中国科学技术大学
University of Science and Technology of China

GAMES 301: 第13讲

参数化应用 – 曲面对应与高阶多项式映射

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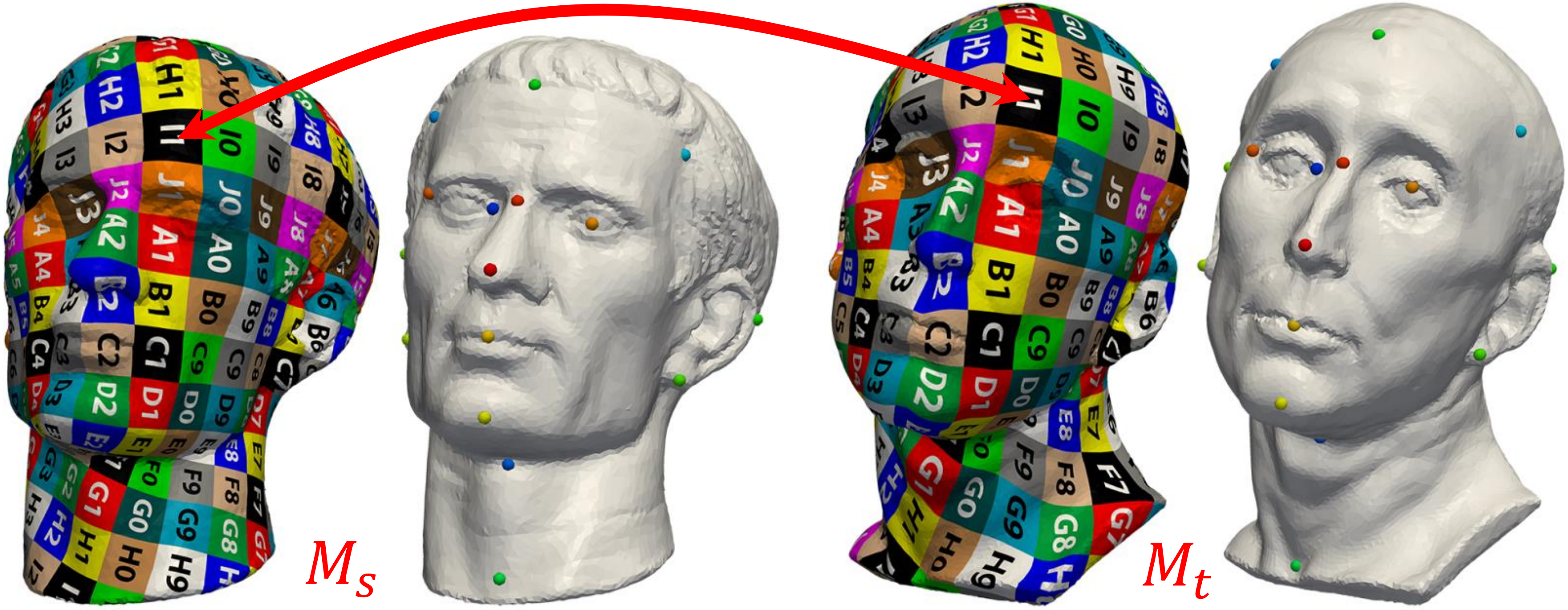
Inter-surface mappings

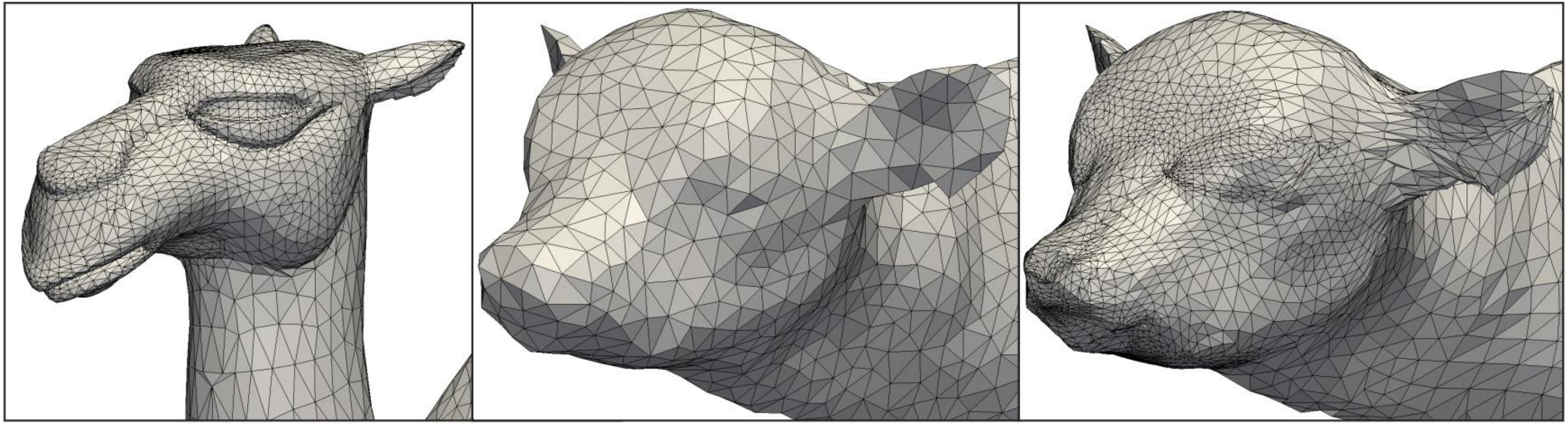
Surface Mapping

- Inter-surface mapping, Cross parameterization
- A one-to-one mapping f between two surfaces M_s and M_t



Compatible meshes

- Meshes with identical connectivity (M_s and $\widehat{M_t}$)



M_s M_t $\widehat{M_t} = f(M_s) \approx M_t$

Applications

- Morphing
- Attribute transfer
- ……

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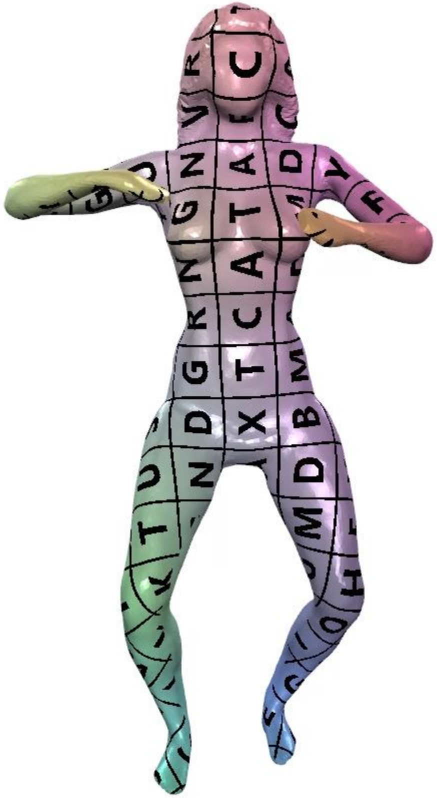
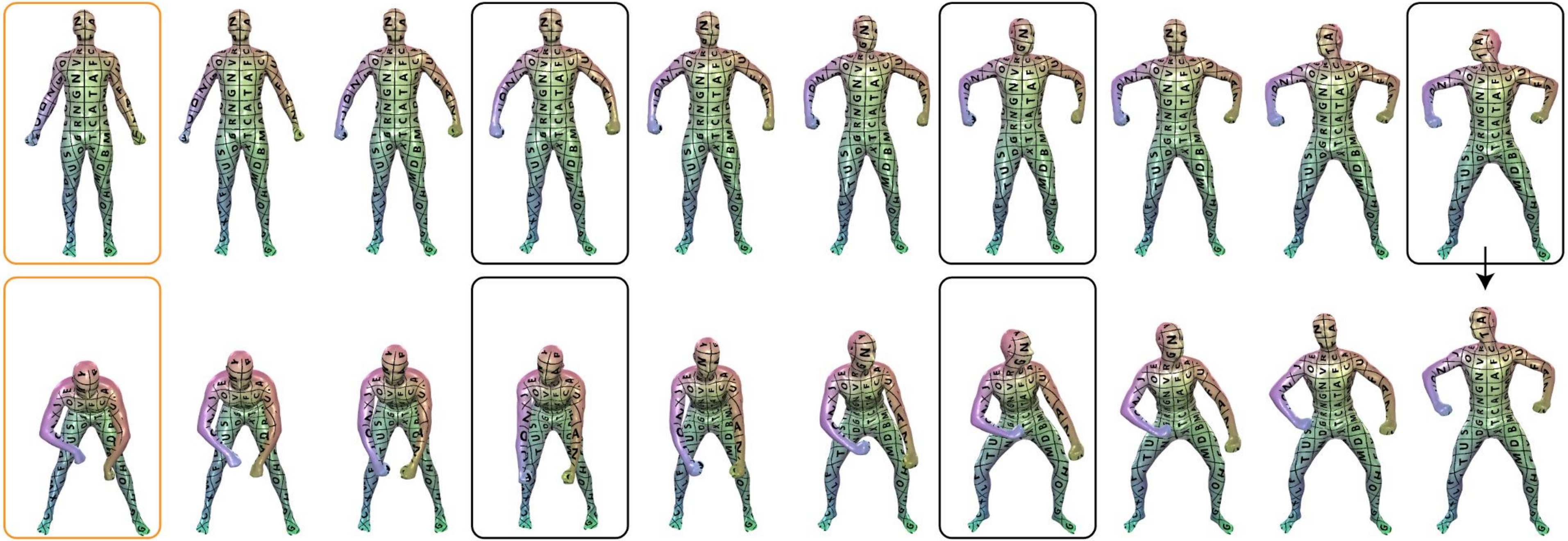
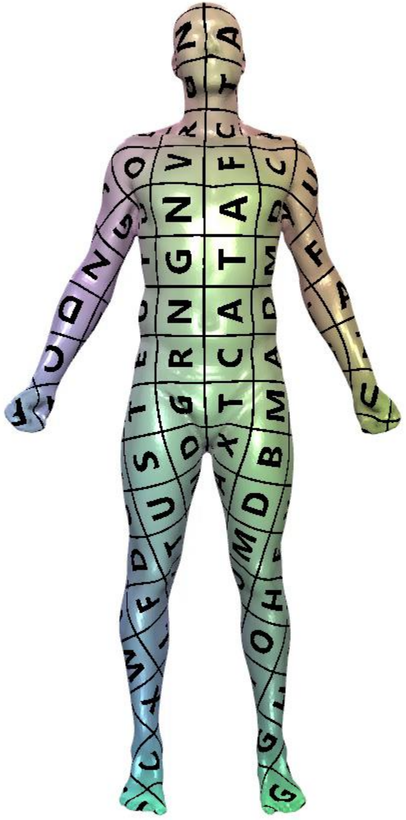
- ……

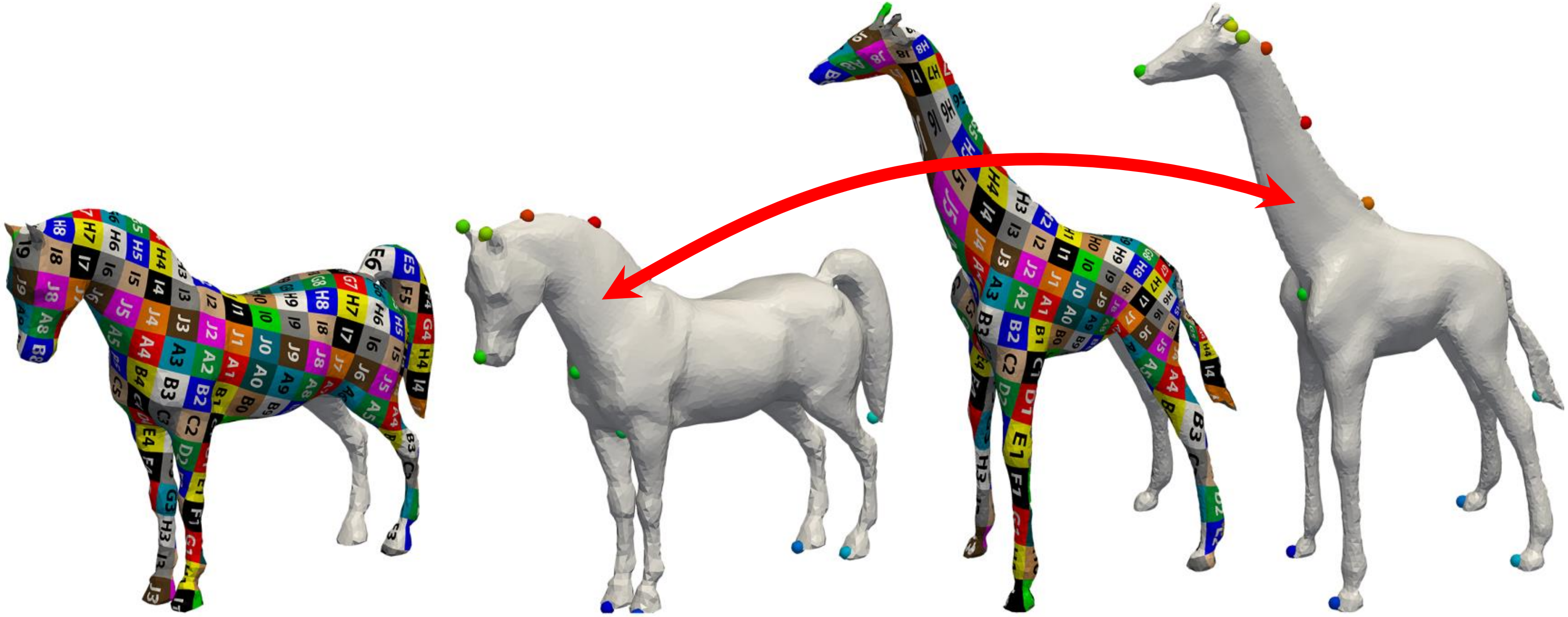






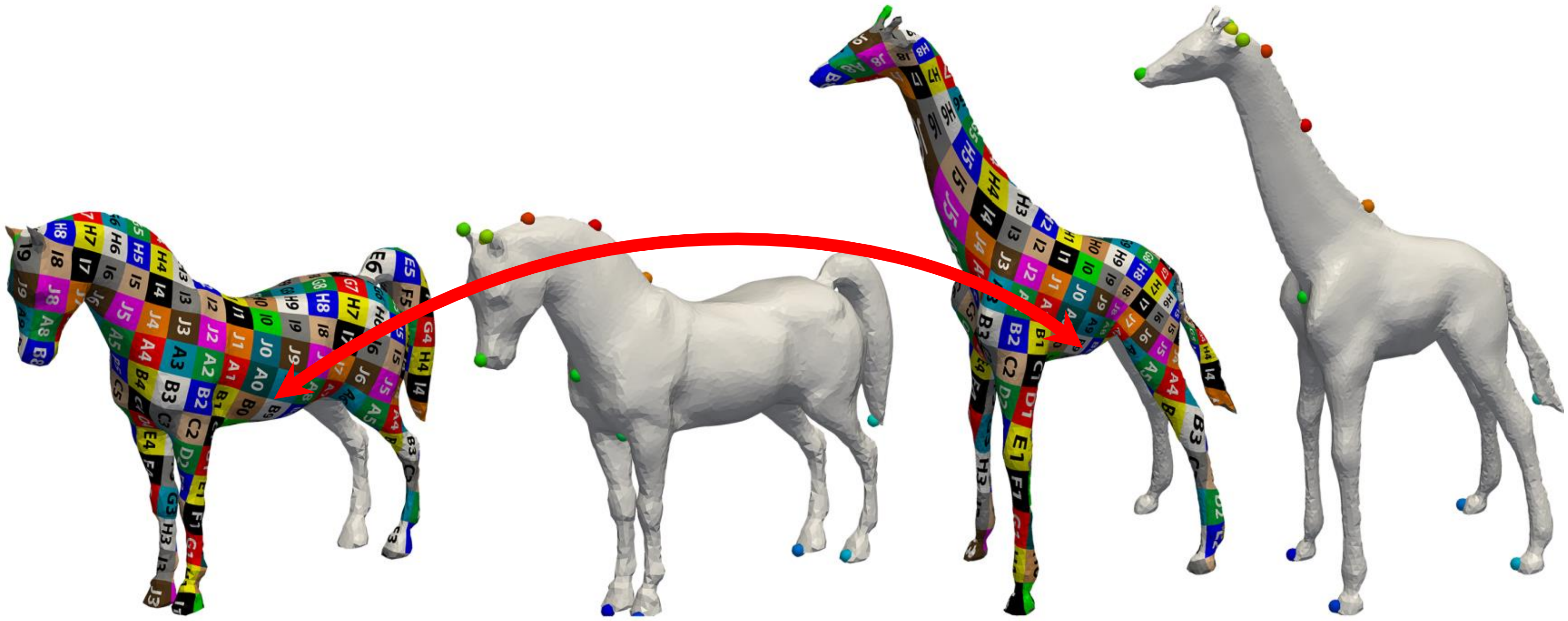
Inputs

- Two (n) models and some corresponding landmarks



Goal

- Bijection and low distortion

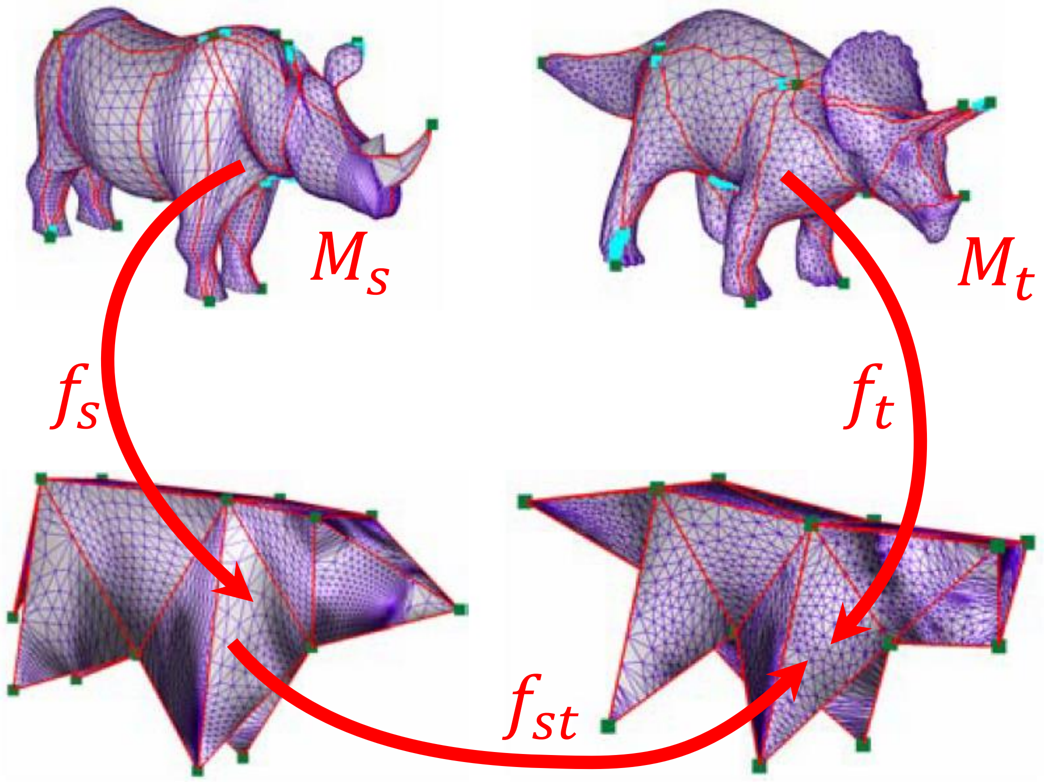


Algorithm stages

- Construct a common base domain
  - Topologically identical triangular layouts of the two meshes.
- Compute a low distortion cross-parameterization
  - Each patch is mapped to the corresponding base mesh triangle.
- Compatibly remesh the input models using the parameterizations

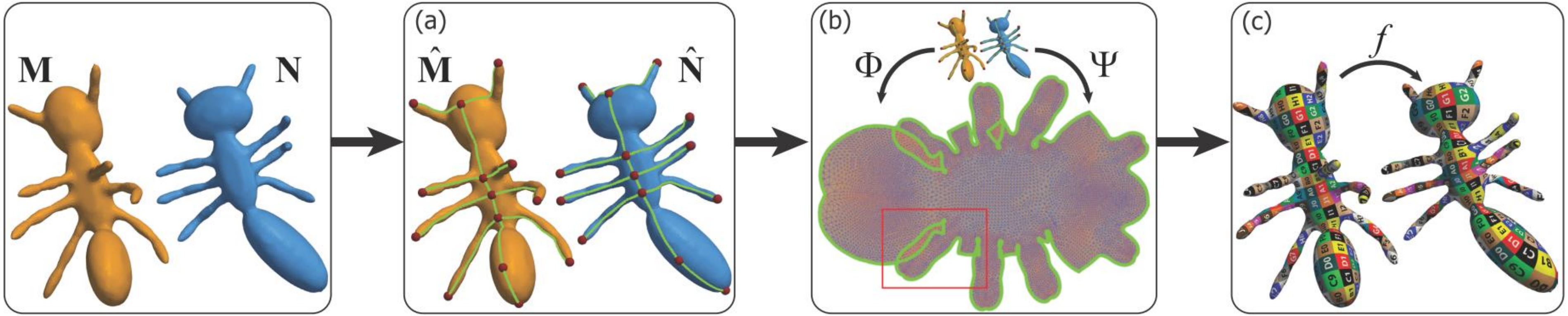
One common base domain

- $f = f_t^{-1} \circ f_{st} \circ f_s$



Algorithm steps

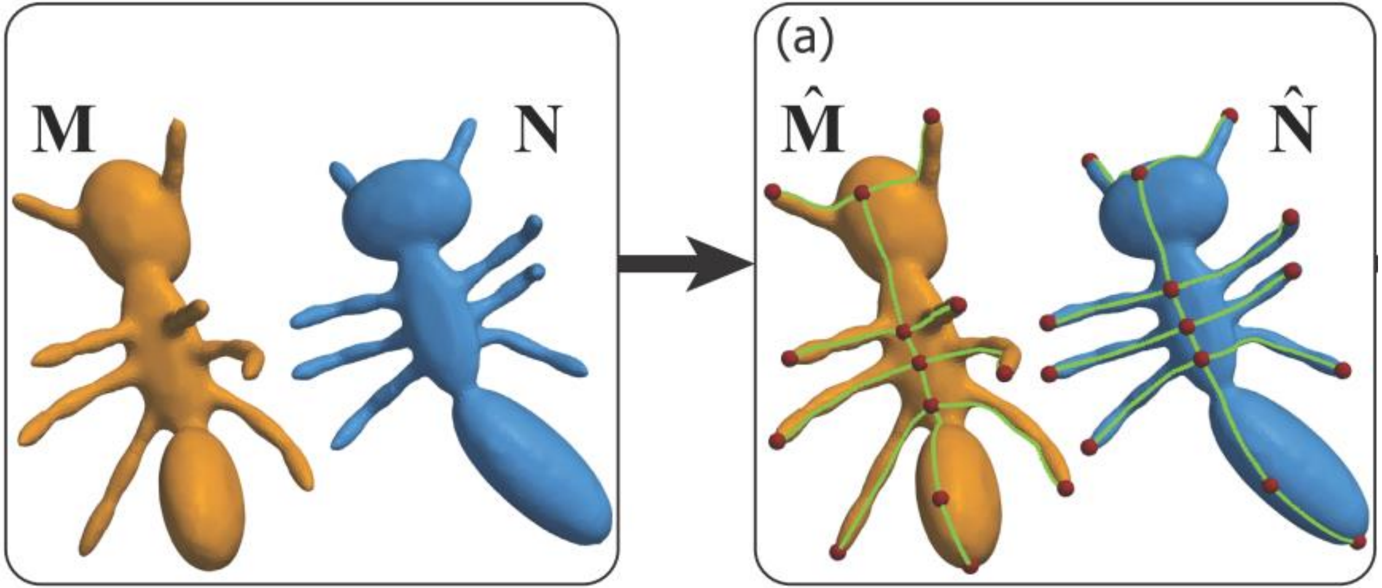
- (a) Cutting to disk topology.
- (b) Computing the joint flattenings Φ, Ψ.
- (c) Bijection Lifting.



Aigerman, Noam, Roi Poranne, and Yaron Lipman. "Lifted bijections for low distortion surface mappings." *ACM Transactions on Graphics (TOG)* 33.4 (2014): 1-12.

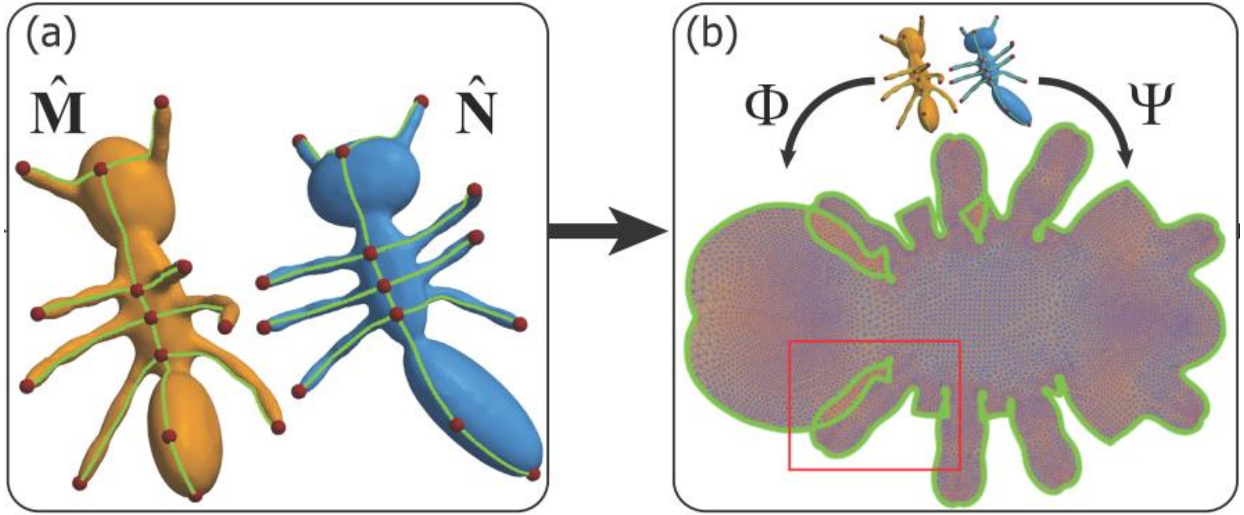
Cutting paths

- Bijective correspondence
  - Shortest path
  - Minimal spanning tree



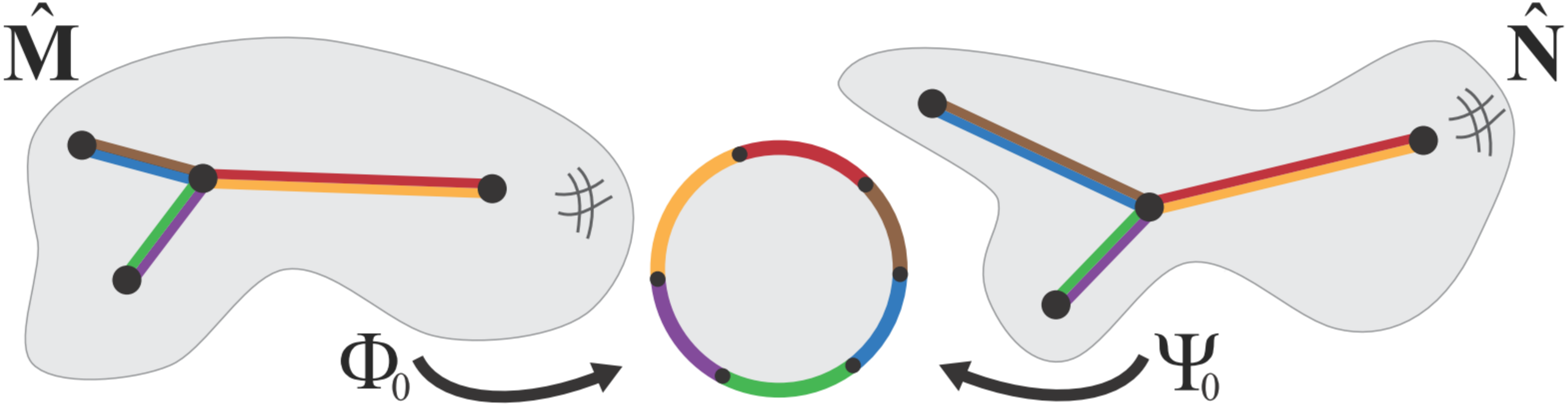
Computing Φ, Ψ

- Constraint
  - **Common boundary condition**
  - Locally injective
- Solvers:
  - Former methods



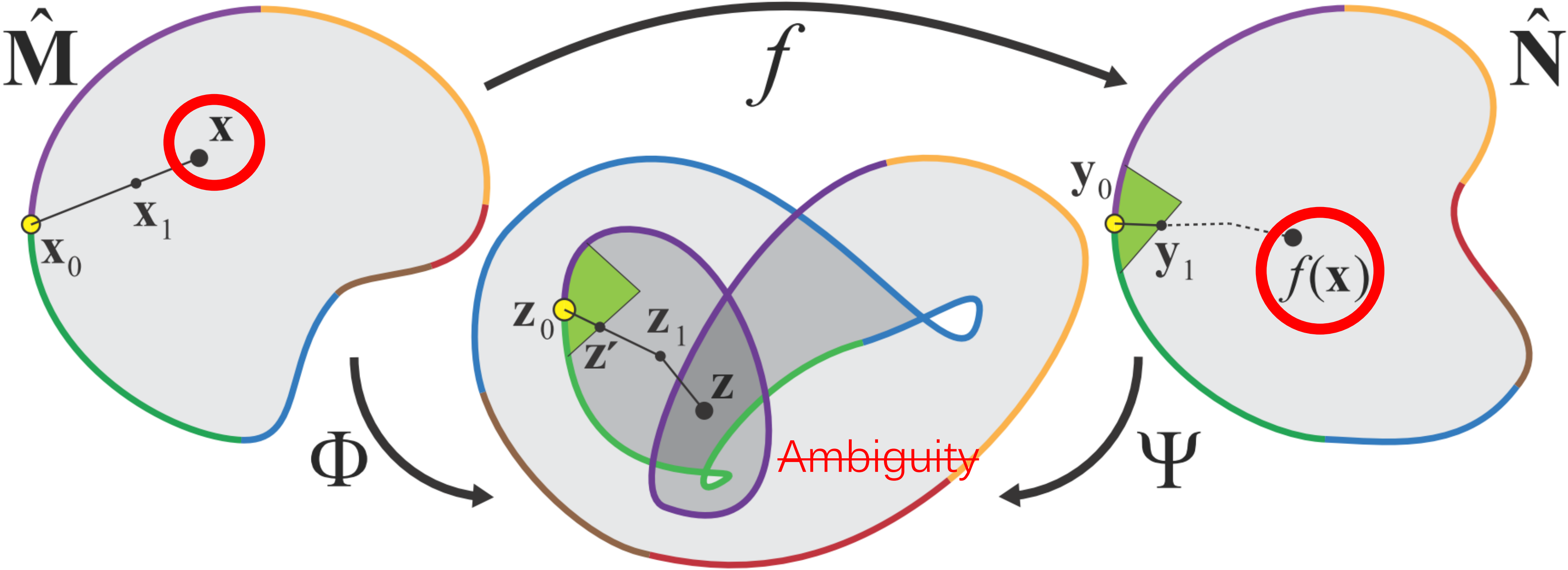
Bijection Lifting

- Bijective parameterizations

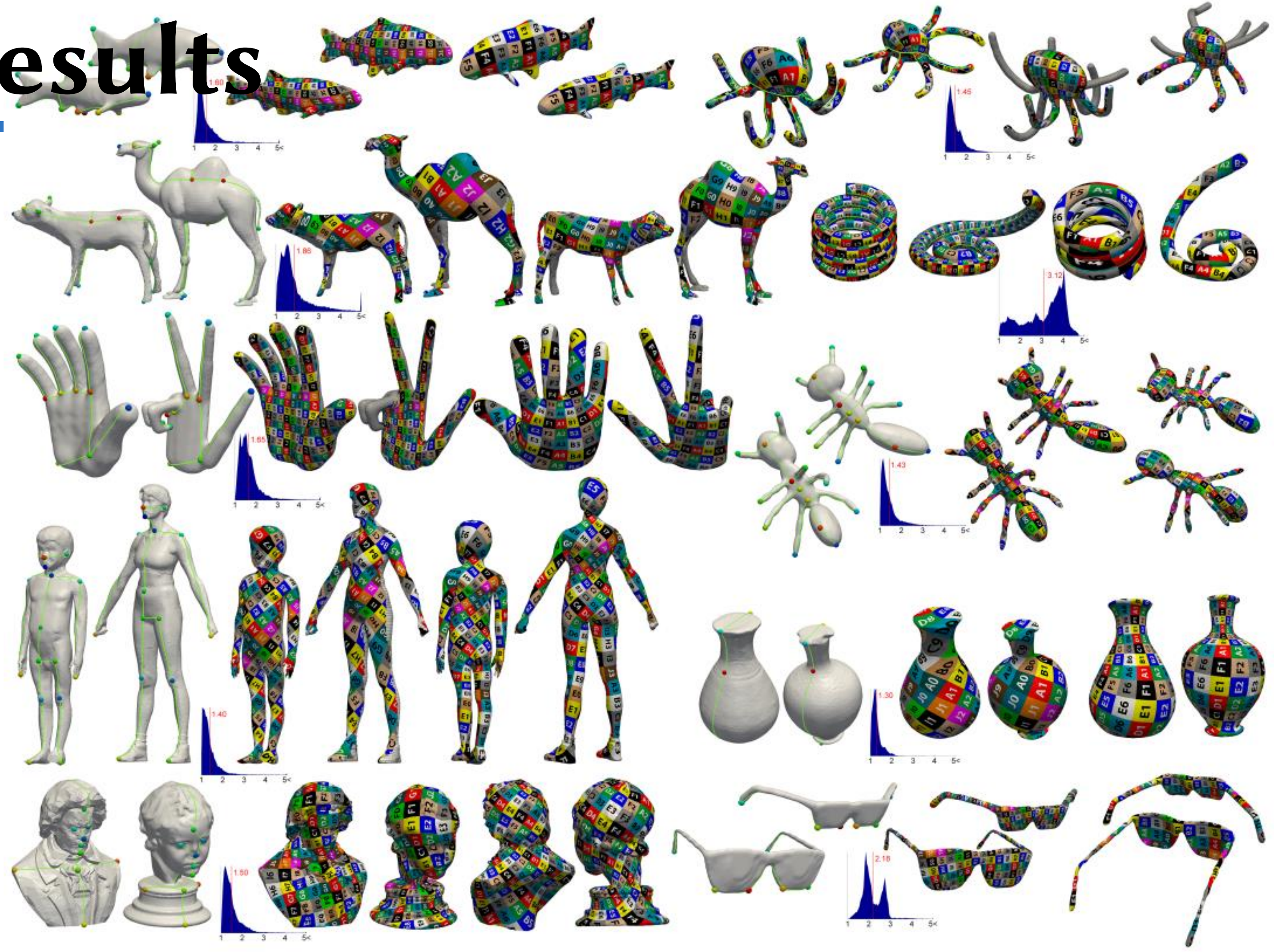


Bijection Lifting

- Only locally injective constrains

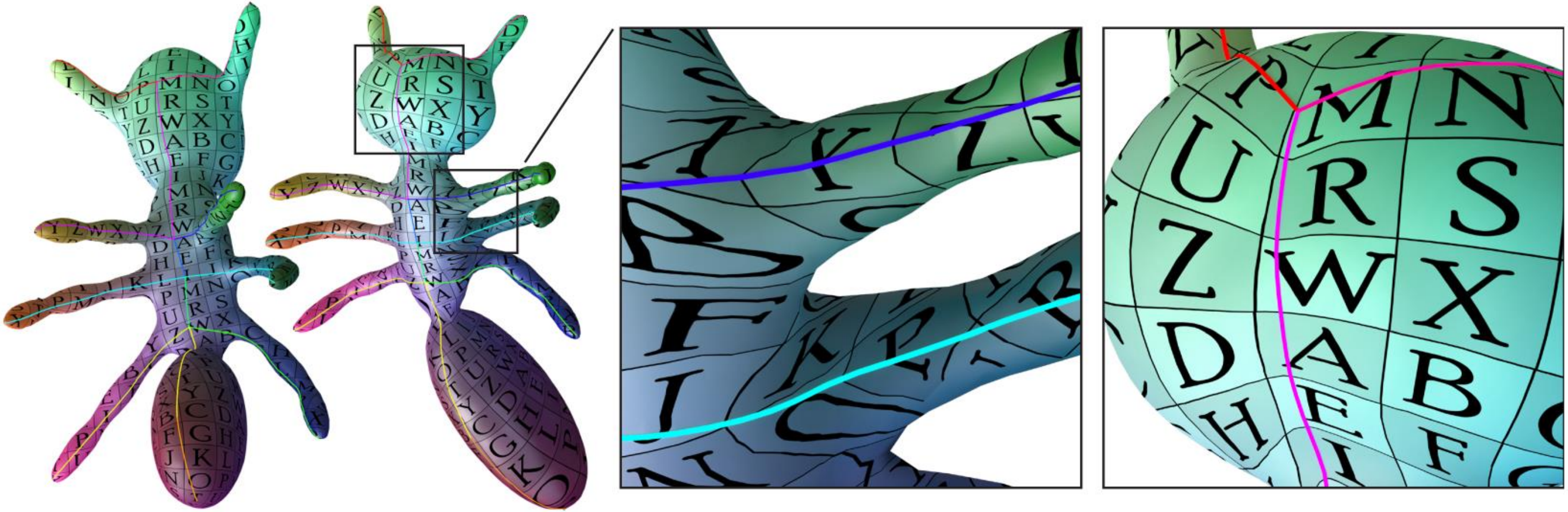


Results



Disadvantages

- Cut-dependent



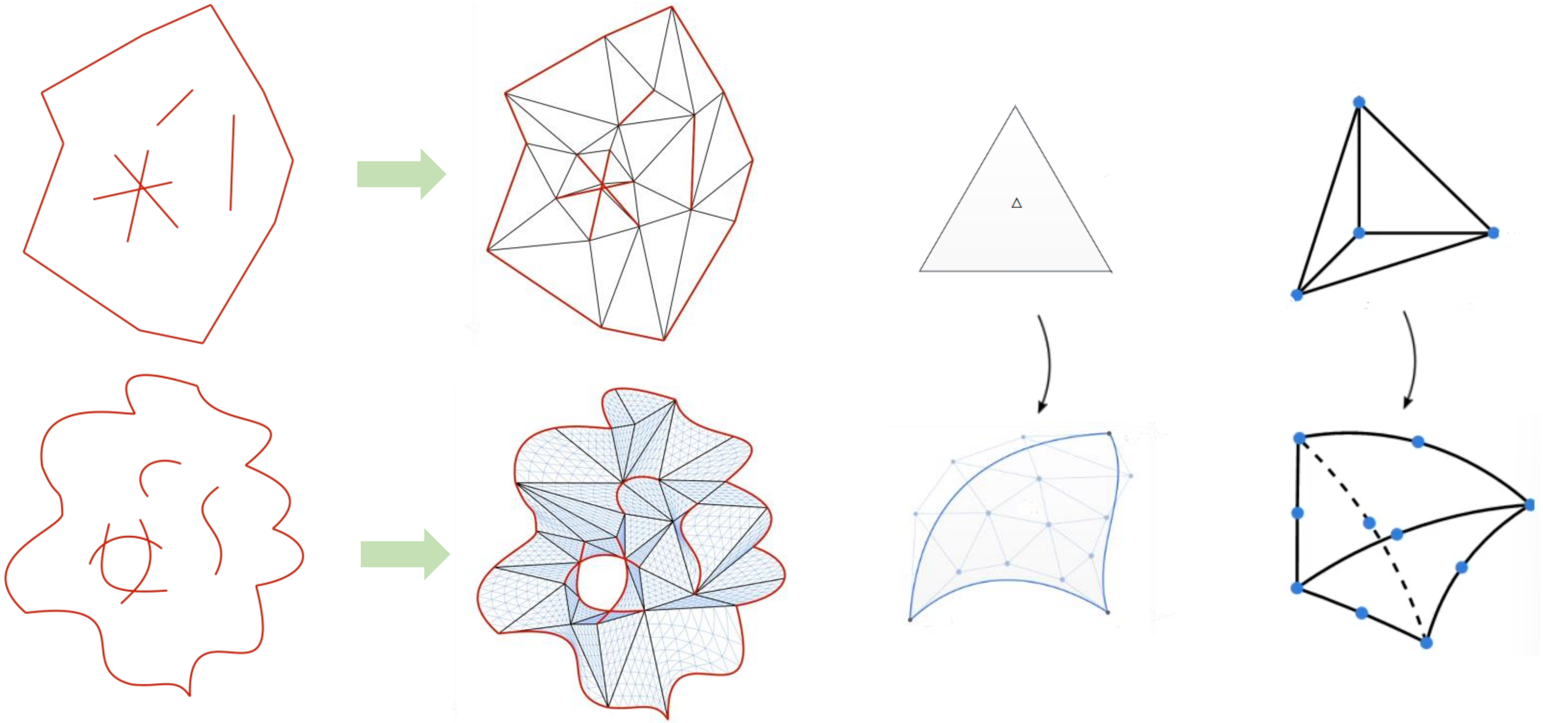
More methods

- Inter-Surface Mapping, 2004
- Functional Maps: A Flexible Representation of Maps Between Shapes, 2012
- Hyperbolic Orbifold Tutte Embeddings, 2016
- Variance-Minimizing Transport Plans for Inter-surface Mapping, 2017
- ……

High-order polynomial mappings

for high-order meshing

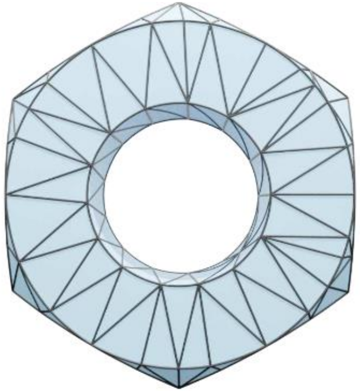
High-order meshes

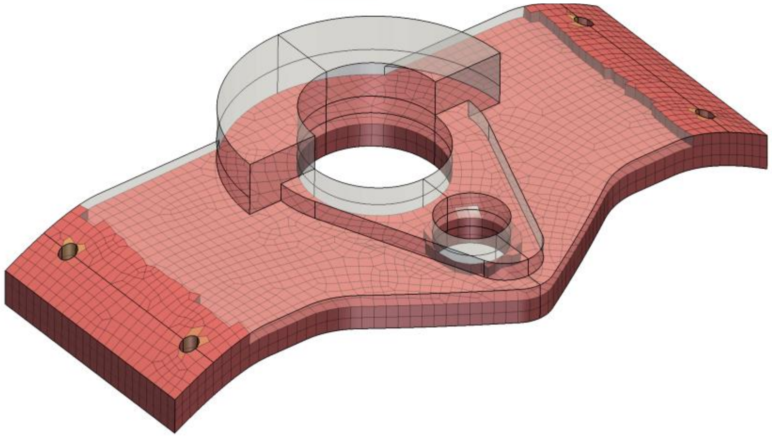


High-order meshes

- Ready to approximate complex curved domains
- Feature low numerical dissipation and dispersion
- Faster than low-order methods

High-order meshes



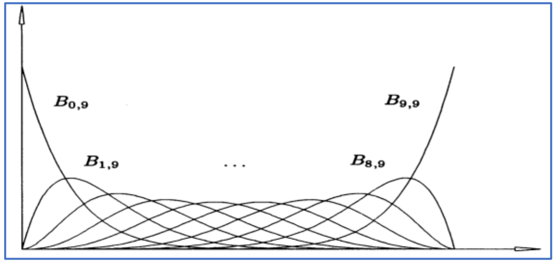



High-order basis

- Bézier basis:

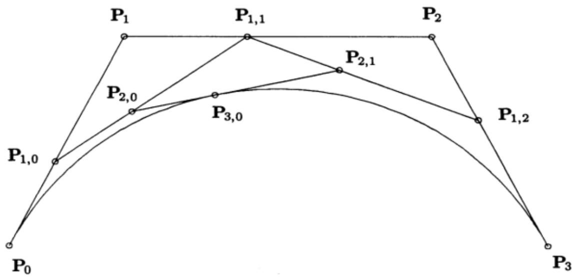
$$B_i^n(t) = C_n^i t^i (1-t)^{n-i}$$

$$B_i^n(t) = (1-t)B_i^{n-1}(t) + tB_{i-1}^{n-1}(t)$$



- Function: $f(t) = \sum_{i=0}^{n} p_i B_i^n(t)$

非负性，归一性，迭代定义



凸包性，端点插值，细分求值

High-order basis

- Lagrange basis: $l_i^n(t) = \prod_{\substack{j=0 \\ i \neq j}}^{n} \frac{t - t_j}{t_i - t_j}$
- Function: $f\left(t_j\right) = \sum_{i=0}^{n} p_i l_i^n(t_i) = p_j$

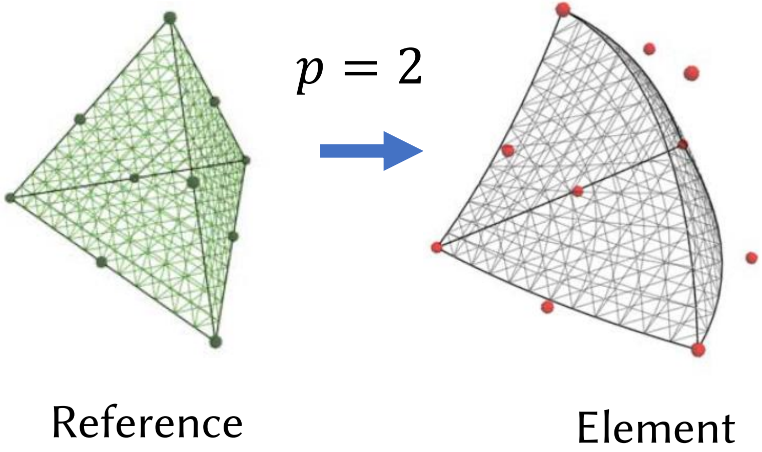
$l_i^n\left(t_j\right) = \delta_{ij}$

节点插值

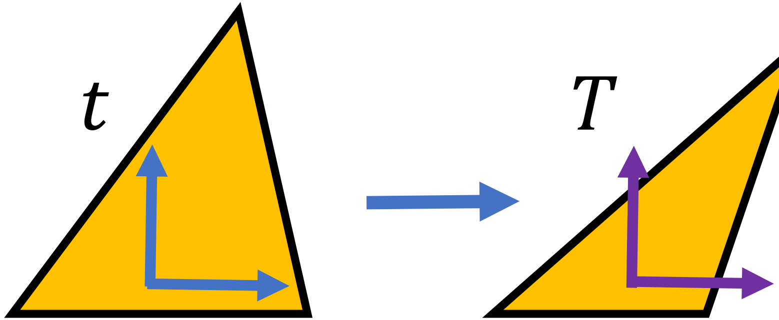
- Others:
  - monomial basis,
  - orthogonal basis of polynomials
  - …

Elements

- Tetrahedron:
  - $f_t(\boldsymbol{\xi})=\sum_{i=1}^{N_p} B_i^p(\boldsymbol{\xi})\,\boldsymbol{P}_i$
  - B_i^p: Bézier basis
  - $\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2, 1 - \xi_0 - \xi_1 - \xi_2)$
  - $\boldsymbol{P}_i$: control points

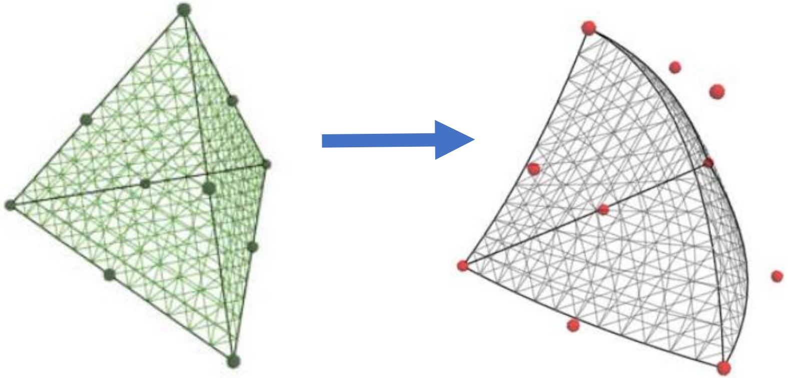


Jacobian



$$f_t(\boldsymbol{\xi}) = J_t\boldsymbol{\xi} + \boldsymbol{b}_t$$

$$det(J_t(\boldsymbol{\xi})) = const$$

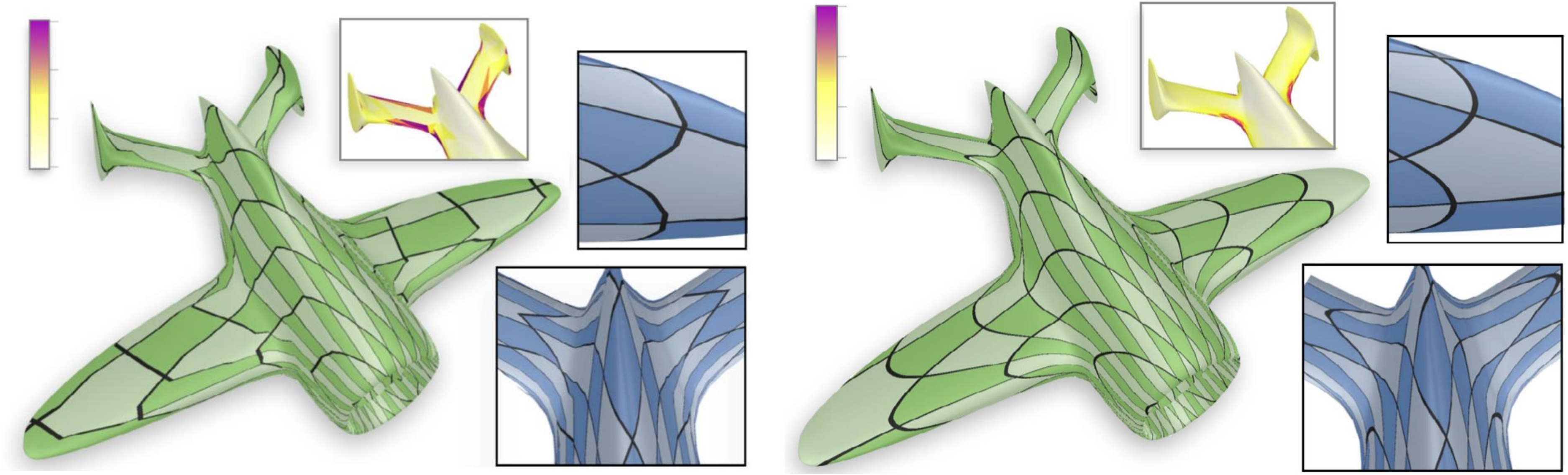


$$f_t(\boldsymbol{\xi}) = \sum_{i=1}^{N_p} B_i^p(\boldsymbol{\xi})\,\boldsymbol{P}_i$$

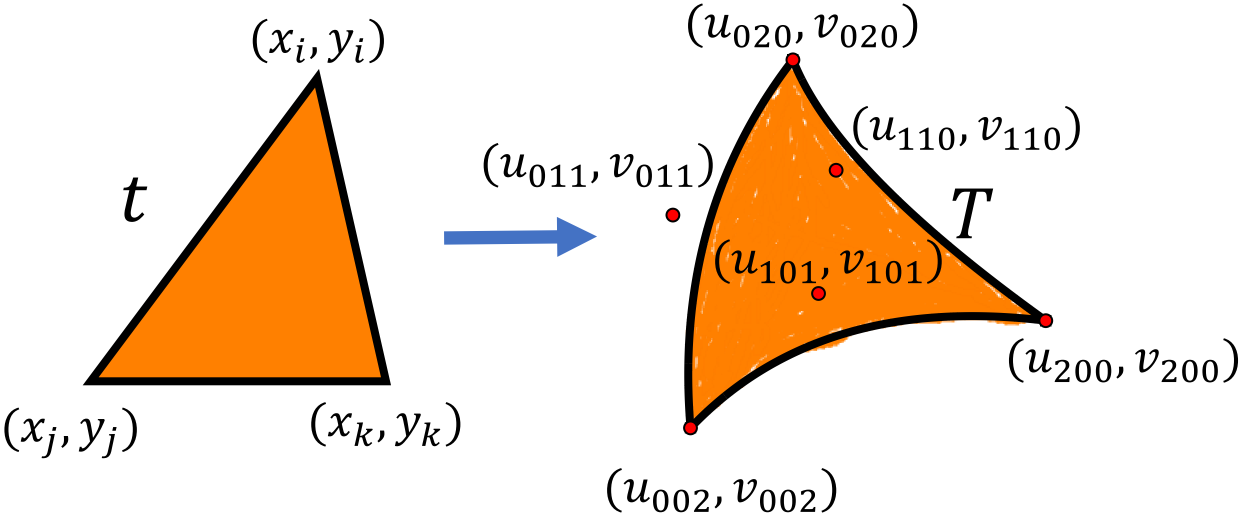
$$f_{t,\xi} = \left(\frac{\partial f_t}{\partial \xi_0}, \frac{\partial f_t}{\partial \xi_1}, \frac{\partial f_t}{\partial \xi_2}\right)$$

$$det(f_{t,\xi}) = poly(\boldsymbol{\xi})$$

High-order parametrization



High-order parametrization



$\frac{n(n+1)}{2}$ control points

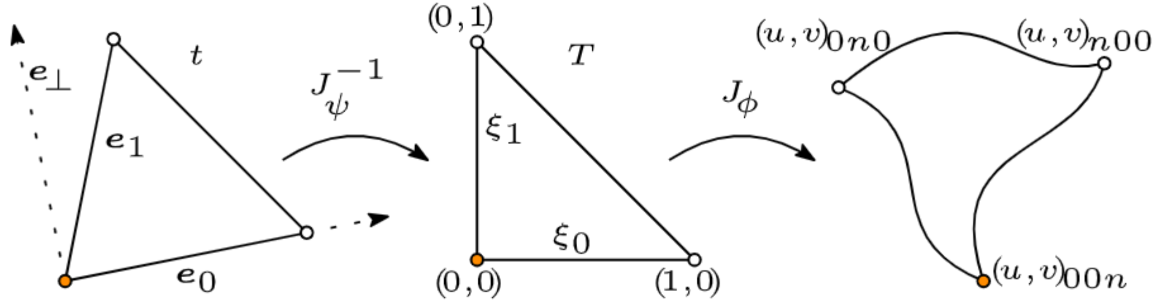
$$P_{ijk} = \begin{pmatrix} u_{ijk} \\ v_{ijk} \end{pmatrix}$$

$$f_t(\boldsymbol{\xi}) = \begin{pmatrix} u(\boldsymbol{\xi}) \\ v(\boldsymbol{\xi}) \end{pmatrix} = \sum_{i+j+k=n} \begin{pmatrix} u_{ijk} \\ v_{ijk} \end{pmatrix} B^n_{ijk}(\boldsymbol{\xi})$$

$$\boldsymbol{\xi} = (\xi_0, \xi_1, \xi_2) \quad B^n_{ijk}(\boldsymbol{\xi}) = \frac{n!}{i!j!k!} \xi_0^i \xi_1^i \xi_2^i$$

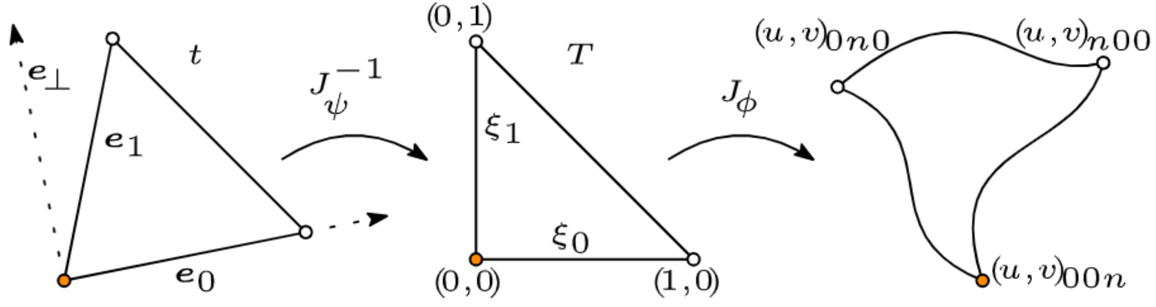
High-order parametrization

$$f_t = \phi \circ \psi^{-1}, \psi: T \to t, \phi: T \to R^2$$



$$J_{f_t}(\xi) = J_\phi(\xi) J_\psi^{-1}, \quad J_\psi = \begin{bmatrix} e_0^T e_0 & e_1^T e_0 \\ 0 & e_1^T e_\perp \end{bmatrix}$$

High-order parametrization



$$J_\phi = \begin{bmatrix} \frac{\partial u}{\partial \xi_0}(\boldsymbol{\xi}) & \frac{\partial u}{\partial \xi_1}(\boldsymbol{\xi}) \\ \frac{\partial v}{\partial \xi_0}(\boldsymbol{\xi}) & \frac{\partial v}{\partial \xi_1}(\boldsymbol{\xi}) \end{bmatrix},$$

$$\frac{\partial u}{\partial \xi_0}(\xi) = \sum_{i+j+k=n-1} \left(u_{(i+1)jk} - u_{ij(k+1)}\right) B_{ijk}^{n-1}(\boldsymbol{\xi})$$

$$\frac{\partial u}{\partial \xi_1}(\xi) = \sum_{i+j+k=n-1} \left(u_{i(j+1)k} - u_{ij(k+1)}\right) B_{ijk}^{n-1}(\boldsymbol{\xi})$$

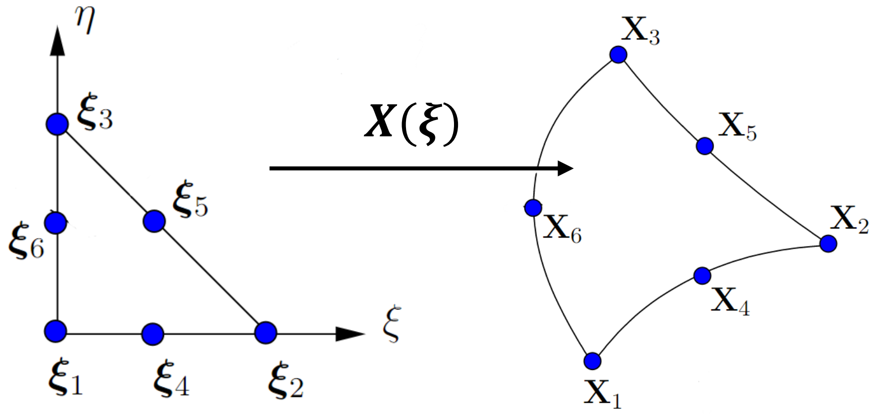
Validity condition

Notation

$$\boldsymbol{X}(\xi) = \sum_{i=1}^{N_p} L_i^p(\xi)\boldsymbol{X}_i$$

$$J(\xi) := \ det\ \boldsymbol{X}_{,\xi}$$

$$J_{min} := \ min_{\xi}\, J(\xi)$$

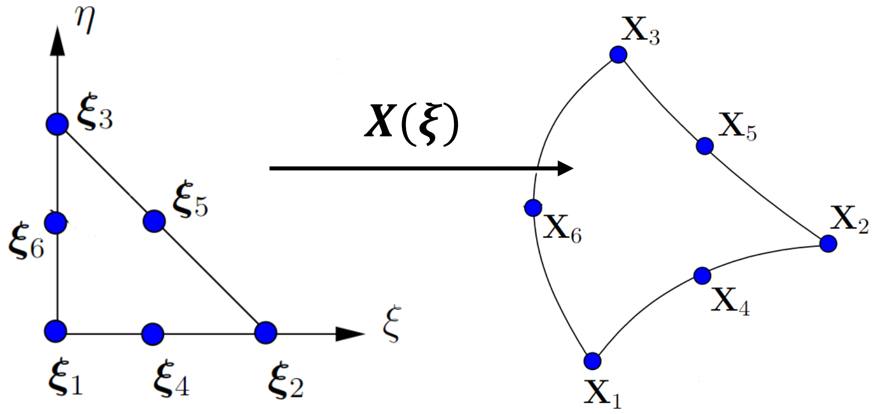


Lagrange basis

$$\boldsymbol{X}(\boldsymbol{\xi}) = \sum_{i=1}^{N_p} L_i^p(\boldsymbol{\xi})\boldsymbol{X}_i$$

$$J(\boldsymbol{\xi}) := \ det\ \boldsymbol{X}_{,\boldsymbol{\xi}}$$

$J_i = J(\xi, \eta)$ at node i



$$J(\xi,\eta) = J_1\underbrace{(1-\xi-\eta)(1-2\,\xi-2\,\eta)}_{L_1^2(\xi,\eta)} + J_2\,\underbrace{\xi(2\,\xi-1)}_{L_2^2(\xi,\eta)} + J_3\,\underbrace{\eta(2\,\eta-1)}_{L_3^2(\xi,\eta)} +$$

$$J_4\,\underbrace{4(1-\xi-\eta)\xi}_{L_4^2(\xi,\eta)} + J_5\ \underbrace{4\xi\eta}_{L_5^2(\xi,\eta)}\ + J_6\,\underbrace{4(1-\xi-\eta)\eta}_{L_6^2(\xi,\eta)}$$

Lagrange basis

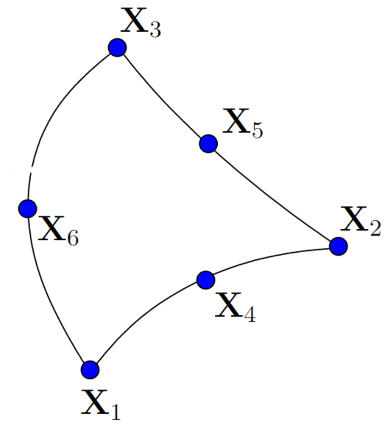
$$J(\xi,\eta) = J_1 \underbrace{(1-\xi-\eta)(1-2\,\xi-2\,\eta)}_{L_1^2(\xi,\eta)} + J_2 \underbrace{\xi(2\,\xi-1)}_{L_2^2(\xi,\eta)} + J_3 \underbrace{\eta(2\,\eta-1)}_{L_3^2(\xi,\eta)} +$$
$$J_4 \underbrace{4(1-\xi-\eta)\xi}_{L_4^2(\xi,\eta)} + J_5 \underbrace{4\xi\eta}_{L_5^2(\xi,\eta)} + J_6 \underbrace{4(1-\xi-\eta)\eta}_{L_6^2(\xi,\eta)}$$

$$\frac{\partial\, J}{\partial\xi} = \frac{\partial\, J}{\partial\eta} = 0 \quad \Rightarrow \quad \begin{bmatrix} 4(J_1+J_2-2J_4) & 4(J_1-J_4+J_5-J_6) \\ 4(J_1-J_4+J_5-J_6) & 4(J_1+J_3-2J_6) \end{bmatrix} \begin{pmatrix} \xi_{sta} \\ \eta_{sta} \end{pmatrix}$$
$$= \begin{pmatrix} -(-3J_1-J_2+4J_4) \\ -(-3J_1-J_3+4J_6) \end{pmatrix}$$

Monomial basis

$$J(\xi,\eta) = J_1 + (-3J_1 - J_2 + 4J_4)\xi + (-3J_1 - J_3 + 4J_6)\eta + 4(J_1 - J_4 + J_5 - J_6)\xi\eta + 2(J_1 + J_2 - 2J_4)\xi^2 + 2(J_1 + J_3 - 2J_6)\eta^2$$

$$4J_4 \geq 3J_1 + J_2, \quad 4J_6 \geq 3J_1 + J_3,$$
$$J_1 + J_5 \geq J_4 + J_6, \quad J_1 + J_2 \geq 2J_4, \quad J_1 + J_3 \geq 2J_6$$



Bézier basis

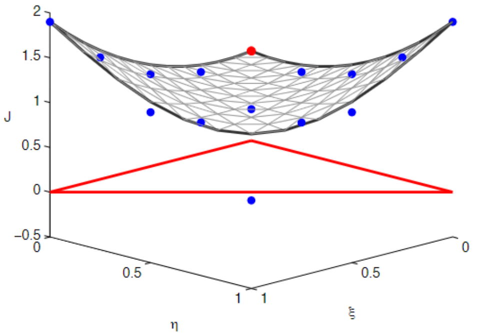
$$J(\xi,\eta) = J_1 \underbrace{(1-\xi-\eta)^2}_{B_1^{(2)}(\xi,\eta)} + J_2 \underbrace{\xi^2}_{B_2^{(2)}(\xi,\eta)} + J_3 \underbrace{\eta^2}_{B_3^{(2)}(\xi,\eta)} + \left(2\,J_4 - \frac{1}{2}(J_2 + J_1)\right) \underbrace{2\,\xi(1-\xi-\eta)}_{B_4^{(2)}(\xi,\eta)}$$
$$+ \left(2J_5 - \frac{1}{2}(J_3 + J_2)\right) \underbrace{2\,\xi\,\eta}_{B_5^{(2)}(\xi,\eta)} + \left(2\,J_6 - \frac{1}{2}(J_1 + J_3)\right) \underbrace{2\,\eta(1-\xi-\eta)}_{B_6^{(2)}(\xi,\eta)}$$

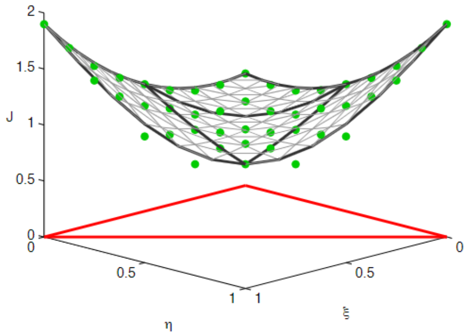
$$min_{\xi,\eta} J(\xi,\eta) = min_{\xi,\eta} \left(\sum_i B_i^{(2)}(\xi,\eta) K_i \right) \geq min_{\xi,\eta} \left(\sum_i B_i^{(2)}(\xi,\eta) \right) min_i\, K_i = min_i\, K_i$$

- Bounds

$$\boldsymbol{J}_{min} \geq \min\left\{J_1, J_2, J_3, 2\,J_4 - \frac{J_2 + J_1}{2}, 2\,J_5 - \frac{J_2 + J_3}{2}, 2\,J_6 - \frac{J_3 + J_1}{2}\right\}$$
$$\leq \min\{J_1, J_2, J_3\}$$

Bézier basis





- Subdivision & General Bounds

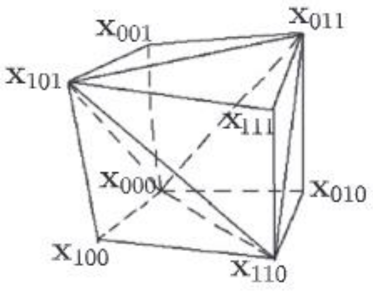
$$b'_{min} = min_{i,q}\, b_i^{[q]} \leq J_{min} \leq min_{i<K_{f,q}} b_i^{[q]}$$

Result

	Curved Element Classification			# elements analysed at given stage	CPU time for given stage [s]
	Valid elements	Invalid elements	Undertermined elements		
First stage	29,715	8,039	1,224	38,978	1.865
1 subdiv.	+787	+0	437	1,224	1.16e-1
2 subdiv.	+285	+17	135	437	8.40e-2
3 subdiv.	+56	+15	64	135	4.02e-2
4 subdiv.	+16	+22	26	64	2.40e-2
5 subdiv.	+5	+15	6	26	1.10e-2
6 subdiv.	+1	+2	3	6	4.34e-3
7 subdiv.	+1	+2	0	3	1.47e-3
Subtotal	30,866	8,112			2.146
Total	38,978				

Extensions

- Bezier bounds + Iterative refinement
  - High-order triangle
  - High order quadrangles
  - High order tetrahedra
  - High-order prisms
  - Hexahedra (trilinear map):



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