any point p on Riemann manifold (M, g), $\exists U(p) \subset M$ and local coordinate (s, t), s.t.

$$\mathrm{g} = e^{2\eta(s,t)}\,(ds^2 + dt^2)$$



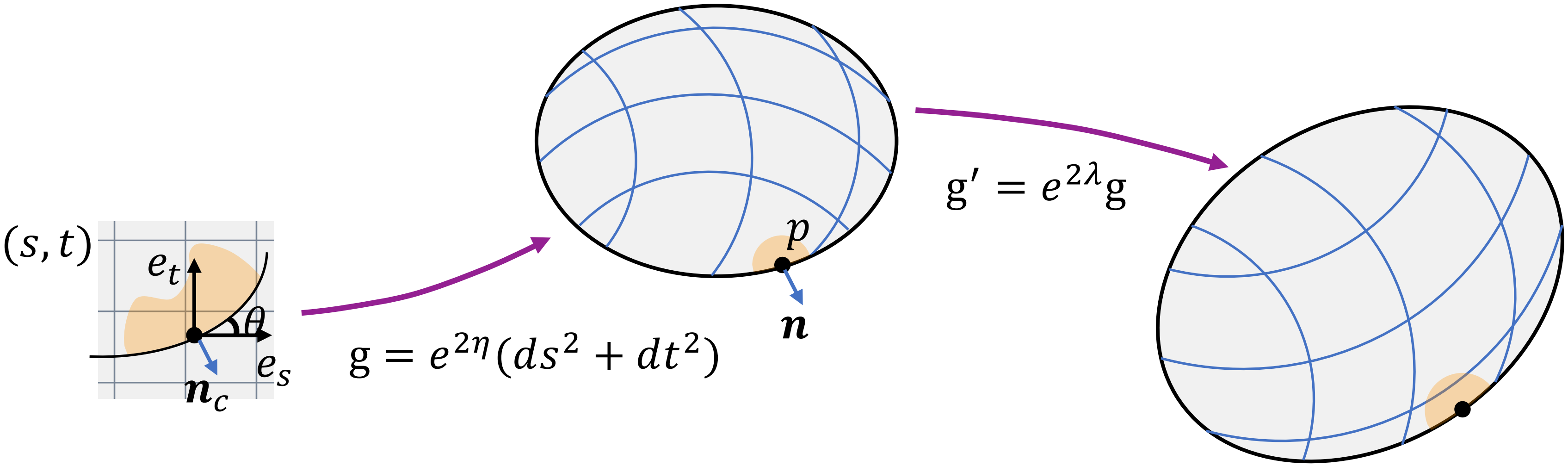
Gaussian curvature

- $K = -\frac{1}{\sqrt{EG}}\left(\left(\frac{(\sqrt{E})_t}{\sqrt{G}}\right)_t + \left(\frac{(\sqrt{G})_s}{\sqrt{E}}\right)_s\right) = -\frac{1}{e^{2\eta}}(\eta_{tt} + \eta_{ss}) = -e^{-2\eta}\Delta\eta$
- Set $\eta \to \eta + \lambda, \quad K' = -e^{-2(\eta+\lambda)}\Delta(\eta + \lambda)$
- $K' = e^{-2\lambda}(-e^{-2\eta}\Delta\eta - e^{-2\eta}\Delta\lambda) = e^{-2\lambda}(K - \Delta_{\mathrm{g}}\lambda)$



Geodesic curvature

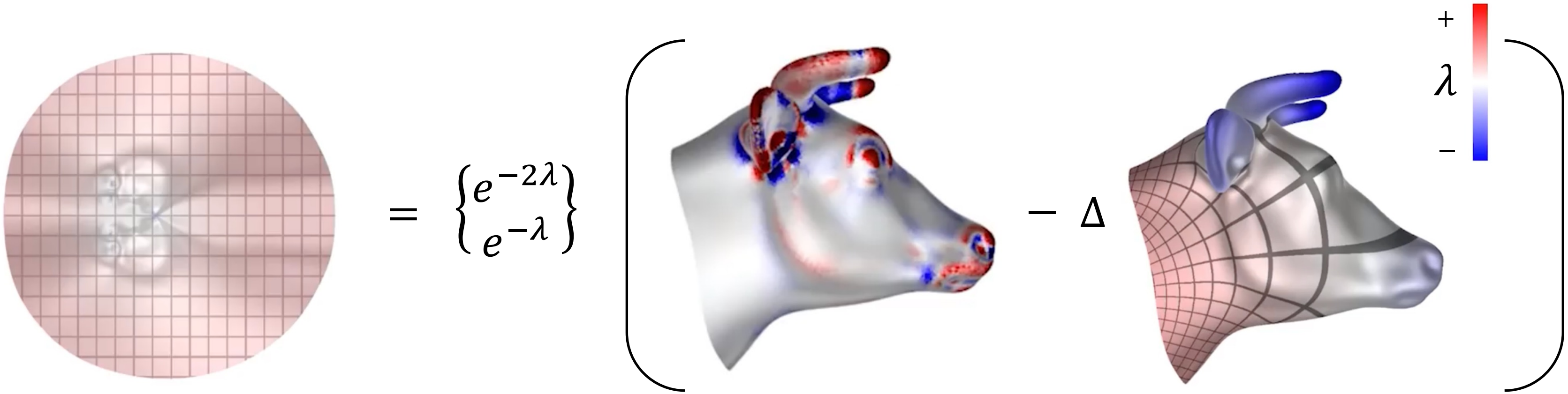
- $\kappa = \frac{d\theta}{dr} - \frac{(\ln E)_t}{2\sqrt{G}}\cos\theta + \frac{(\ln G)_s}{2\sqrt{E}}\sin\theta = \frac{d\theta}{e^{\eta}dr} - \frac{\eta_t}{e^{\eta}}\cos\theta + \frac{\eta_s}{e^{\eta}}\sin\theta = e^{-\eta}\left(k_c - \partial_{\boldsymbol{n}_c}\eta\right)$
- Set $\eta \to \eta + \lambda,\ \ \kappa' = e^{-(\eta+\lambda)}\left(k_c - \partial_{\boldsymbol{n}_c}(\eta+\lambda)\right)$
- $\kappa' = e^{-\lambda}\left(e^{-\eta}\left(k_c - \partial_{\boldsymbol{n}_c}\eta\right) - e^{-\eta}\partial_{\boldsymbol{n}_c}\lambda\right) = e^{-\lambda}\left(\kappa - \partial_{\boldsymbol{n}}^{M}\lambda\right)$



Yamabe equation

- Non linear differential equation

$$-\begin{cases} K' = e^{-2\lambda}(K - \Delta_{\mathrm{g}}\, \lambda) \\ \kappa' = e^{-\lambda}(\kappa - \partial_{\boldsymbol{n}}^{M} \lambda) \end{cases}$$



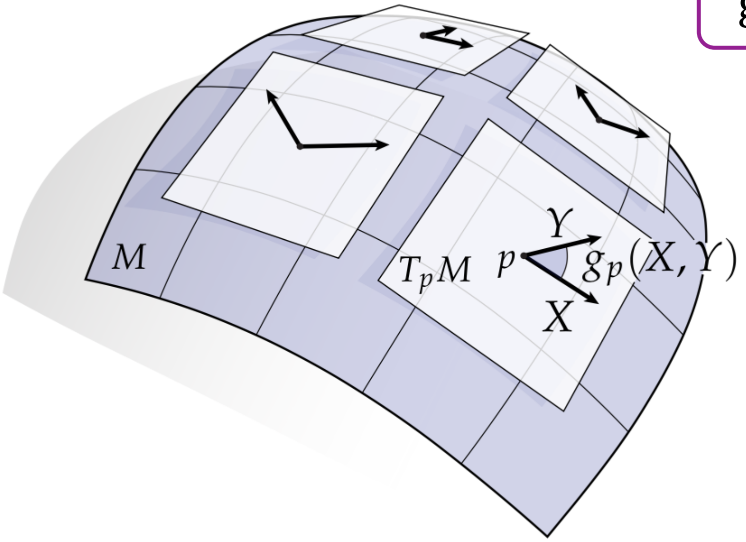
2

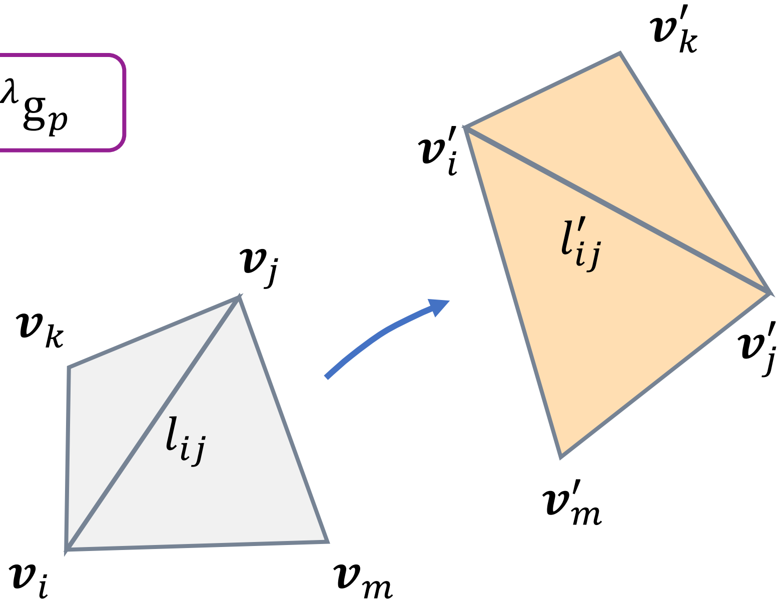
Conformal equivalence of triangle meshes

Discrete conformal metric

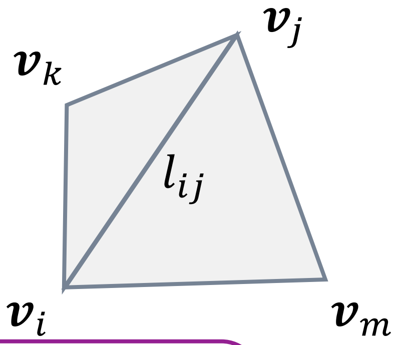
- Smooth Riemann metric
  - $|X| = \sqrt{g_p(X,X)}, \forall X \in T_pM$
  - $|X'| = e^{\lambda}|X|, \forall X \in T_pM$
- Discrete metric
  - $l : E \to \mathbb{R}^+ \Longrightarrow e_{ij} \to l_{ij}$
  - $l'_{ij} = e^{(\lambda_i+\lambda_j)/2} l_{ij}, \quad \lambda : V \to \mathbb{R}$

$$\boxed{g'_p = e^{2\lambda} g_p}$$





Conformal equivalence of triangle meshes



- From log conformal factor: $l'_{ij} = e^{(\lambda_i+\lambda_j)/2} l_{ij}$
- From length cross ratio: $c_{ij} = \frac{l_{ki} l_{mj}}{l_{im} l_{jk}} \Longrightarrow c'_{ij} = c_{ij}$

$$c'_{ij} = \frac{l'_{ki}}{l'_{im}} \frac{l'_{mj}}{l'_{jk}} = \frac{l_{ki} e^{(\lambda_k+\lambda_i)/2}}{l_{im} e^{(\lambda_i+\lambda_m)/2}} \frac{l_{mj} e^{(\lambda_m+\lambda_j)/2}}{l_{jk} e^{(\lambda_j+\lambda_k)/2}} = \frac{l_{ki}}{l_{im}} \frac{l_{mj}}{l_{jk}} = c_{ij}$$

For ijk, $\lambda_i^{jk} = \log\left(\frac{l'_{ij} l'_{ik}}{l'_{jk}} \Big/ \frac{l_{ij} l_{ik}}{l_{jk}}\right)$; for imj, $\lambda_i^{mj} = \log\left(\frac{l'_{im} l'_{ij}}{l'_{mj}} \Big/ \frac{l_{im} l_{ij}}{l_{mj}}\right)$

Springborn, B. et al. (2008). Conformal equivalence of triangle meshes. *ACM SIGGRAPH.*

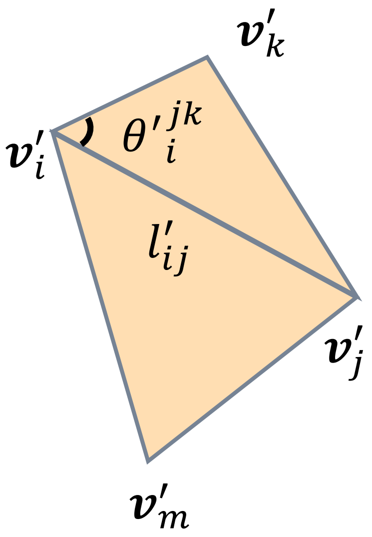
Optimizing log conformal factor

- Treating angle as function of $\lambda : V \rightarrow \mathbb{R}$

$$\theta'^{jk}_{i} = \arccos \frac{l'^2_{ij} + l'^2_{ik} - l'^2_{jk}}{2 l'_{ij} l'_{ik}}$$

- Parameterizing to a planar shape
  - $\sum_{ijk \in St(i)} \theta'^{jk}_{i} = 2\pi, \ \forall\, i$ interior vertex
  - $\sum_{ijk \in St(i)} \theta'^{jk}_{i} = \beta_i$, for i boundary vertex

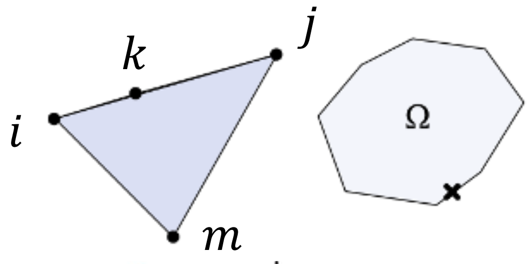
- Optimizing a convex energy
  - $E(\lambda) = \sum_{ijk \in T} f(t_{ij}, t_{jk}, t_{ki}) + \frac{1}{2} \sum_i \alpha_i \lambda_i\,, \ \ t_{ij} = \log l'_{ij}$
  - $\frac{\partial E}{\partial \lambda_i} = \frac{1}{2}\left(\alpha_i - \sum_{ijk \in St(i)} \theta'^{jk}_{i}\right) = 0 \Longrightarrow \alpha_i = \sum_{ijk \in St(i)} \theta'^{jk}_{i}$



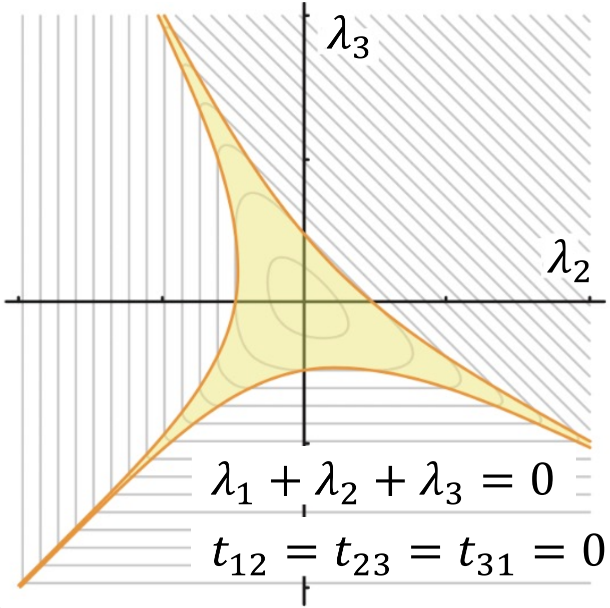
Springborn, B. et al. (2008). Conformal equivalence of triangle meshes. *ACM SIGGRAPH.*

Optimizing log conformal factor

- Feasible set of $\lambda : V \to \mathbb{R}$
  - Triangle inequality: $\begin{cases} l'_{jk} + l'_{ki} > l'_{ij} \\ l'_{ki} + l'_{ij} > l'_{jk} \\ l'_{ij} + l'_{jk} > l'_{ki} \end{cases}$
  - Extending: If $l'_{jk} + l'_{ki} \leq l'_{ij}$, $\theta'^{ij}_{k} = \pi,\ \theta'^{jk}_{i} = \theta'^{ki}_{j} = 0$



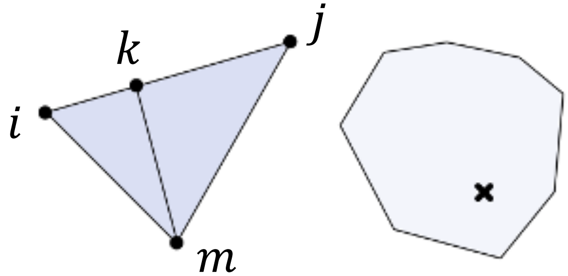
- Optimizing method
  - Gradient descent $\delta\lambda_i = -\frac{\partial E}{\partial \lambda_i}$
  - Newton method $(\text{Hess } E \cdot \delta\lambda)_i = -\frac{\partial E}{\partial \lambda_i}$

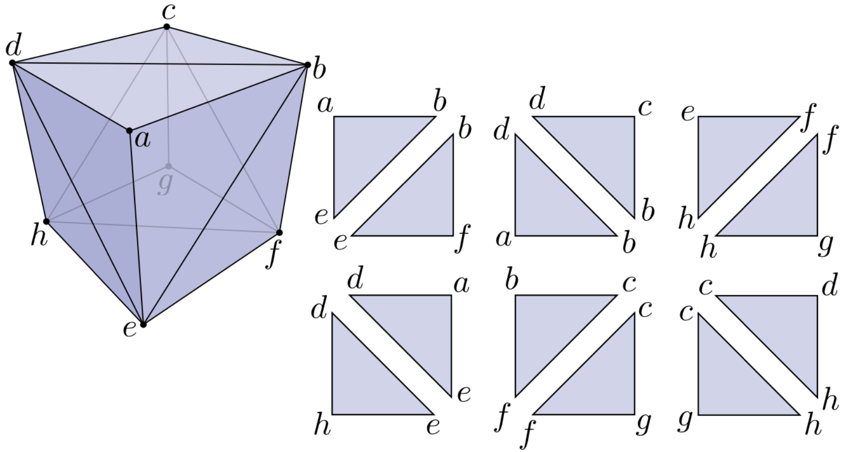


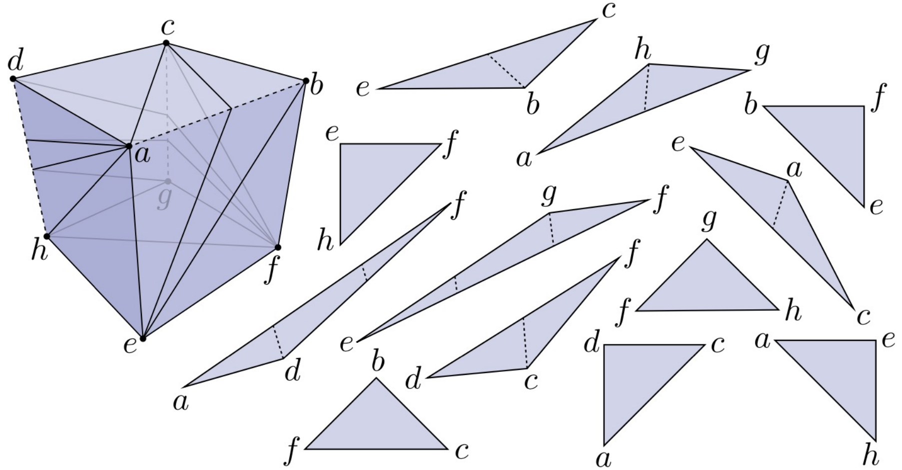
Springborn, B. et al. (2008). Conformal equivalence of triangle meshes. *ACM SIGGRAPH.*

Geodesic distance

- Edge flip for global minimum λ^*
  - $l'_{jk} + l'_{ki} \leq l'_{ij}$
  - $l'_{ij} \rightarrow l'_{km} = e^{(\lambda_k+\lambda_m)/2} l_{km}$
- From planar to $\mathbb{R}^3$





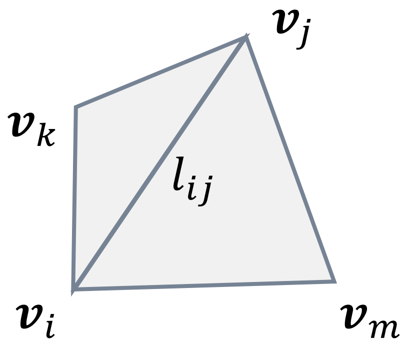


Constraining length cross ratio

- Linearizing constraint:

$$\log c_{ij} = \log(\frac{l_{ki}}{l_{im}}\frac{l_{mj}}{l_{jk}}) = t_{ki} + t_{mj} - t_{im} - t_{jk} \equiv const$$

- Mesh conformal deformation
  - Optimizing the vertex location $\boldsymbol{v}_i \in \mathbb{R}^3$
  - Treating the t_{ij} as the function of $\boldsymbol{v}_\mathrm{i} \Longrightarrow \delta \boldsymbol{t} = J\delta\boldsymbol{v}$
  - Constraint $L\boldsymbol{t} \equiv const \Longrightarrow L\delta\boldsymbol{t} = LJ\delta\boldsymbol{v} = \mathbf{0}$



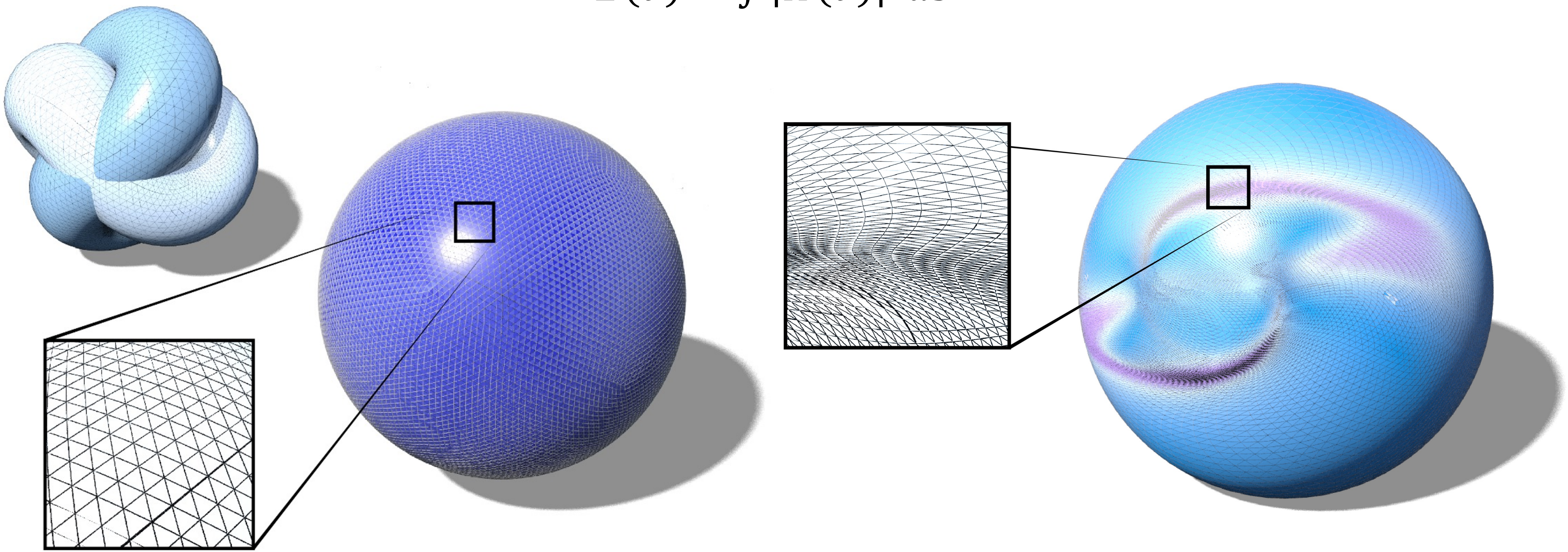
- Minimizing energy $E(\boldsymbol{v})$ under conformal mapping
  - Local minima: $< \frac{\partial E}{\partial \boldsymbol{v}}, \delta\boldsymbol{v} > = 0, \;\; \forall\, \delta\boldsymbol{v} \in \{LJ\delta\boldsymbol{v} = \mathbf{0}\}$
  - Projected gradient descent

Soliman, Yousuf, et al. "Constrained willmore surfaces." ACM Transactions on Graphics (TOG) 40.4 (2021): 1-17.

Constraining length cross ratio

- Optimizing Willmore energy:

$$E(\boldsymbol{v}) = \int |H(\boldsymbol{v})|^2 dS$$



Soliman, Yousuf, et al. "Constrained willmore surfaces." ACM Transactions on Graphics (TOG) 40.4 (2021): 1-17.

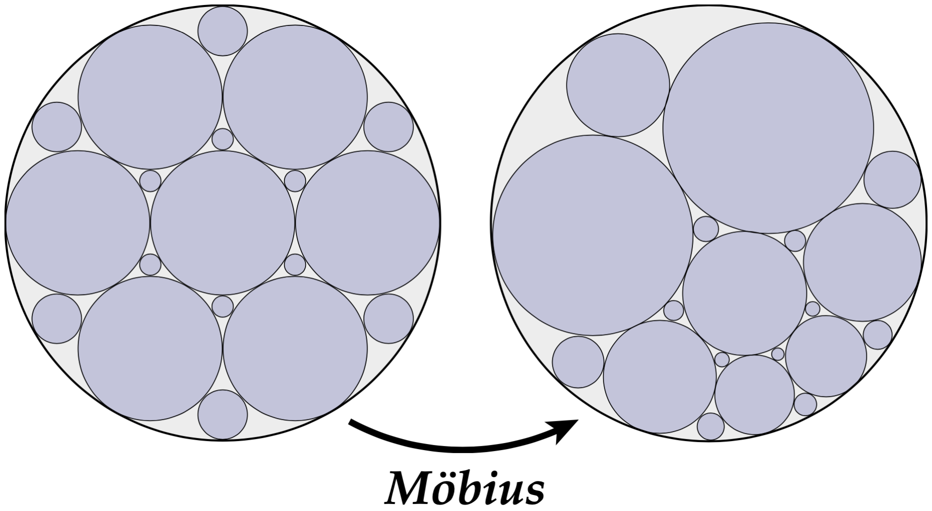
3

Piecewise Möbius transformation

Möbius transformation

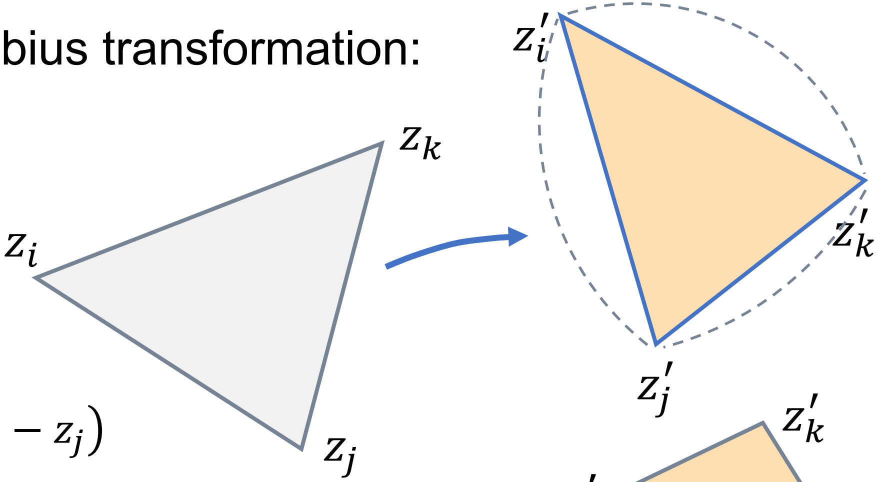
- $f\colon \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\}$
  - $f(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$. (If $ad = bc$, then $f(z) = \frac{c(az+b)}{c(cz+d)} = \frac{caz+ad}{c(cz+d)} = \frac{a}{c}$)
  - Translation $z \mapsto z + t_1$, dilation $z \mapsto t_2 z$, inversion $z \mapsto \frac{1}{z}$

- Circle preservation (Line as circle with radius ∞)



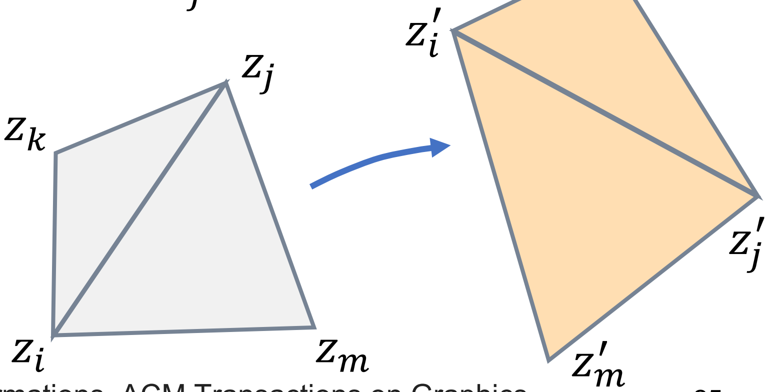
Piecewise Möbius transformation

- For the triangle $ijk \in T$, consider a Möbius transformation:
  - $ijk \in T, \{z_i, z_j, z_k\} \subset \mathbb{C}$
  - $f(z) = \frac{az+b}{cz+d}, ad - bc = 1$



- Mapping of edges:
  - Edge line to circle arc
  - $z_i' - z_j' = \frac{az_i+b}{cz_i+d} - \frac{az_j+b}{cz_j+d} = \frac{1}{(cz_i+d)(cz_j+d)}(z_i - z_j)$

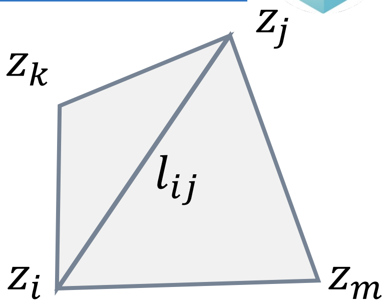
- Piecewise-compatible
  - Denote $D_i^{jk} \Leftarrow \frac{1}{cz_i+d}$
  - Constraint on edge ij: $D_i^{jk} D_j^{ki} = D_i^{mj} D_j^{im}$



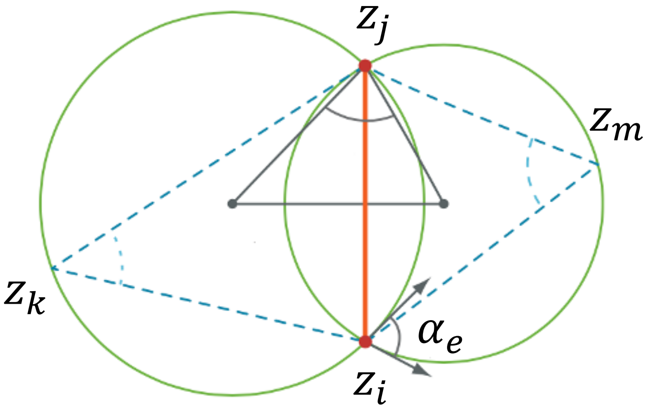
Vaxman, A. et al. (2015). Conformal mesh deformations with Möbius transformations. ACM Transactions on Graphics.

Piecewise Möbius transformation

- Preserving length cross ratio:
  - $w_{ij} = \frac{(z_k - z_i)(z_m - z_j)}{(z_i - z_m)(z_j - z_k)} \Longrightarrow |w_{ij}| = \left|\frac{(z_k - z_i)(z_m - z_j)}{(z_i - z_m)(z_j - z_k)}\right| = \frac{l_{ki} l_{mj}}{l_{im} l_{jk}}$
  - $|w'_{ij}| = \left|\frac{D_k^{ij} D_i^{jk} D_m^{ji} D_j^{im}}{D_i^{mj} D_m^{ji} D_j^{ki} D_k^{ij}}\right| |w_{ij}| = \left|\frac{D_i^{jk} D_j^{im}}{D_i^{mj} D_j^{ki}}\right| |w_{ij}|$
  - Combining $D_i^{jk} D_j^{ki} = D_i^{mj} D_j^{im} \Longrightarrow |D_i^{jk}| = |D_i^{mj}|, \ \forall\, i$



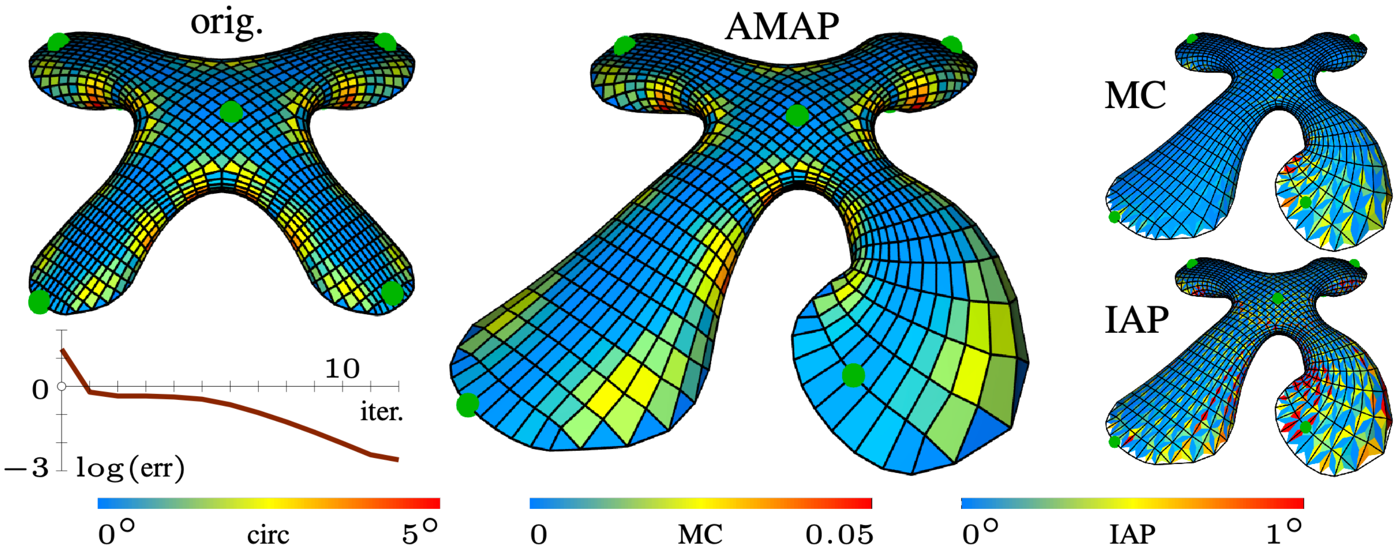
- Preserving circle intersection angles:
  - $\cos \alpha_e = -\frac{Re(w_{ij})}{|w_{ij}|}$
  - Combining $D_i^{jk} D_j^{ki} = D_i^{mj} D_j^{im} \Longrightarrow D_i^{jk} \overline{D}_i^{mj} \in \mathbb{R}, \ \forall\, i$



Vaxman, A. et al. (2015). Conformal mesh deformations with Möbius transformations. ACM Transactions on Graphics.

Piecewise Möbius transformation

- Conformal constraint:
  - Preserving length cross ratio: $\left|D_i^{jk}\right| = \left|D_i^{mj}\right|, \forall\, i$
  - Preserving circle intersection angles: $D_i^{jk}\overline{D}_i^{mj} \in \mathbb{R},\ \forall\, i$
  - Preserving both: $D_i^{jk} = D_i^{mj}, \forall\, i$ (as Möbius as possible)



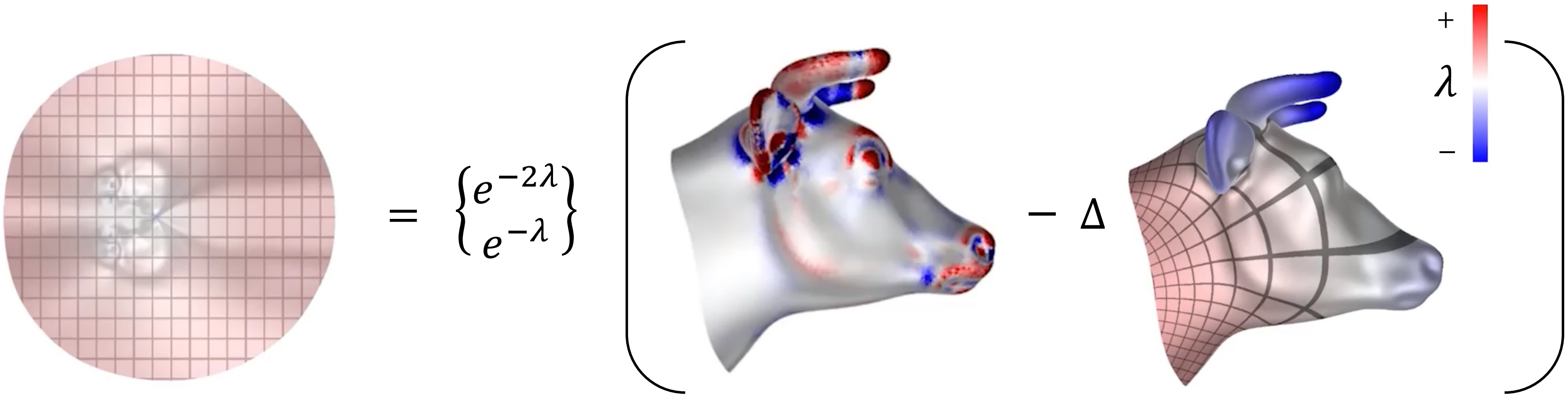
Vaxman, A. et al. (2015). Conformal mesh deformations with Möbius transformations. ACM Transactions on Graphics.

4

Ricci flow and Calabi flow

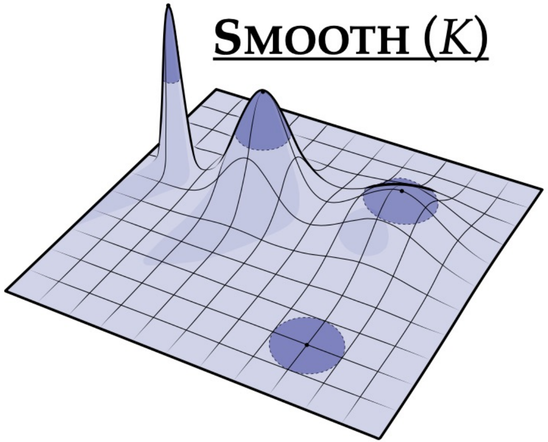
Ricci flow

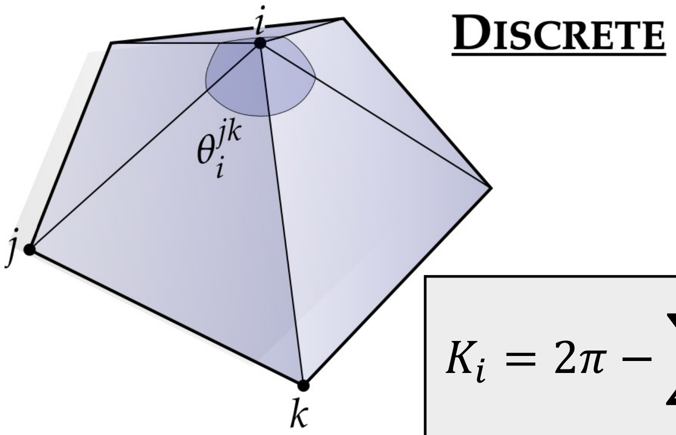
- Ricci energy: $E(\mathrm{g}) = \int (K(\mathrm{g}) + |\nabla f|^2) e^{-f} d\mu$, f dilaton function
- Ricci flow (gradient flow): $\frac{\partial \mathrm{g}}{\partial t} = -\nabla E = -2(K(\mathrm{g}) - K')\mathrm{g}$
- Conformal metric: $\mathrm{g} = e^{2\lambda}\mathrm{g}^0 \implies E(\lambda)$ convex and $\frac{\partial \lambda}{\partial t} = K' - K(\lambda)$



Discrete Ricci flow

- Log conformal factor: $\lambda_i : \boldsymbol{v}_i \in V \rightarrow \mathbb{R}, \forall\, i$
- Discrete Ricci flow: $\frac{\partial \lambda_i}{\partial t} = K_i' - K_i$



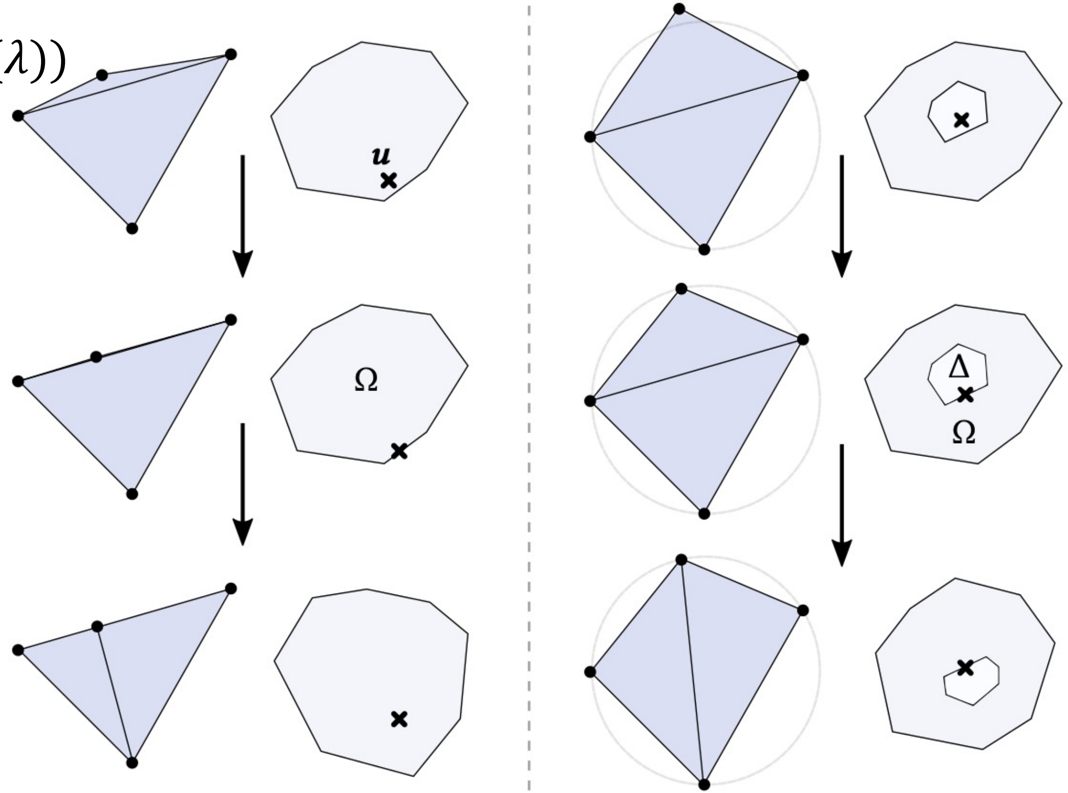


$$K_i = 2\pi - \sum_{t_{ijk} \in St(i)} \theta_i^{jk}$$

Yang, Yong-Liang, et al. "Generalized discrete Ricci flow." *Computer Graphics Forum*. Vol. 28. No. 7. Oxford, UK: Blackwell Publishing Ltd, 2009.

Gradient descent

- Update : $\lambda_i \leftarrow \lambda_i + t(K' - K(\lambda))$
- From λ_i to compute K_i
  - $\theta_i^{jk} = \arccos\frac{l_{ij}^2+l_{ik}^2-l_{jk}^2}{2l_{ij}l_{ik}}$
  - $K_i = 2\pi - \sum_{t_{ijk}\in St(i)} \theta_i^{jk}$
- Dynamic triangulation
  - Flip-on-degeneration
  - Flip-on-Delaunay-violation



Yang, Yong-Liang, et al. "Generalized discrete Ricci flow." *Computer Graphics Forum*. Vol. 28. No. 7. Oxford, UK: Blackwell Publishing Ltd, 2009.

Gradient descent

- Update : $\lambda_i \leftarrow \lambda_i + t(K' - K(\lambda))$
- From λ_i to compute K_i
  - $\theta_i^{jk} = \arccos \frac{l_{ij}^2 + l_{ik}^2 - l_{jk}^2}{2 l_{ij} l_{ik}}$
  - $K_i = 2\pi - \sum_{t_{ijk} \in St(i)} \theta_i^{jk}$
- Dynamic triangulation
  - Flip-on-degeneration
  - Flip-on-Delaunay-violation

1. Initialize: $\lambda_i = 0, \quad l_{ij} = l_{ij}^0$
2. Computing θ_i^{jk} and K_i using l_{ij}
3. If $\|K' - K\| < \epsilon$, terminate
4. Update $\lambda_i \leftarrow \lambda_i + t(K' - K)$
5. Update $l_{ij} = e^{\frac{\lambda_i + \lambda_j}{2}} l_{ij}^0$ and dynamic triangulation
6. Repeat 2-5

Yang, Yong-Liang, et al. "Generalized discrete Ricci flow." *Computer Graphics Forum*. Vol. 28. No. 7. Oxford, UK: Blackwell Publishing Ltd, 2009.

Newton method

- Gradient:

$$\nabla E(\lambda_i) = K_i - K_i'$$

- Hessian matrix

$$H_{ij} = \frac{\partial K_i}{\partial \lambda_j} = \Delta_{ij}$$

- Update :

$$\lambda_i \leftarrow \lambda_i + t\big(\Delta^{-1}(K' - K)\big)_i$$

1. Initialize： $\lambda_i = 0,\quad l_{ij} = l_{ij}^0$
2. Computing θ_i^{jk}, K_i and Δ using l_{ij}
3. If $\|K' - K\| < \epsilon$ ， terminate
4. Update $\lambda_i \leftarrow \lambda_i + t\big(\Delta^{-1}(K' - K)\big)_i$
5. Update $l_{ij} = e^{\frac{\lambda_i + \lambda_j}{2}} l_{ij}^0$ and dynamic triangulation
6. Repeat 2-5

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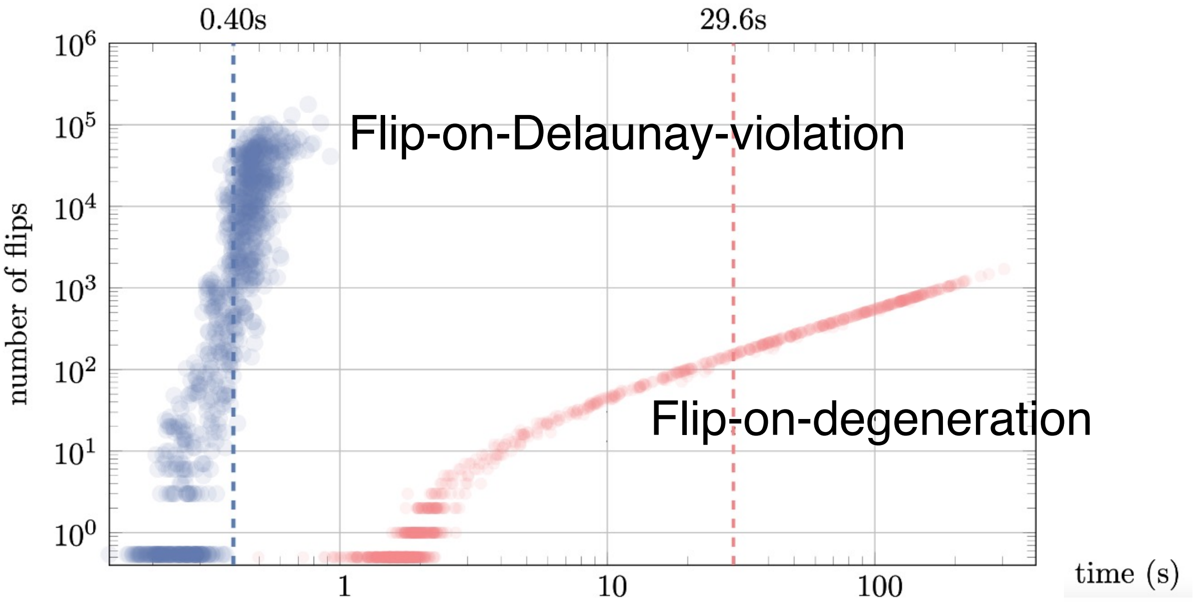
Dynamic triangulation

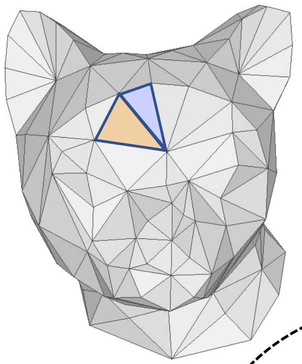
- Dynamic triangulation
  - Flip-on-degeneration
  - Flip-on-Delaunay-violation

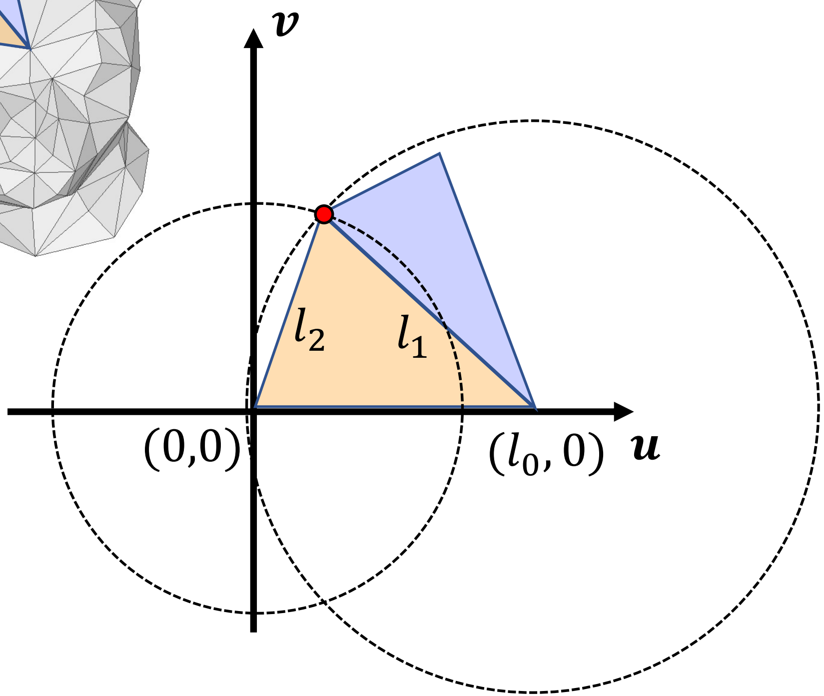
- Newton method
  - Local flip
  - Global solver

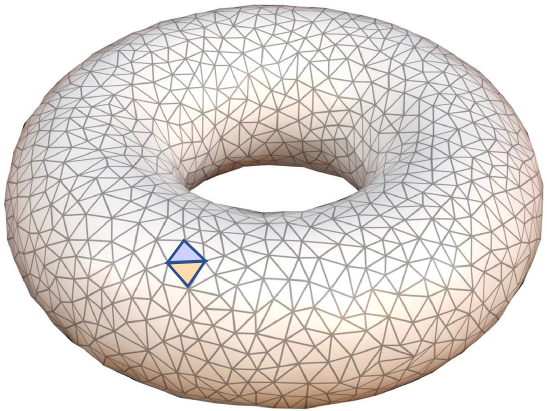


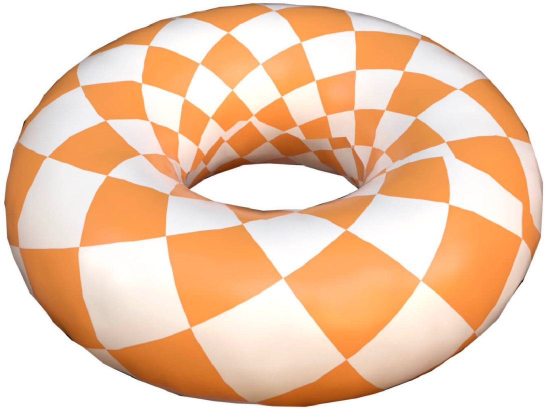
Campen, Marcel, et al. "Efficient and robust discrete conformal equivalence with boundary." ACM Transactions on Graphics (TOG) 40.6 (2021): 1-16.

Plane embedding





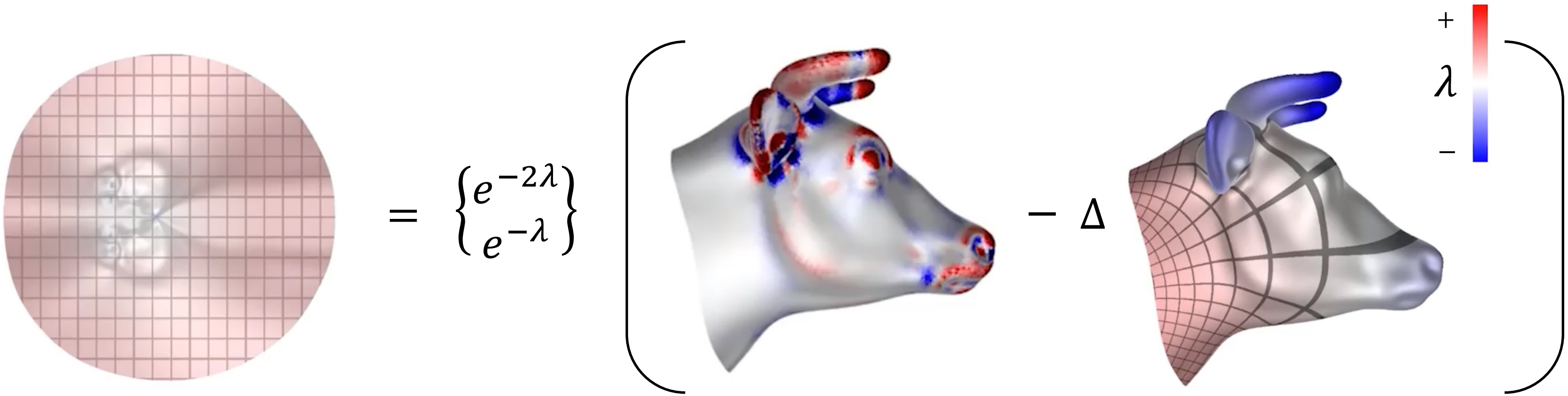




Yang, Yong-Liang, et al. "Generalized discrete Ricci flow." *Computer Graphics Forum*. Vol. 28. No. 7. Oxford, UK: Blackwell Publishing Ltd, 2009.

Calabi flow

- Calabi energy: $E(\mathrm{g}) = \int (K(\mathrm{g}) - K')^2 dA_{\mathrm{g}}$
- Calabi flow (gradient flow): $\frac{\partial \mathrm{g}}{\partial t} = -\nabla E = -2\Delta(K(\mathrm{g}) - K')\mathrm{g}$
- Conformal metric: $\mathrm{g} = e^{2\lambda}\mathrm{g}^0 \implies E(\lambda)$ convex and $\frac{\partial \lambda}{\partial t} = \Delta(K' - K(\lambda))$



Discrete Calabi flow

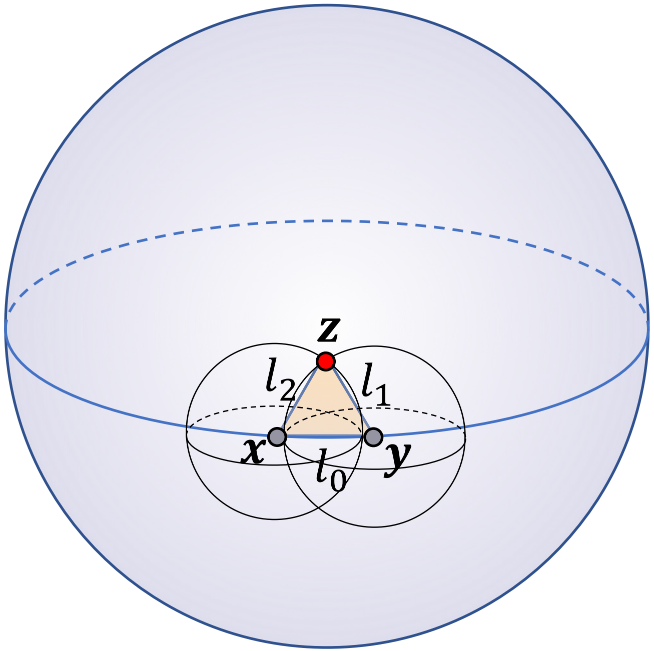
- Log conformal factor:
$$\lambda_i : \boldsymbol{v}_i \in V \to \mathbb{R}, \forall\ i$$
- Discrete Calabi flow:
$$\frac{\partial \lambda_i}{\partial t} = \left(\Delta(K' - K)\right)_i$$
- Gradient descent
- Approximate Newton method
$$H = \frac{\partial(\Delta(K-K'))}{\partial \lambda}$$
$$= \Delta^2 + \frac{\partial \Delta}{\partial \lambda}(K - K')$$
$$\approx \Delta^2$$

1. Initialize：$\lambda_i = 0, \quad l_{ij} = l_{ij}^0$
2. Computing θ_i^{jk}, K_i and Δ using l_{ij}
3. If $\|K' - K\| < \epsilon$, terminate
4. Update $\lambda_i \leftarrow \lambda_i + t\left((\Delta \text{ or } \Delta^{-1})(K' - K)\right)_i$
5. Update $l_{ij} = e^{\frac{\lambda_i + \lambda_j}{2}} l_{ij}^0$ and dynamic triangulation
6. Repeat 2-5

Su, K. H., et al. "Discrete calabi flow: A unified conformal parameterization method." *Computer Graphics Forum*. Vol. 38. No. 7. 2019.

Sphere embedding

- Intersection of spheres
  - $$\begin{cases} \|\boldsymbol{z}-\boldsymbol{y}\|^2 = l_1^2 \\ \|\boldsymbol{z}-\boldsymbol{x}\|^2 = l_2^2 \\ \|\boldsymbol{z}\|^2 = r^2 \end{cases}$$
- Intersection of line and sphere
  - $$\begin{cases} <\boldsymbol{z},\boldsymbol{y}> = r^2 - \frac{l_1^2}{2} \\ <\boldsymbol{z},\boldsymbol{x}> = r^2 - \frac{l_2^2}{2} \\ \|\boldsymbol{z}\|^2 = r^2 \end{cases}$$



Su, K. H., et al. "Discrete calabi flow: A unified conformal parameterization method." *Computer Graphics Forum*. Vol. 38. No. 7. 2019.

HW

Conjugate harmonic functions

Conjugate harmonic coordinates

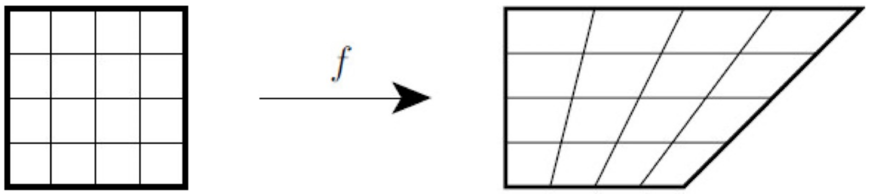
- Solving Laplacian equations:
  - For interior vertices $\begin{cases} \Delta u = 0 \\ \Delta v = 0 \end{cases}$
  - Boundary control

- Dirichlet boundary condition:
  - Boundary curve $\gamma: \partial M \to \mathbb{R}^2$

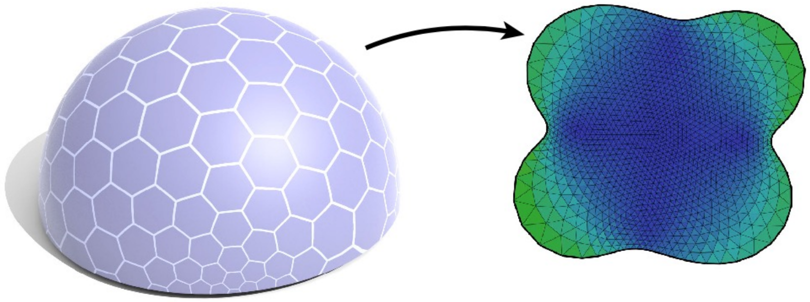
$$u\Big|_{\partial M} = \gamma_u, v\Big|_{\partial M} = \gamma_v$$

- Neumann boundary condition:
  - Boundary gradients $\mathrm{h}: \partial M \to \mathbb{R}^2$

$$\partial_M u = h_u, \partial_M v = h_v$$



Hamonic, not conformal

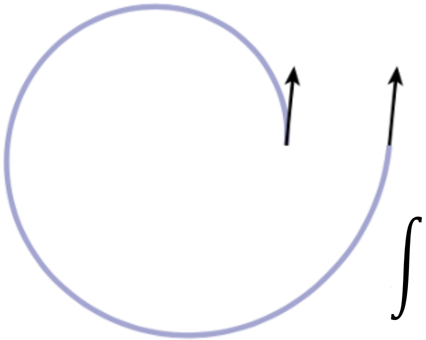


Conformal, conjugate gradients

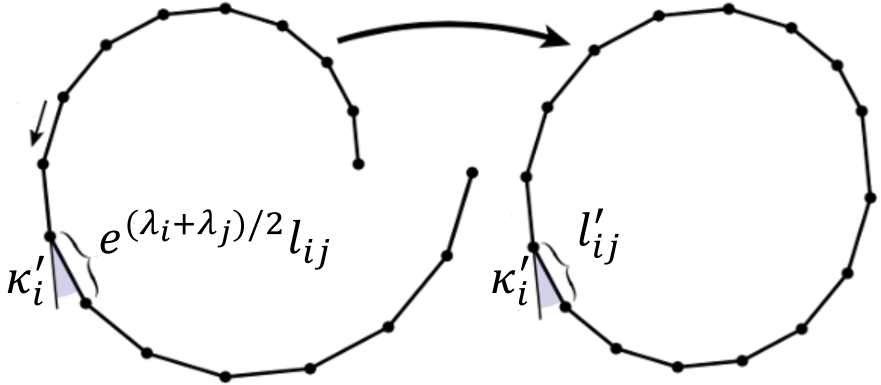
Sawhney, R., & Crane, K. (2017). Boundary first flattening. ACM Transactions on Graphics.

Boundary condition

- Yamabe equations:
  - $\begin{cases} K' = e^{-2\lambda}(K - \Delta_g \lambda) \\ \kappa' = e^{-\lambda}(\kappa - \partial_{\boldsymbol{n}}^M \lambda) \end{cases}$
- Integration:
  - $\begin{cases} \Delta_g \lambda dA = K dA - K' e^{2\lambda} dA \\ \partial_{\boldsymbol{n}}^M \lambda ds = \kappa - \kappa' e^{\lambda} ds \end{cases}$
- Discretization:
  - $(\Delta\lambda)_i = K_i - K_i'$
  - $l_{ij}' = e^{(\lambda_i+\lambda_j)/2} l_{ij}$?



$$\int (I + \delta) \frac{\partial \gamma}{\partial s} ds$$



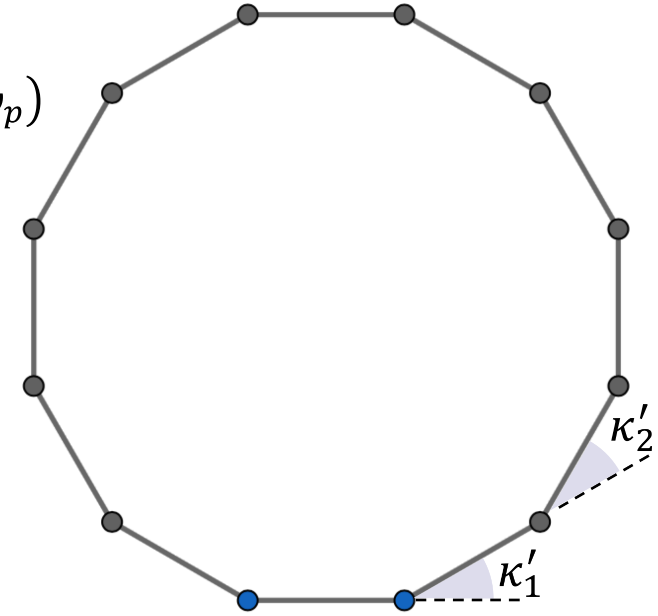
Sawhney, R., & Crane, K. (2017). Boundary first flattening. ACM Transactions on Graphics.

Boundary optimization

- Geodesic curvature κ_i'
  - Cumulative angle: $\psi_p = \sum_{i=1}^{p-1} \kappa_i'$
  - Unit tangent vector: $T_{ij=\gamma_{p\to p+1}} = \left(\cos\psi_p, \sin\psi_p\right)$

- Formulation:
  - Energy: $\sum_{ij\in\partial M} l_{ij}^{-1}\left(l_{ij}' - e^{\frac{\lambda_i+\lambda_j}{2}} l_{ij}\right)^2$
  - Constraint: $\sum_{ij\in\partial M} l_{ij}' T_{ij} = \mathbf{0}$

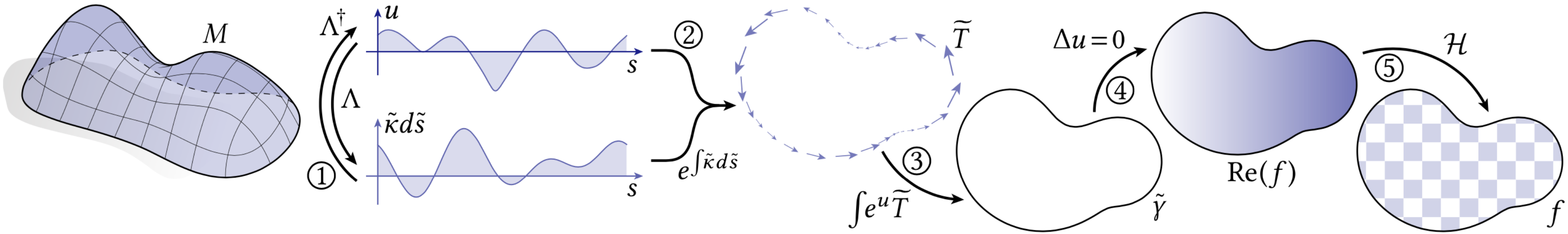
- Boundary curve:
  - $(u_1, v_1) = (0, 0)$
  - $(u_p, v_p) = \sum_{ij\in\gamma_{1\to p}} l_{ij}' T_{ij}$



Sawhney, R., & Crane, K. (2017). Boundary first flattening. ACM Transactions on Graphics.

Boundary first flattening

- For boundary, specify either length (or curvature) of target curve
- Solve Yamabe problem to get complementary data
- Optimize boundary data to get close boundary curve
- Solve conjugate harmonic coordinates



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