中国科学技术大学
University of Science and Technology of China

数学建模
Mathematical Modeling

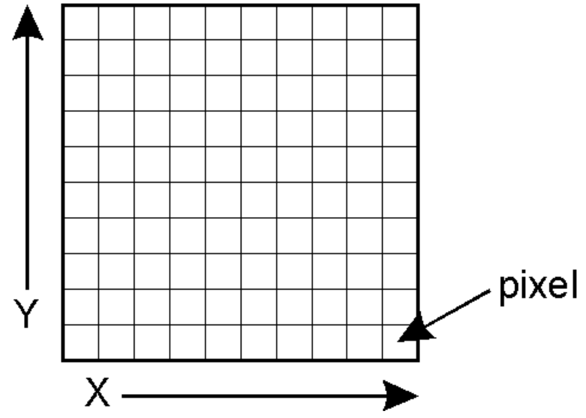
陈仁杰
中国科学技术大学

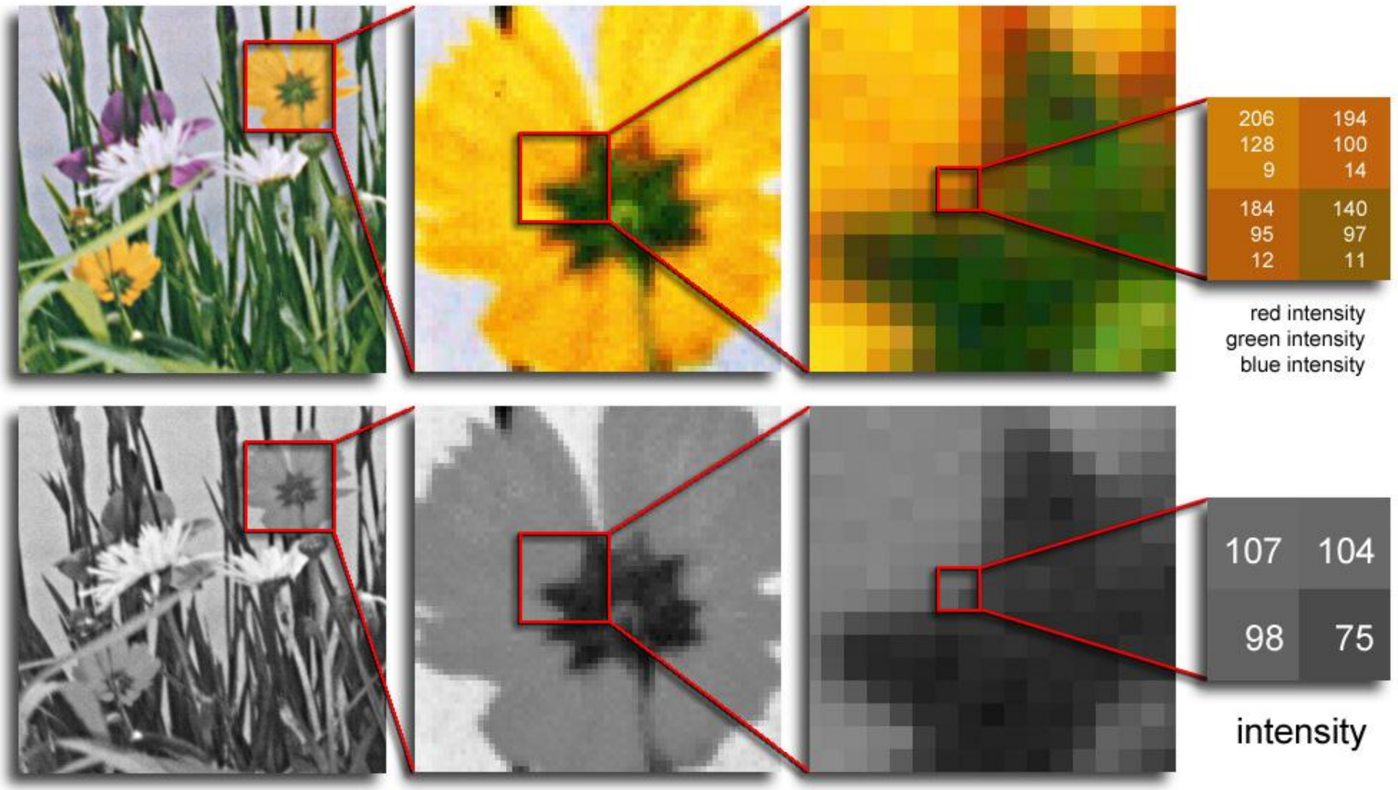
矩阵分解模型

协方差矩阵

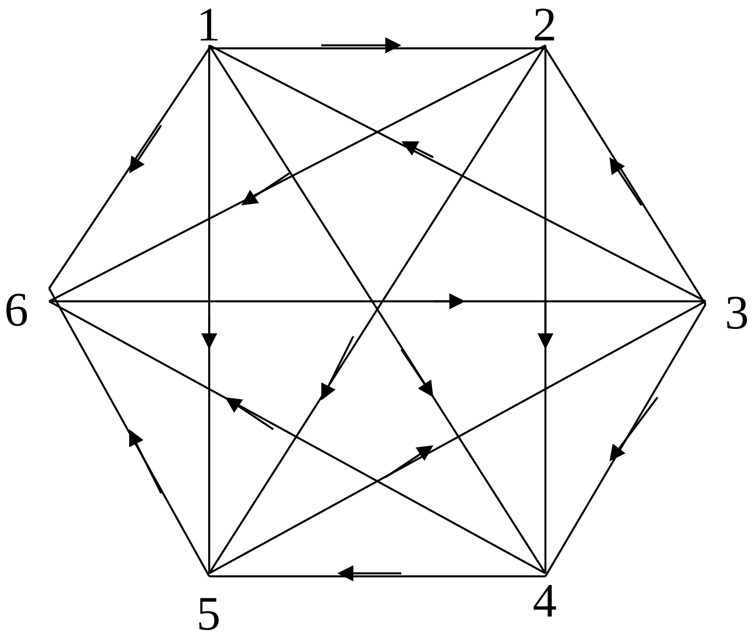
- 方差：单个随机变量的离散程度
- 协方差：两个随机变量的相似程度
- 协方差矩阵

光栅图像： 矩阵


Y
X
pixel


206
128
9
194
100
14
184
95
12
140
97
11
red intensity
green intensity
blue intensity
107
104
98
75
intensity

图的邻接矩阵



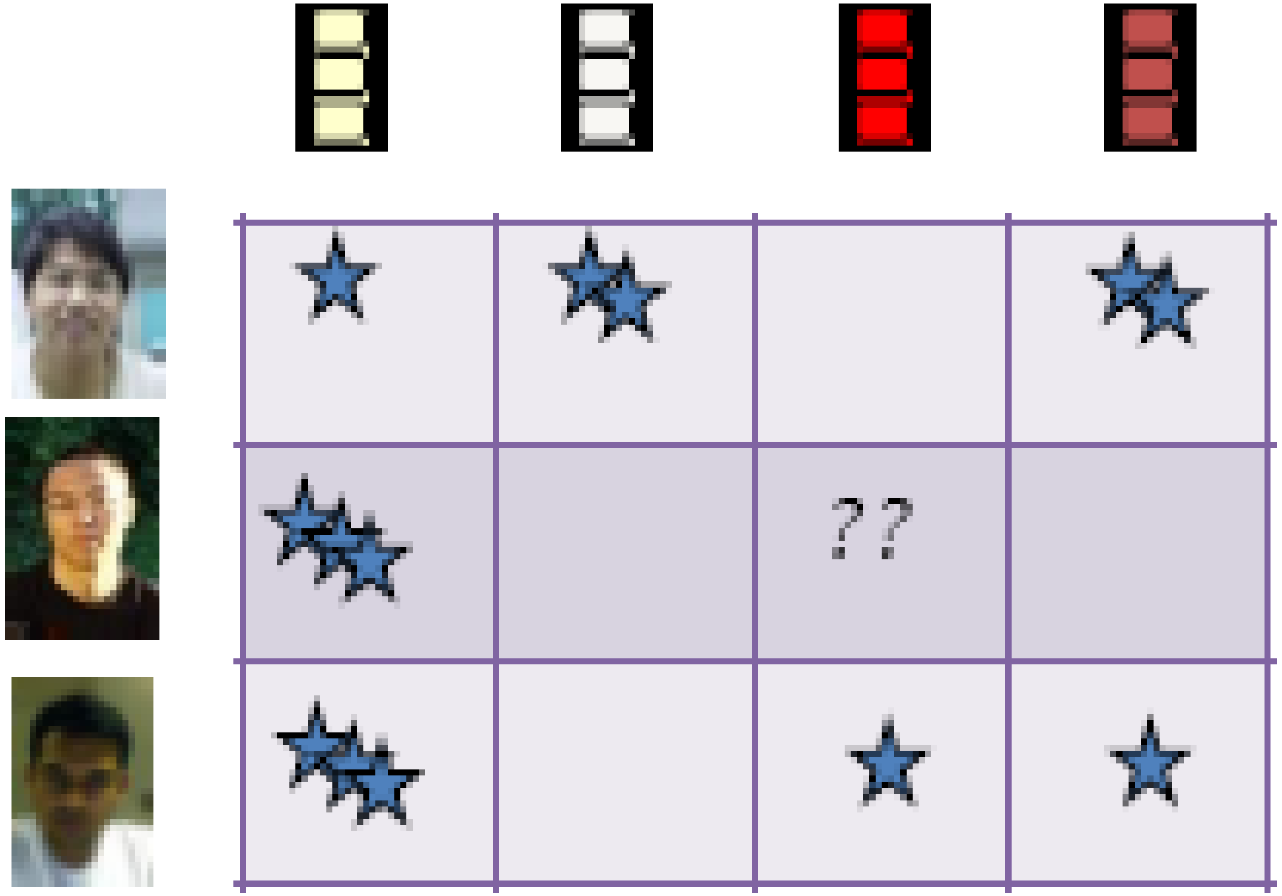
$$A=\begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

问题： 用户-物品评分

User /item	1	2	3	4	5	......
1	5	4	4.5	?	3.9	......
2	?	4.5	?	4.5	?	......
3	4.5	?	4.4	4	4	......
4	?	4.8	?	?	4.5	......
5	4	?	4.5	5	?	......
......	......	......	......	......	......	......

https://zhuanlan.zhihu.com/p/34497989

Netflix Challenge



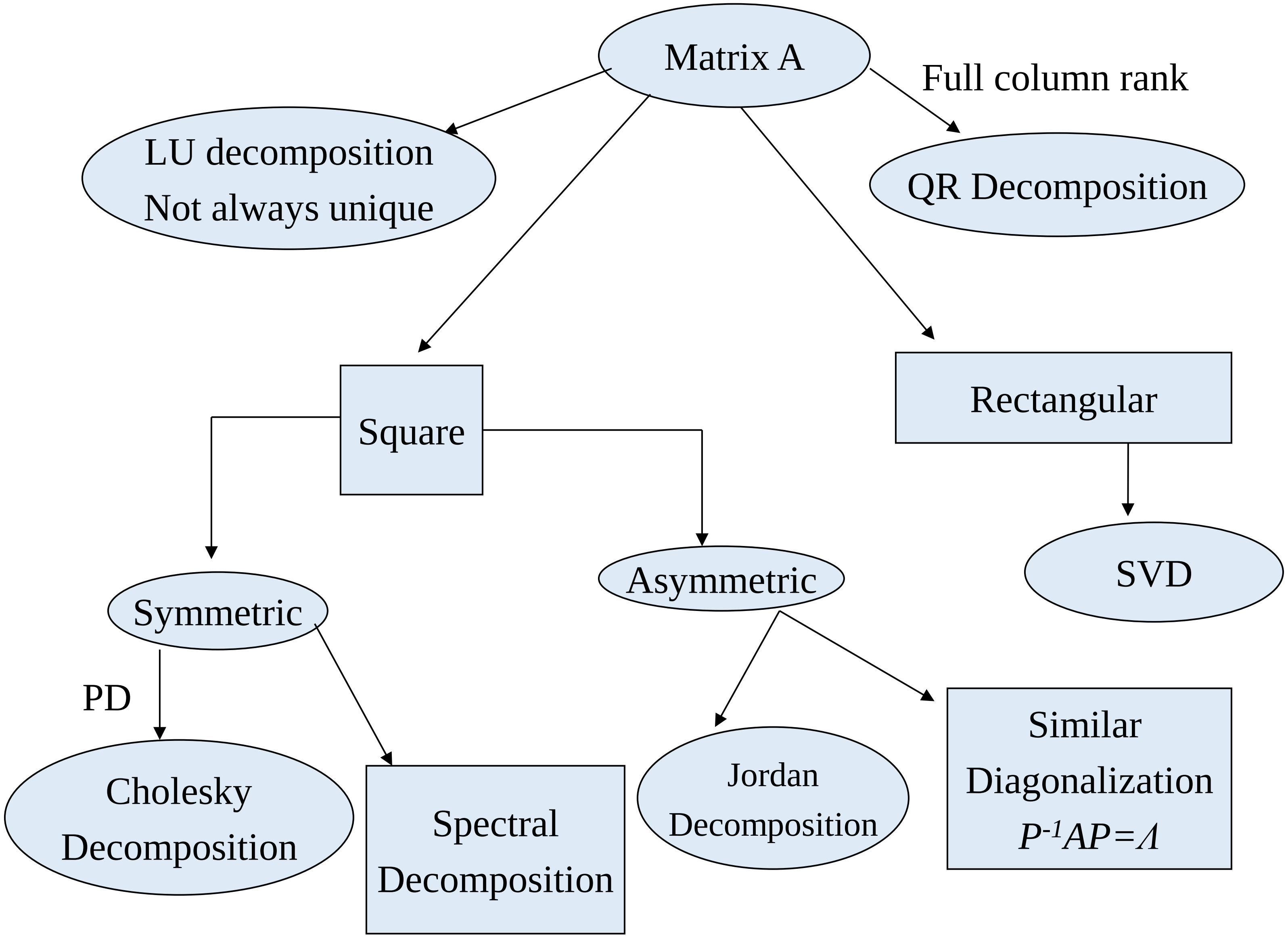
理解矩阵

- 变换
- 线性变换
- 基函数的转换

Matrix Decomposition (Factorization)

- LU decomposition
- QR decomposition
- Cholesky decomposition
- Jordan Decomposition
- Spectral decomposition (Eigendecomposition)
- Singular value decomposition (SVD)
- Low rank decomposition
- …

矩阵分解

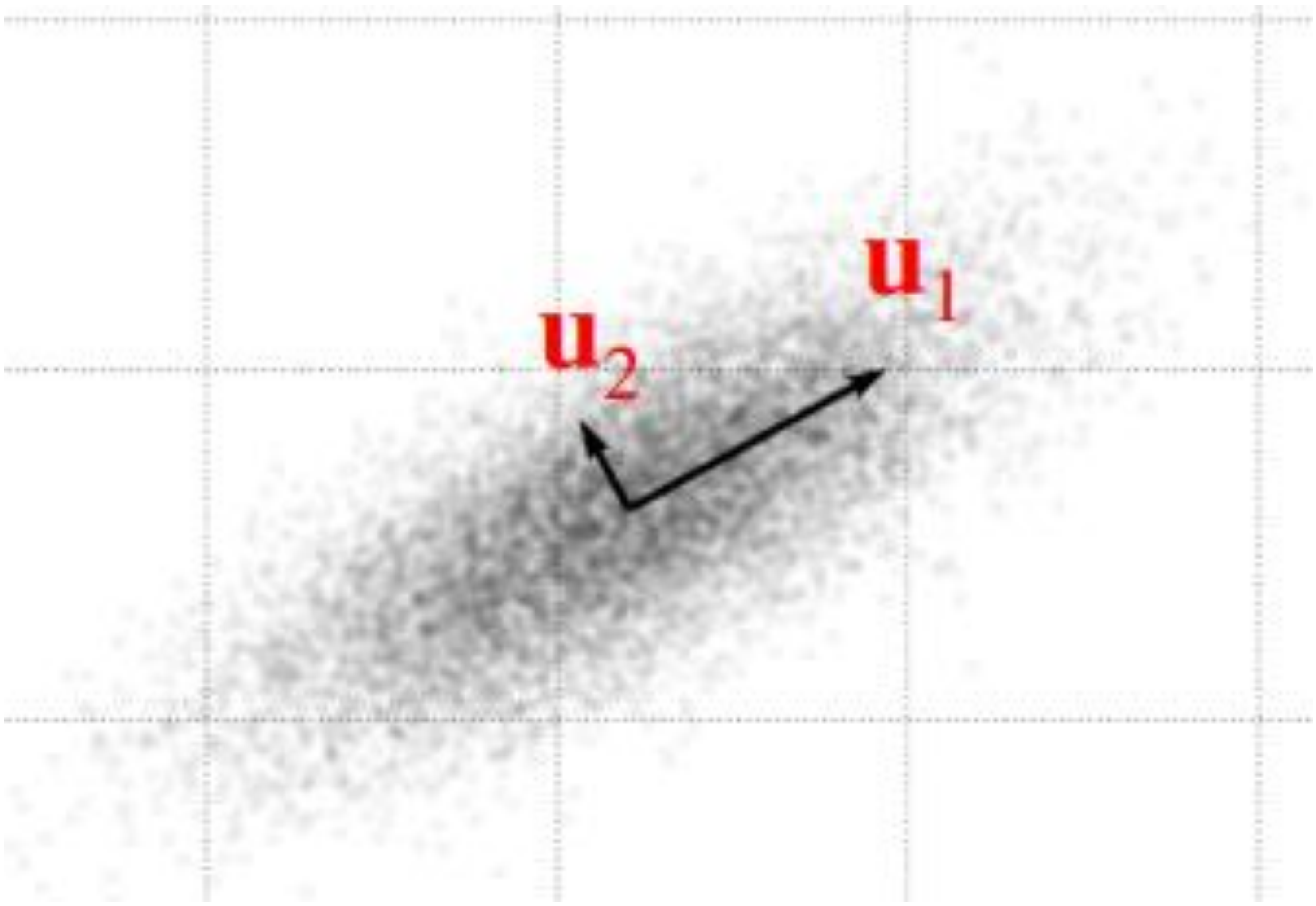


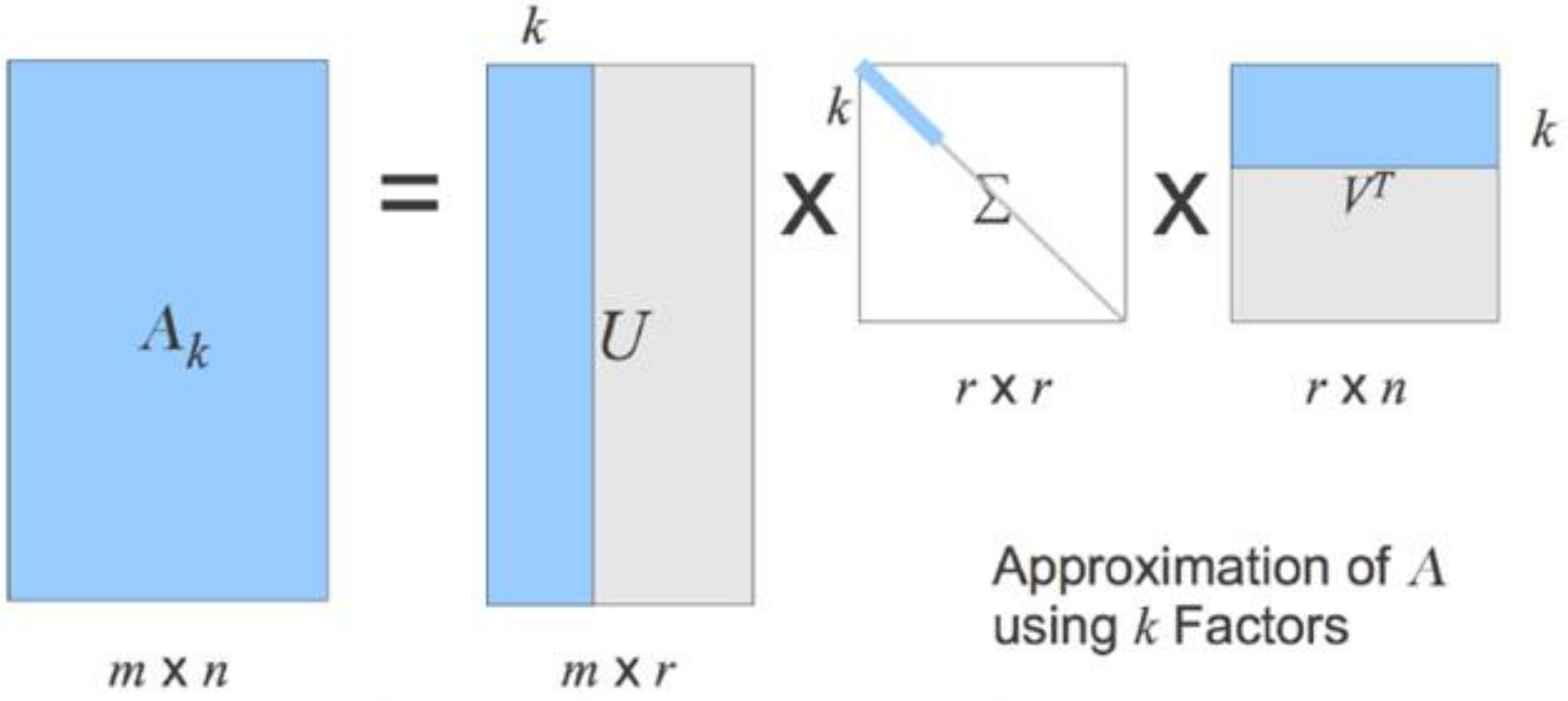
Matrix A
Full column rank
LU decomposition
Not always unique
QR Decomposition
Square
Rectangular
Symmetric
Asymmetric
SVD
PD
Cholesky Decomposition
Spectral Decomposition
Jordan Decomposition
Similar Diagonalization
$P^{-1}AP=\Lambda$

SVD

图像压缩

SVD



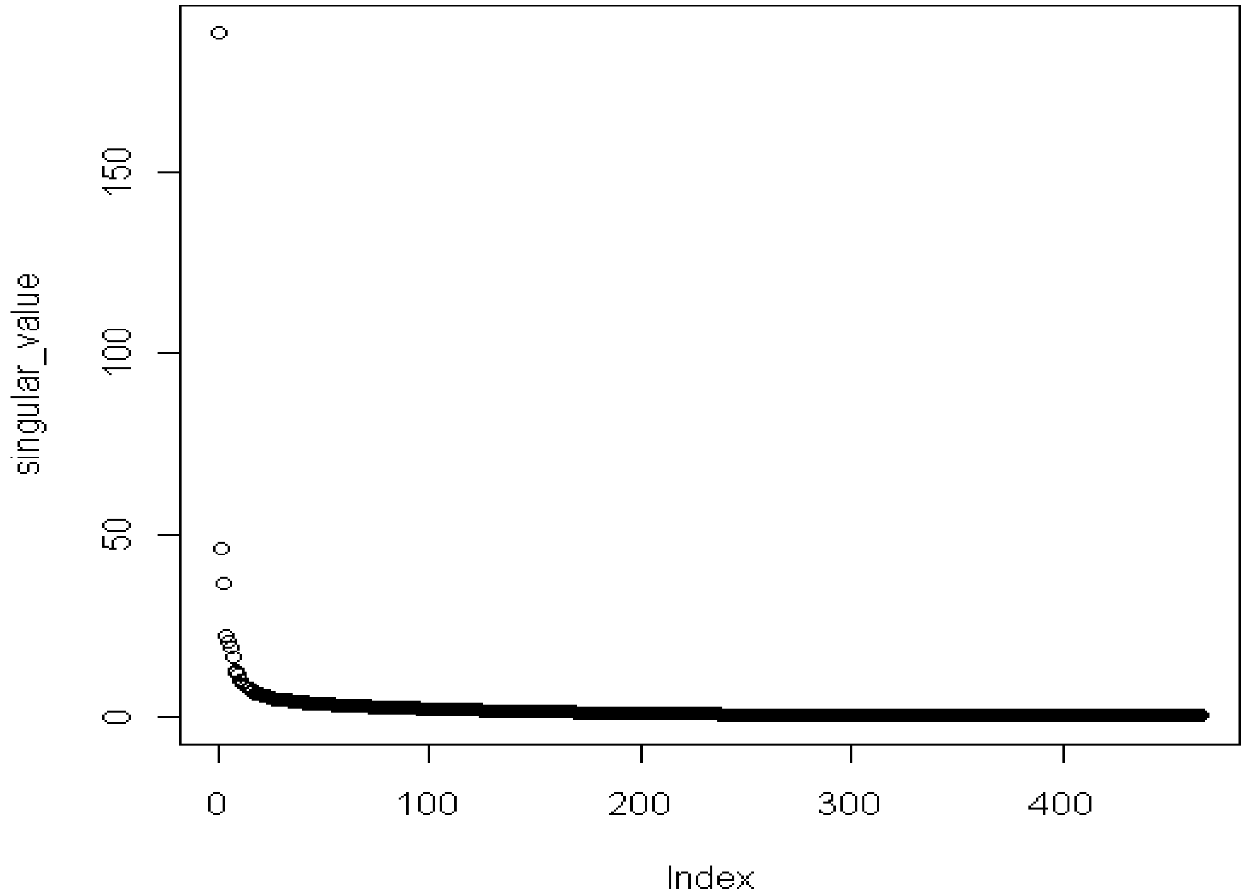


Approximation of A using k Factors

Image Compression Example

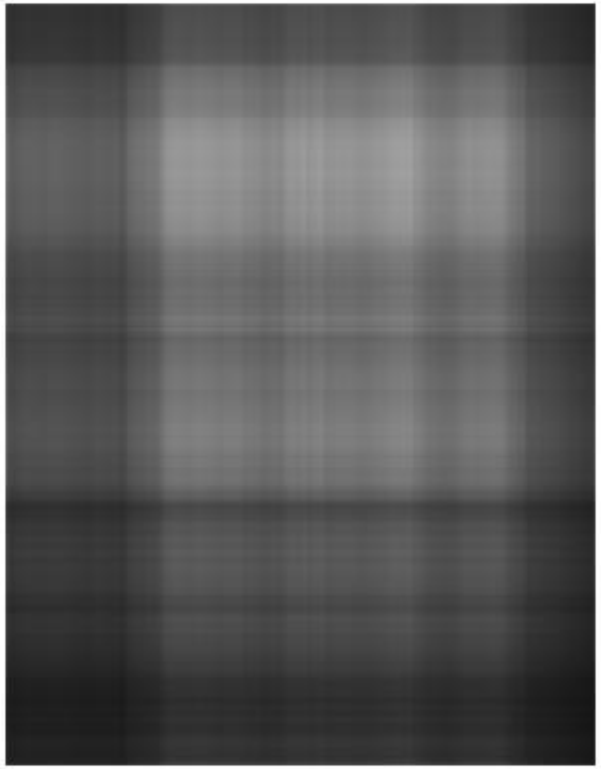
This image is 600×465 pixels

Singular values of flowers image

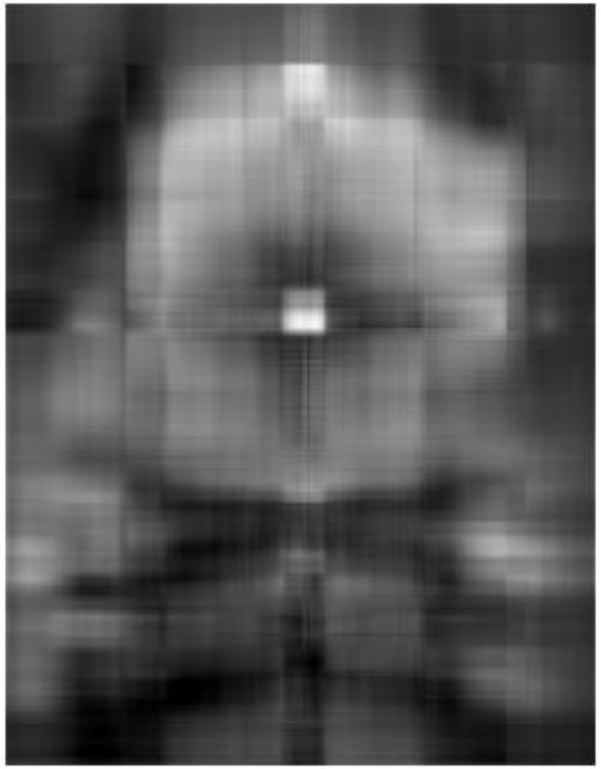


Plot of the singular values

Low rank Approximation to flowers image



Rank-1 approximation



Rank- 5 approximation

Low rank Approximation to flowers image

Rank-20 approximation

Rank-30 approximation

Low rank Approximation to flowers image

Rank-50 approximation

Rank-80 approximation

Low rank Approximation to flowers image

Rank-100 approximation

Rank-120 approximation

Low rank Approximation to flowers image

Rank-150 approximation

True Image

推荐系统

用户-物品评分

User /item	1	2	3	4	5	……
1	5	4	4.5	?	3.9	……
2	?	4.5	?	4.5	?	……
3	4.5	?	4.4	4	4	……
4	?	4.8	?	?	4.5	……
5	4	?	4.5	5	?	……
……	……	……	……	……	……	……

https://zhuanlan.zhihu.com/p/34497989

	item 1	item 2	item 3	item 4
user 1	R11	R12	R13	R14
user 2	R21	R22	R23	R24
user 3	R31	R32	R33	R34

R

=

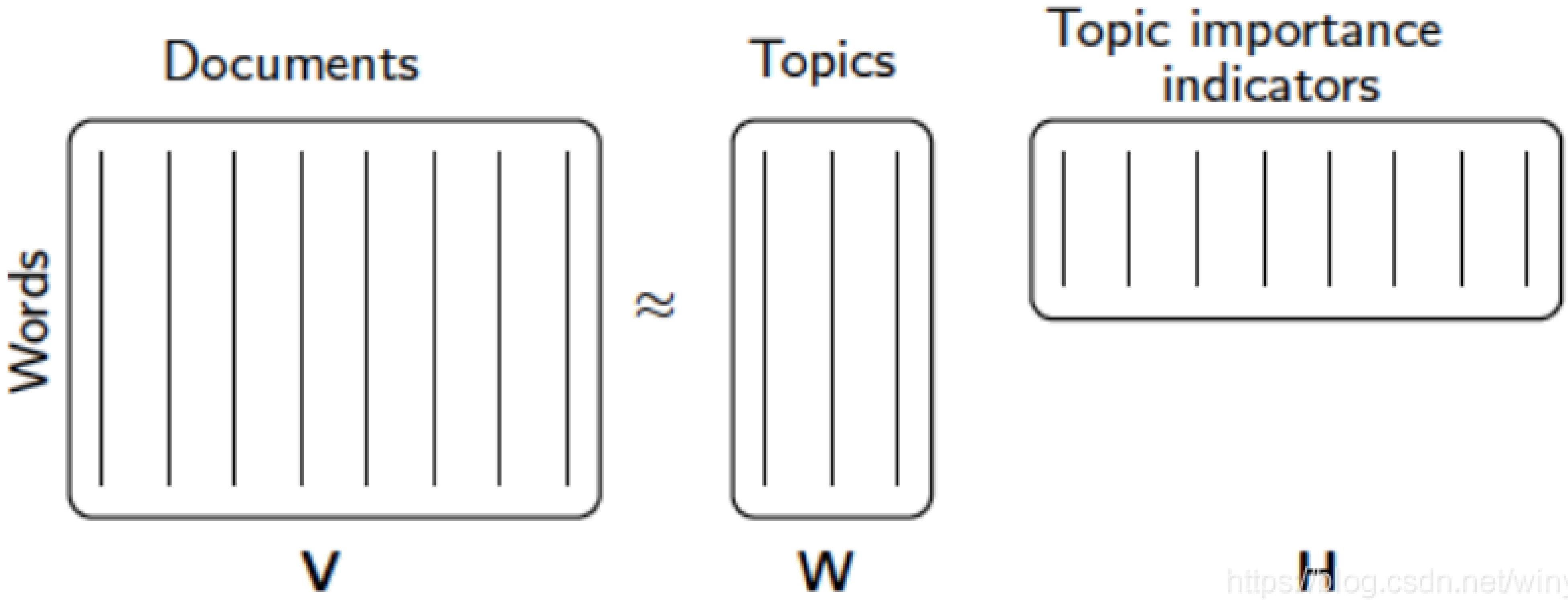
	class 1	class 2	class 3
user 1	P11	P12	P13
user 2	P21	P22	P23
user 3	P31	P32	P33

P

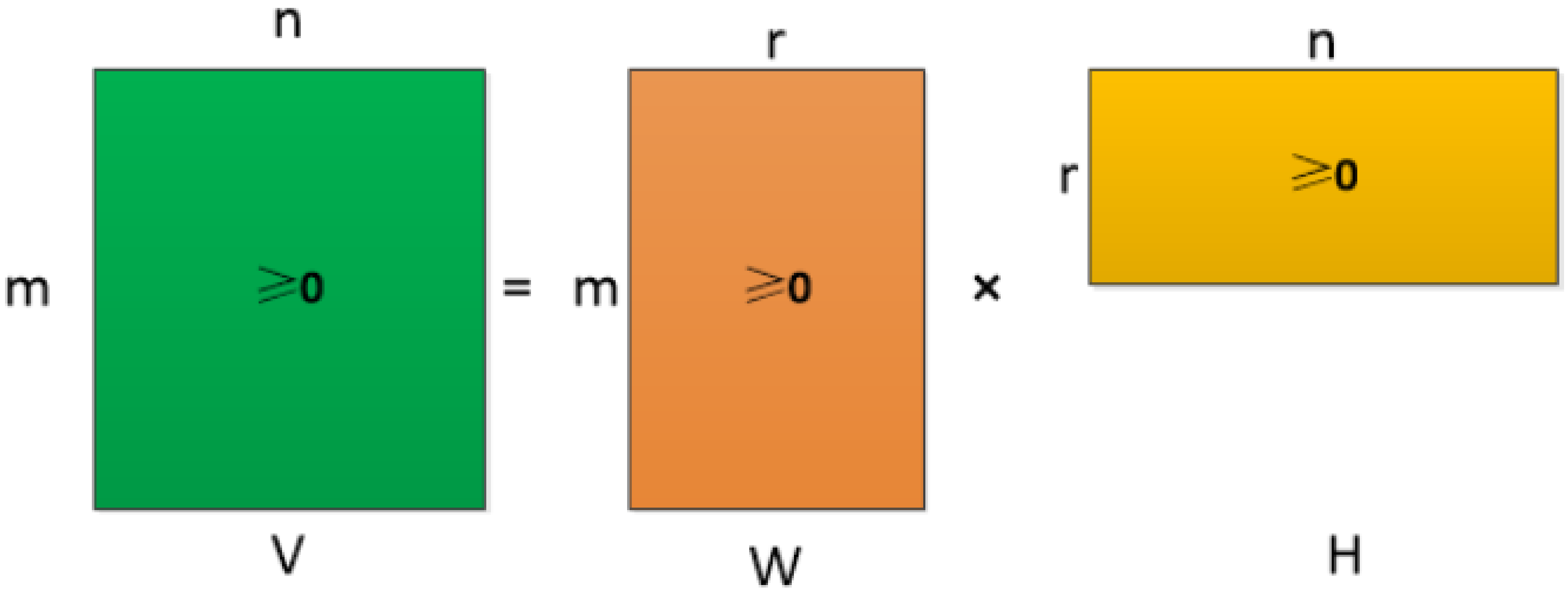
×

	item 1	item 2	item 3	item 4
class 1	Q11	Q12	Q13	Q14
class 2	Q21	Q22	Q23	Q24
class 3	Q31	Q32	Q33	Q34

Q


Documents
Topics
Topic importance
indicators
Words
≈
V
W
H

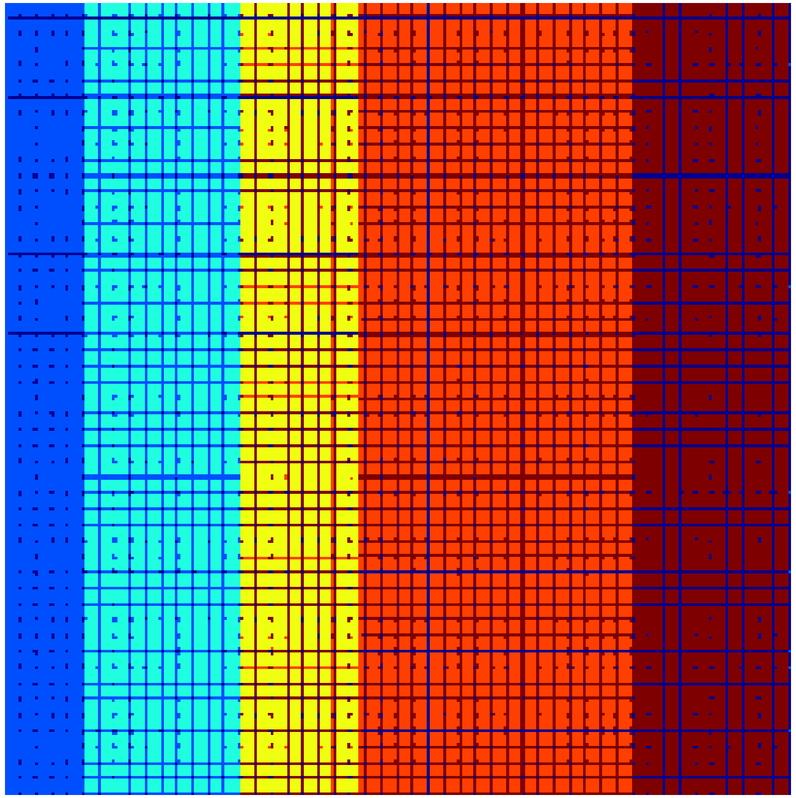
非负矩阵分解
(Non-negative Matrix Factorization， NMF)



https://www.biaodianfu.com/matrix-factorization.html

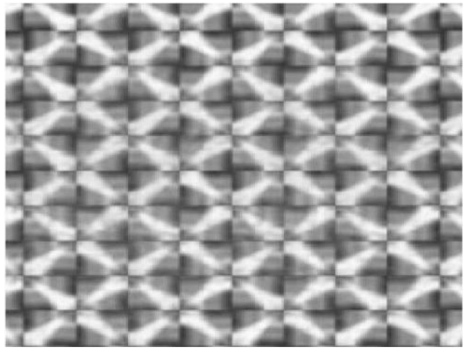
在图像处理中的应用

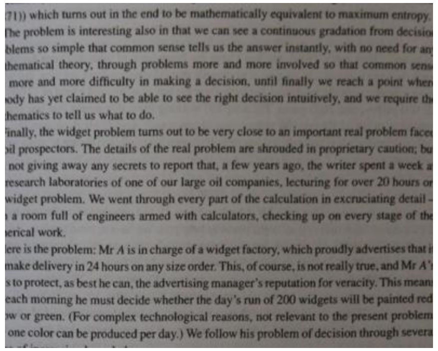
Low Rank Matrix



$$A = [\mathbf{a}_1 \mid \cdots \mid \mathbf{a}_n] \in \mathbb{R}^{m \times n}, \qquad \mathrm{rank}(A) \ll m.$$

高维数据往往具有低维结构

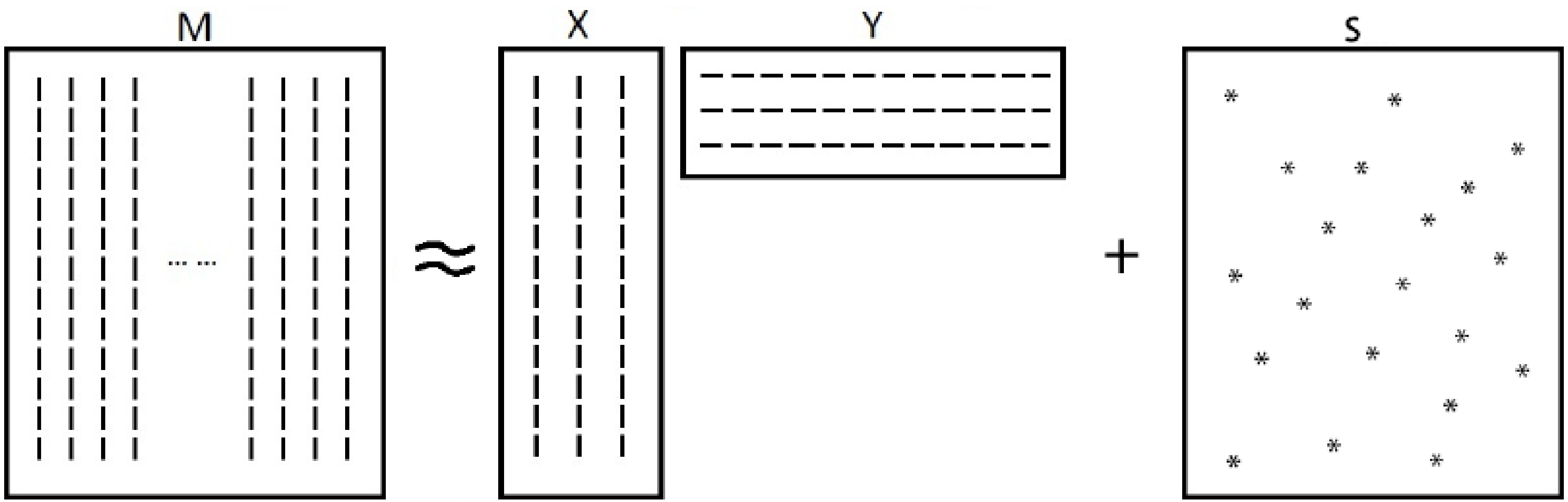




71)) which turns out in the end to be mathematically equivalent to maximum entropy.
he problem is interesting also in that we can see a continuous gradation from decisio
blems so simple that common sense tells us the answer instantly, with no need for an
hematical theory, through problems more and more involved so that common sens
more and more difficulty in making a decision, until finally we reach a point wher
ody has yet claimed to be able to see the right decision intuitively, and we require th
hematics to tell us what to do.
inally, the widget problem turns out to be very close to an important real problem face
il prospectors. The details of the real problem are shrouded in proprietary caution; bu
not giving away any secrets to report that, a few years ago, the writer spent a week a
research laboratories of one of our large oil companies, lecturing for over 20 hours o
widget problem. We went through every part of the calculation in excruciating detail –
a room full of engineers armed with calculators, checking up on every stage of the
erical work.
ere is the problem: Mr *A* is in charge of a widget factory, which proudly advertises that i
make delivery in 24 hours on any size order. This, of course, is not really true, and Mr *A*'
s to protect, as best he can, the advertising manager's reputation for veracity. This mean
each morning he must decide whether the day's run of 200 widgets will be painted red
w or green. (For complex technological reasons, not relevant to the present problem
one color can be produced per day.) We follow his problem of decision through severa

Visual data exhibit ***low-dimensional structures*** due to rich ***local*** regularities, ***global*** symmetries, ***repetitive*** patterns, or ***redundant*** sampling.

Low Rank + Sparse

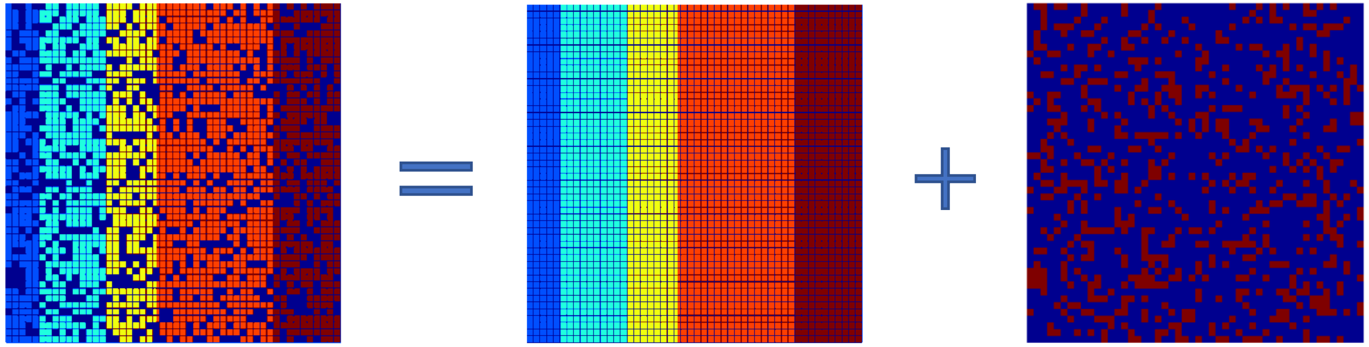


$$M = \text{low-rank matrix } XY + \text{sparse matrix } S$$

(robust PCA)

ROBUST PCA – *Problem Formulation*

D - observation $\quad$ A_0 – low-rank $\quad$ E_0 – sparse



Problem: Given $D = A_0 + E_0$, recover A_0 and E_0.

Low-rank component $\quad$ Sparse component (gross errors)

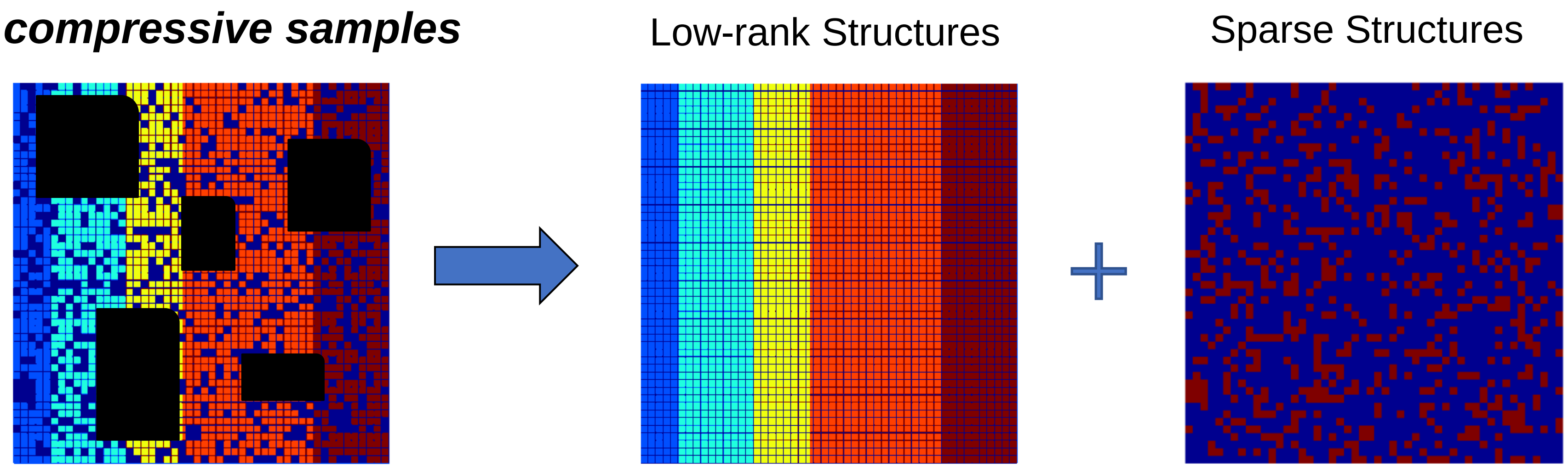
Numerous approaches in the literature:

- Multivariate trimming [Gnanadesikan and Kettering ‘72]
- Power Factorization [Wieber’70s]
- Random sampling [Fischler and Bolles ‘81]
- Alternating minimization [Shum & Ikeuchi’96, Ke and Kanade ‘03]
- Influence functions [de la Torre and Black ‘03]

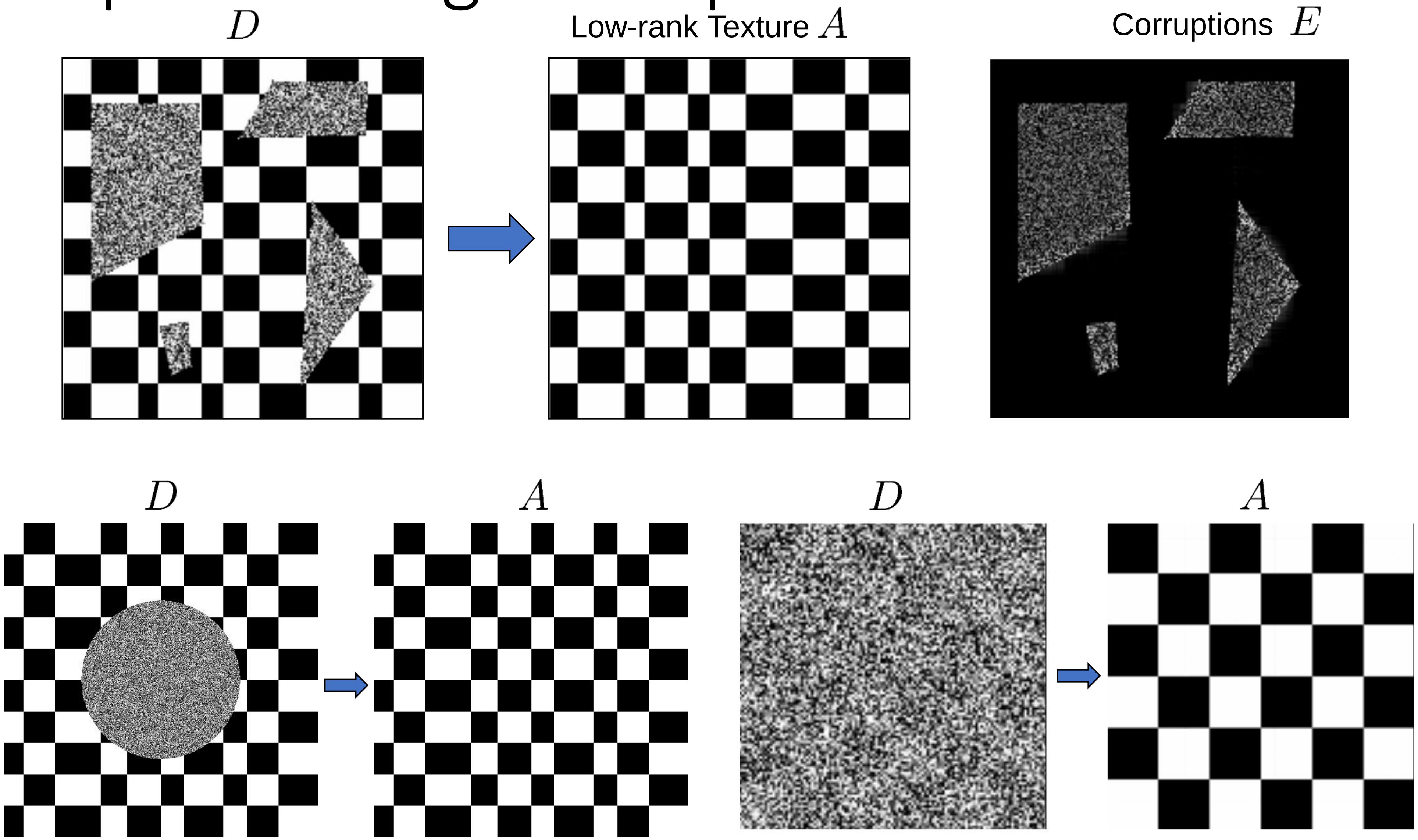
Key question: ***guarantee correctness with an efficient algorithm?***

Implications: Highly Compressive Sensing of Structured Information!

Recover low-dimensional structures from a fraction of missing measurements with structured support.



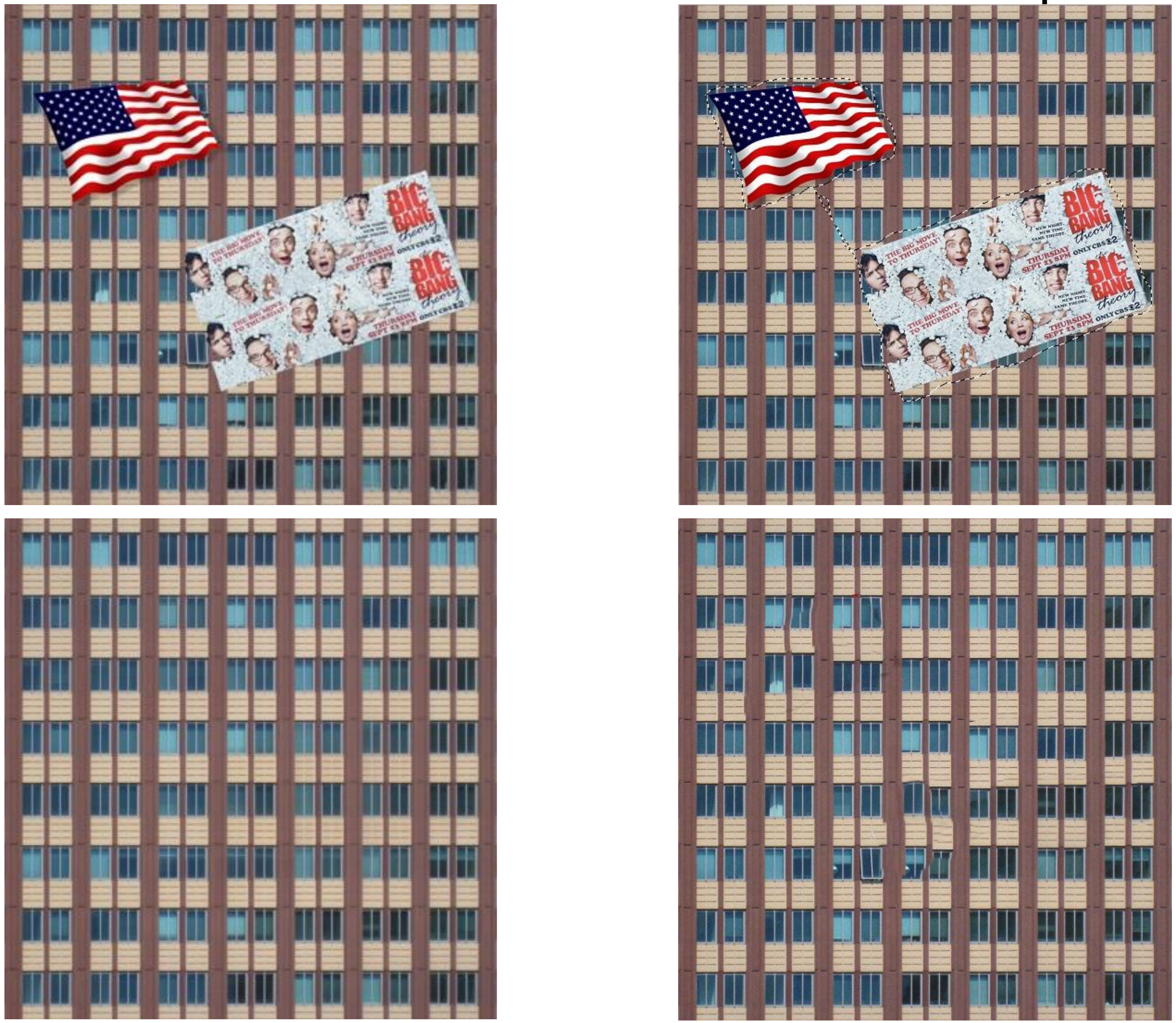
Example 1: Image Completion

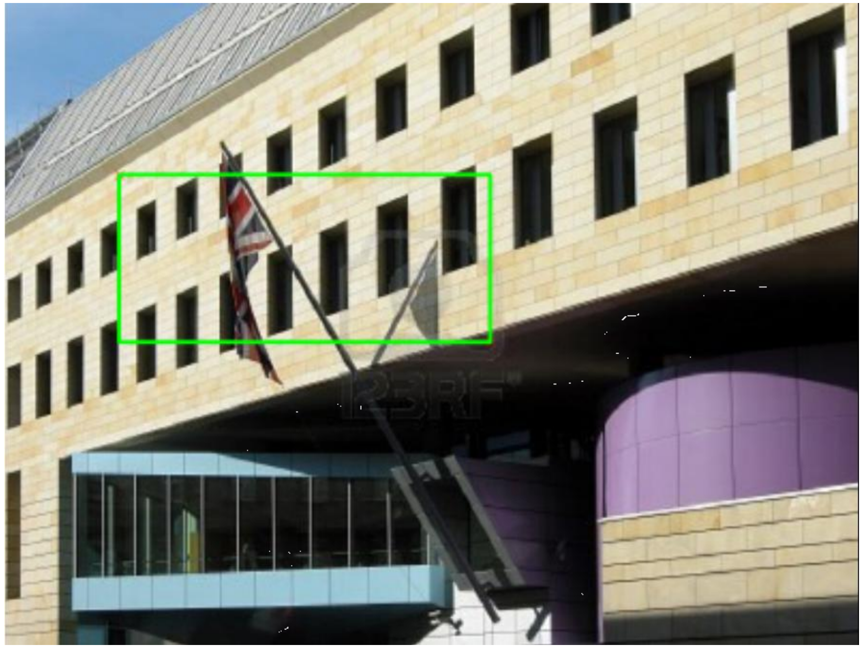
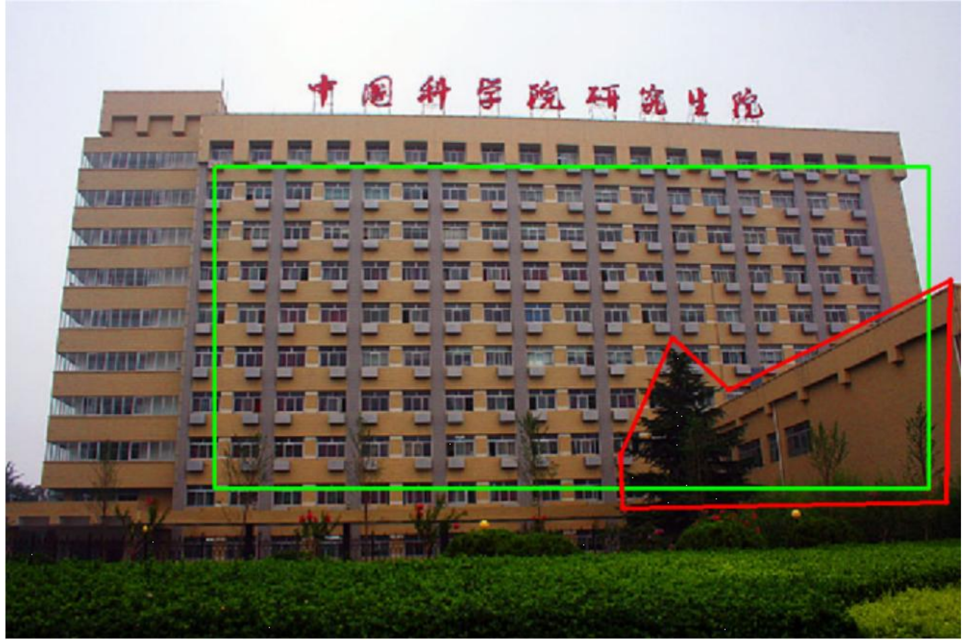


Low-rank Method | Photoshop

Input

Output





中国科学院研究生院


中国科学院研究生院

Repairing (Distorted) Low-rank Textures

Low-rank Method | Photoshop

Input

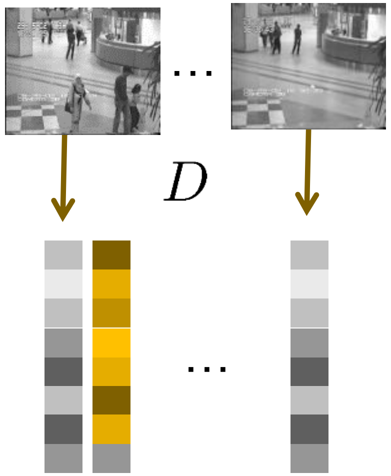
Output

Repairing Video Frames: *background modeling from video*

Surveillance video

200 frames,
144 x 172 pixels,

Significant foreground motion



Video D = Low-rank appx. A + Sparse error E

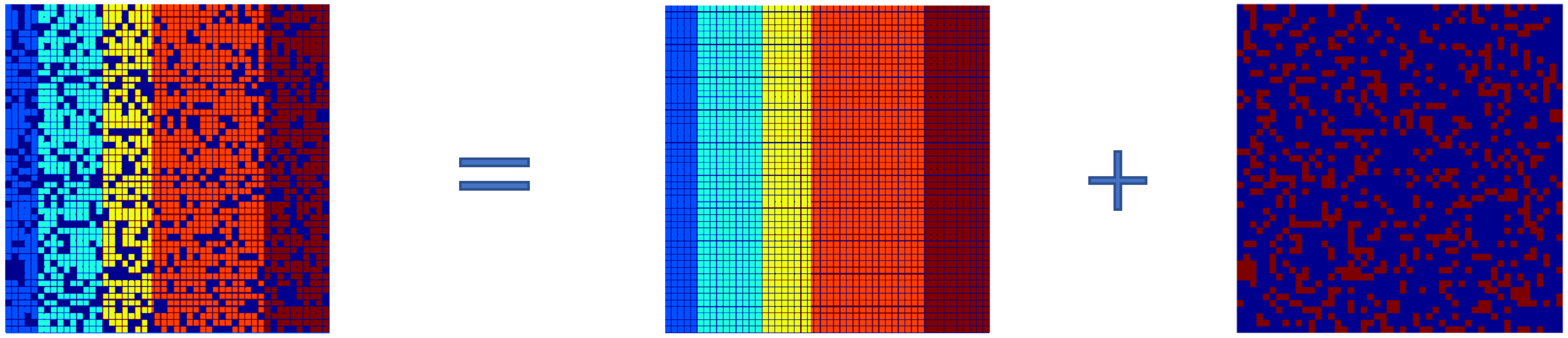
Candès, Li, Ma, and W., JACM, May 2011.

Sensing or Imaging of Low-rank and Sparse Structures

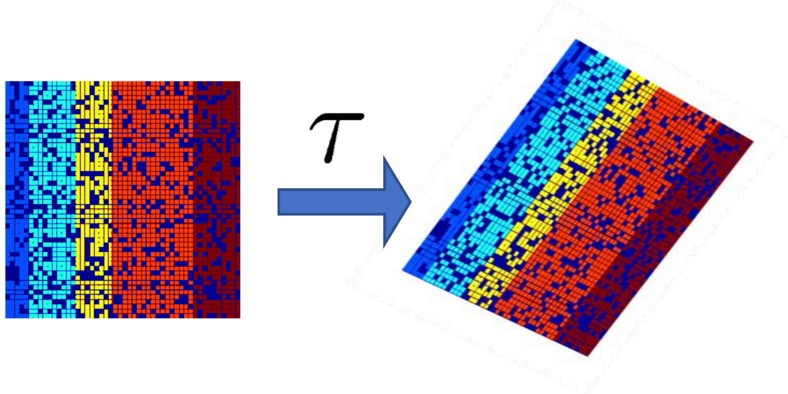
Fundamental Problem: ***How to recover low-rank and sparse structures from corrupted data***

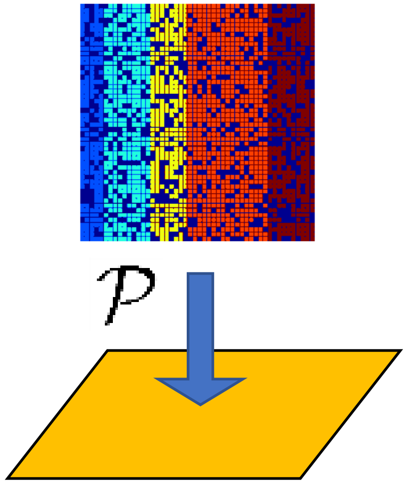
Low-rank Structures

Sparse Structures



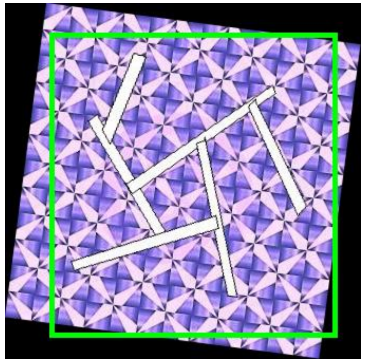
subject to either nonlinear deformation τ or linear compressive sampling $\mathcal{P}$?



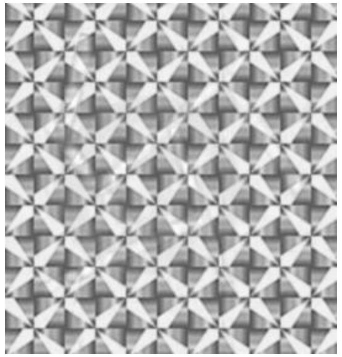


Reconstructing 3D Geometry and Structures

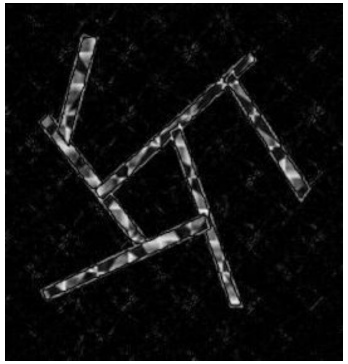
D – deformed observation A – low-rank structures E – sparse errors



$\circ\, \tau \;=$



$+$



Problem: Given $D \circ \tau = A_0 + E_0$, recover τ, A_0 and E_0 simultaneously.

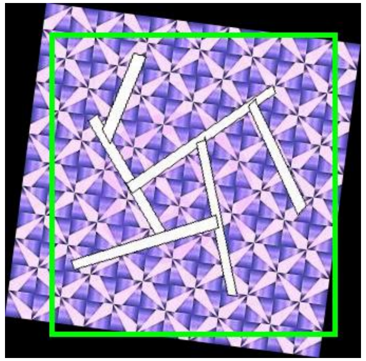
Low-rank component (regular patterns…)

Sparse component (occlusion, corruption, foreground…)

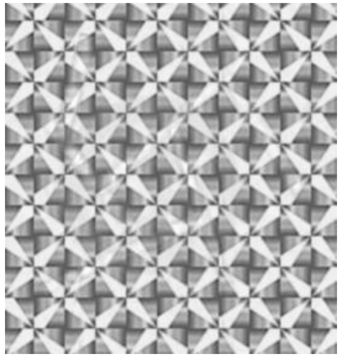
Parametric deformations (affine, projective, radial distortion, 3D shape…)

Transform Invariant Low-rank Textures (TILT)

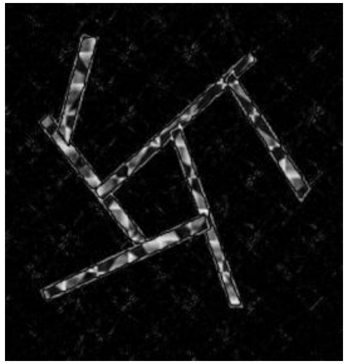
D – deformed observation $\qquad$ A – low-rank structures $\qquad$ E – sparse errors



$\circ\, \tau \;=$



$+$



Objective: *Principal Component Pursuit:*

$$\min \; \|A\|_* + \lambda \|E\|_1 \quad \text{subj} \quad A + E = D \circ \tau$$

Solution: *Iteratively solving the linearized convex program:*

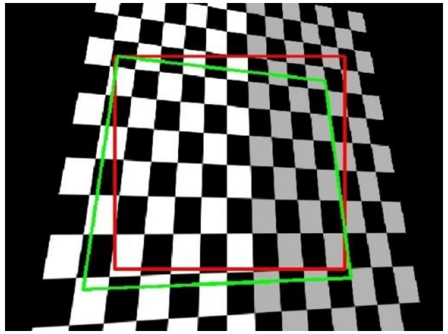
$$\min \; \|A\|_* + \lambda \|E\|_1 \quad \text{subj} \quad A + E = D \circ \tau_k + J \cdot \Delta\tau$$

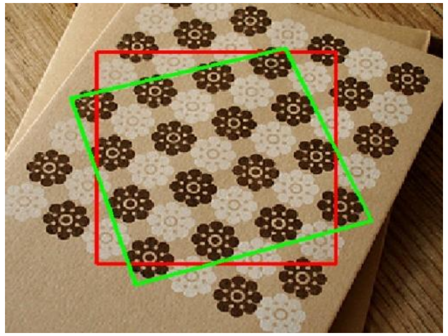
Or reduced version: $\quad \text{subj} \quad \mathcal{P}_Q[A + E] = \mathcal{P}_Q[D \circ \tau_k], \; \mathcal{P}_Q[J] = 0$

Zhang, Liang, Ganesh, Ma, ACCV'10, IJCV'12

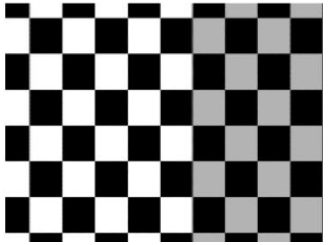
TILT: *Shape from texture*

Input (red window D)



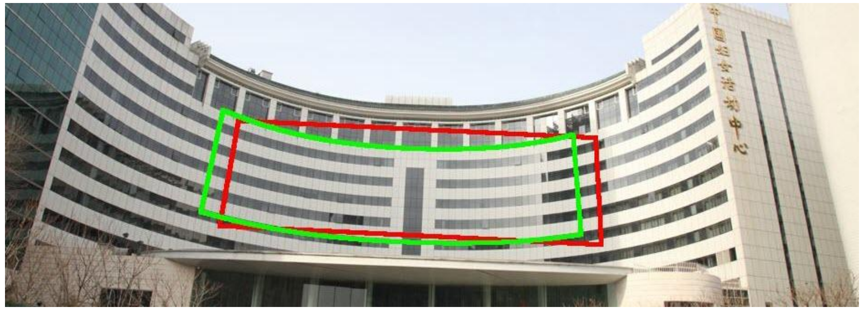


Output (rectified green window A)



Zhang, Liang, Ganesh, Ma, ACCV'10, IJCV'12

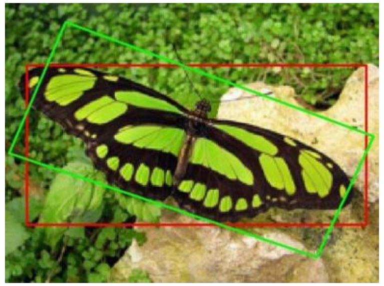
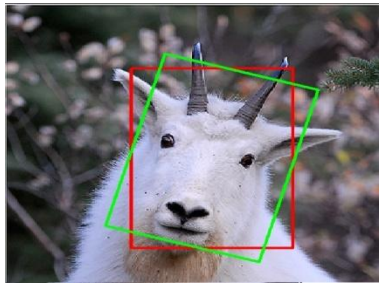
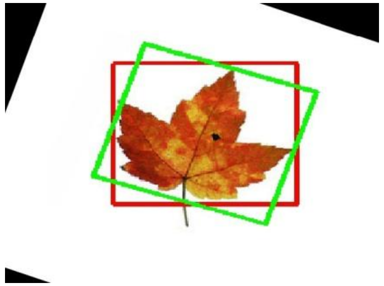
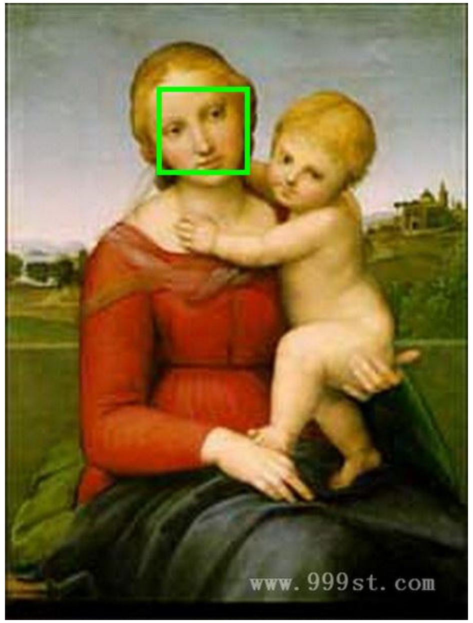
TILT: *Virtual reality*



Zhang, Liang, and Ma, in ICCV 2011

Object Recognition: *Rectifying Pose of Objects*

Input (red window D)

Output (rectified green window A)




Zhang, Liang, Ganesh, Ma, ACCV’10 and IJCV’12

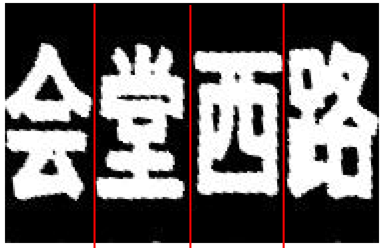
Object Recognition: *Regularity of Texts at All Scales!*

Input (red window D)

Output (rectified green window A)

Zhang, Liang, Ganesh, Ma, ACCV’10 and IJCV’12

Recognition: Street Sign Rectification



A_1 A_2 A_3 A_4

$$\min \sum_{i=1}^{4} \|A_i\|_* + \lambda \|E_i\|_1$$
$$\text{subj} \quad D \circ \tau = [A_1 \cdots A_4] + [E_1 \cdots E_4].$$

Xin Zhang, Zhouchen Lin, and Ma, PR 2012

参考文献: SVD

- https://zhuanlan.zhihu.com/p/36546367
- https://zhuanlan.zhihu.com/p/29846048
- https://www.zhihu.com/question/22237507
- https://zhuanlan.zhihu.com/p/360980054
- https://math.mit.edu/classes/18.095/2016IAP/lec2/SVD_Notes.pdf

参考文献: Low rank

- Robust PCA
  - Emmanuel J. Candès, Xiaodong Li, Yi Ma, John Wright. Robust principal component analysis? Journal of the ACM, 58 (3), 2011.
  - https://arxiv.org/pdf/0912.3599.pdf
  - https://dl.acm.org/doi/10.1145/1970392.1970395
  - https://proceedings.neurips.cc/paper/2009/file/c45147dee729311ef5b5c3003946c48f-Paper.pdf
- Repairing Sparse Low-rank Texture
  - Xiao Liang, Xiang Ren, Zhengdong Zhang, and Yi Ma, European Conference on Computer Vision (ECCV), October 2012.
  - https://people.csail.mit.edu/zhangzd/papers/recover_low-rank_texture_final.pdf
- TILT: Transform-Invariant Low-rank Textures,
  - Zhengdong Zhang, Arvind Ganesh, Xiao Liang, and Yi Ma, Volume 99, Number 1, page 1-24, the International Journal of Computer Vision (IJCV), August 2012.
  - https://arxiv.org/pdf/1012.3216.pdf

作业2：基于矩阵分解的图像处理

- 要求：如下任务2选1
  - 任务1：基于矩阵SVD分解的图像压缩
  - 任务2：基于矩阵低秩分解的图像修复

- 提交
  - 作业报告，包括：数学模型、测试结果、实验分析等

- 提交时间：2025年3月29日周六晚

中国科学技术大学
University of Science and Technology of China

谢　谢　！