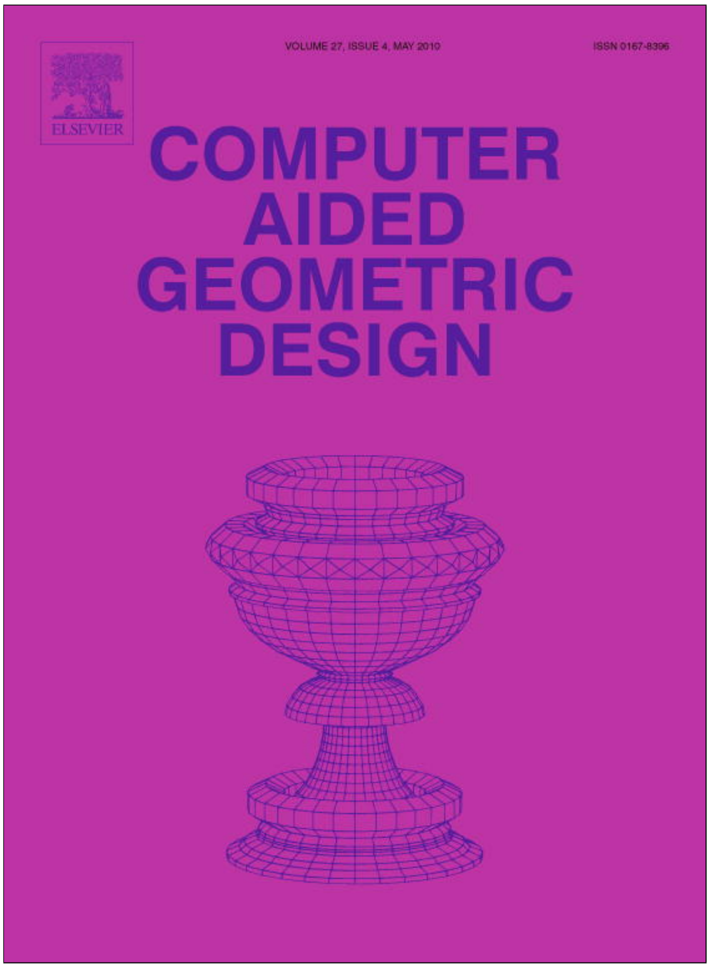
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A spectral characterization of the Delaunay triangulation

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ABSTRACT

The Delaunay triangulation of a planar point set is a fundamental construct in computational geometry. A simple algorithm to generate it is based on flips of diagonal edges in convex quads. We characterize the effect of a single edge flip in a triangulation on the geometric Laplacian of the triangulation, which leads to a simpler and shorter proof of a theorem of Rippa that the Dirichlet energy of any piecewise-linear scalar function on a triangulation obtains its minimum on the Delaunay triangulation. Using Rippa's theorem, we provide a spectral characterization of the Delaunay triangulation, namely that the spectrum of the geometric Laplacian is minimized on this triangulation. This spectral theorem then leads to a simpler proof of a theorem of Musin that the harmonic index also obtains its minimum on the Delaunay triangulation.

1. Introduction

1.1. The Delaunay triangulation

The Delaunay triangulation of a set of points in the plane, and its dual – the Voronoi diagram – are probably one of the most basic spatial structures in computational geometry. Their underlying theory has been extensively developed, and a vast number of practical applications are based on them. The interested reader is referred to one of the many books and surveys on the topic (e.g. Aurenhammer, 1991; Okabe et al., 1992) for more details.

The Delaunay triangulation has been shown to possess a number of optimality properties. On an intuitive level, the triangles in this triangulation are the "fattest" possible. On a quantitative level, it maximizes the minimal angle in the triangulation. A more geometric characterization is the "empty circle" property, namely, that any circle through three points forming a Delaunay triangle does not contain any other points. Lawson (1972) showed that any convex quadrilateral formed by two adjacent triangles which fail to satisfy the empty circle property may be corrected by "flipping" the diagonal edge of the quadrilateral, common to the two triangles, to the opposite diagonal. This operation is called a *Delaunay flip* and the resulting edge is called a *Delaunay edge*. Furthermore, Delaunay flipping will always converge to a Delaunay triangulation. It is easy to see that the empty circle property implies that an (interior) edge of a triangulation is Delaunay if and only if $\alpha + \beta \leqslant \pi$, where α and β are the two unique angles opposite the edge in the triangulation. This is equivalent to each of the conditions $\cot\alpha + \cot\beta \geqslant 0$ or $\sin(\alpha + \beta) \geqslant 0$.

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1.2. Dirichlet energy

The *Dirichlet energy* of a scalar function f on a planar domain Ω is:

$$E_D(f) = \frac{1}{2}\int_{\Omega} |\nabla f|^2 \, \mathrm{d}\Omega$$

A triangulation T of a set of n points in the plane, having vertex set V and edge set E, and a set of scalar values $f = (f_1, \ldots, f_n)^t$ on V, defines a piecewise-linear function over the triangulation in a natural way. Pinkall and Polthier (1993) provided an explicit formula for the Dirichlet energy of this function as a sum over the *triangles* of T:

$$E_D(f) = \frac{1}{4}\sum_{(i,j,k)\in T} \left[\cot\alpha_{ij}(f_i - f_j)^2 + \cot\alpha_{jk}(f_j - f_k)^2 + \cot\alpha_{ki}(f_k - f_i)^2\right] \tag{1}$$

where α_{ij} is the angle opposite the oriented edge (i, j). Rearranging the sum in (1) results in the same Dirichlet energy expressed as a sum over the *edges* of T:

$$E_D(f) = \frac{1}{2}\sum_{(i,j)\in E} w_{ij}(f_i - f_j)^2$$

$$w_{ij} = \frac{1}{2}(\cot\alpha_{ij} + \cot\alpha_{ji}) \tag{2}$$

For a boundary edge, only one cotangent is used in the definition of w_{ij}.

The quadratic energy (2) may be also written in matrix notation using the $n \times n$ *geometric Laplacian* matrix L (which is the piecewise-linear approximation to the continuous Laplacian operator):

$$E_D(f) = \frac{1}{2} f^t L f$$

where

$$L_{ij} = \begin{cases} 0 & j \notin N(i) \\ -w_{ij} & j \in N(i) \\ \sum_{k\in N(i)} w_{ik} & j = i \end{cases}$$

and $N(i)$ is the set of vertices neighboring to vertex i. Equivalently:

$$L = B^t D B$$

where B is the $m \times n$ *incidence matrix* of the triangulation:

$$B_{ev} = \begin{cases} 0 & e \notin E \\ 1 & e \in E, v = l(e) \\ -1 & e \in E, v = r(e) \end{cases}$$

e represents an edge, and v a vertex of the triangulation. $l(e)$ and $r(e)$ are the "left" and "right" vertices of the edge e in some consistent orientation of the triangles. D is the $m \times m$ diagonal matrix $D = \mathrm{diag}(w_{ii})$ and m is the number of edges in the edge set E of the triangulation.

L is known to be symmetric and positive semi-definite (SPSD) (even though the w_{ij} have mixed signs). It is the geometric analog of the combinatorial Laplacian commonly used in spectral graph theory (Biggs, 1993; Chung, 1997), where the weights are taken to be $w_{ij} = 1$. The geometric Laplacian is particularly appealing since the products Lx and Ly for the vectors of x and y coordinates of the triangulation vanish at the interior vertices, namely, they satisfy the *Laplace equation* or are *harmonic* subject to so-called *Dirichlet boundary conditions.* Harmonicity means the coordinates strike a delicate weighted balance, or, equivalently, minimize the Dirichlet energy subject to the boundary conditions. For this reason the weights w_{ij} as defined in (2) may be used as barycentric coordinates in many mesh processing scenarios. It is also noteworthy that

$$A = x^t L x = y^t L y$$

where A is the area of the triangulation.

The spectrum of the combinatorial Laplacian has been shown to reflect many basic properties of its underlying graph. In particular, the smallest eigenvalue of the combinatorial Laplacian is zero (with eigenvector $(1, \ldots, 1)^t$), and the magnitude of the second smallest eigenvalue is related to the "mixing" properties of the graph (Chung, 1997). Namely, a smaller second eigenvalue indicates that a random walk on the graph will converge more rapidly to its stationary distribution. The eigenvectors of the Laplacian have also been used for embedding and clustering applications (Hall, 1970; Koren, 2003). It is interesting to note that, as opposed to the two coordinate vectors which satisfy $Lx = Ly = 0$ on interior vertices, the eigenvector corresponding to the smallest non-zero eigenvalue is typically not harmonic on all vertices, yet manages to attain a smaller Dirichlet energy than these coordinate vectors.

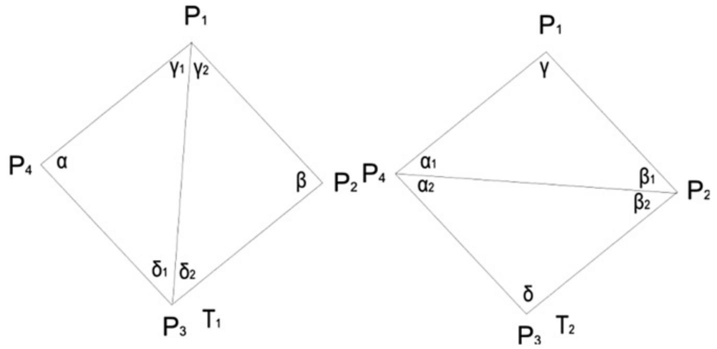

Fig. 1. Delaunay edge flip in a convex quadrilateral. P_2–P_4 is a Delaunay edge.

2. Rippa's theorem

Rippa (1990) proved the following important property of the Delaunay triangulation: The Delaunay triangulation minimizes (among all possible triangulations) the "roughness" of the Piecewise Linear Interpolating Surface (PLIS) resulting from *any* fixed set of scalar function values associated with the points. This "roughness" of the PLIS is none other than its Dirichlet energy. Rippa gave quite an elaborate proof, which was subsequently shortened by Powar (1992). Glickenstein (2007) generalized the theorem to regular triangulations (of which the Delaunay triangulation is a special case), and again provided a lengthy trigonometric proof. We now prove a Key Lemma which characterizes the effect of a Delaunay edge flip on the geometric Laplacian matrix of a triangulation. This will lead to a much simpler proof of Rippa's theorem.

Key Lemma. *Let T_1 be a non-Delaunay triangulation of a set of points and T_2 the same triangulation after one Delaunay edge flip. Let L_1 and L_2 be the geometric Laplacians of T_1 and T_2 respectively. Then $\Delta L = L_1 - L_2$ is a symmetric positive semi-definite (SPSD) matrix with unit rank.*

Proof. Consider the convex quadrilateral $P = (P_1, P_2, P_3, P_4)$, within which the Delaunay edge flip occurred, as in Fig. 1.

From (1) it is evident that the difference between the Dirichlet energies of T_1 and T_2 is due *only* to the contributions of the four triangles related to P. Thus ΔL contains only a 4×4 non-zero sub-matrix determined by P, while all other elements vanish. So we may focus our attention on P, and restrict T_1 and T_2 to two triangles each, as in Fig. 1. For any scalar function $f = (f_1, \ldots, f_4)^t$ defined on the 4 vertices of P, let E_1 and E_2 be the Dirichlet energies of the triangulations T_1 and T_2, respectively. Thus $\Delta E = E_1 - E_2 = f^t(\Delta L)f$, where ΔL is a 4×4 matrix.

We first observe that if no three of the four vertices of P are co-linear (i.e. none of the triangles are degenerate), then for any f_2, f_3, f_4, there always exists some f_1, such that the four points $(P_i.x, P_i.y, f_i)$, $1 \leqslant i \leqslant 4$, are co-planar in R^3. For this configuration obviously $\Delta E = 0$ (since the edge flip does not change the planarity of the quad). Hence the co-rank of ΔL is 3, and ΔL has unit rank. Thus, by simple linear algebra, there must exist a vector $a = (a_1, a_2, a_3, a_4)^t$ and scalar b such that $\Delta L = b(a \cdot a^t)$. Note that $a \cdot a^t$ is always SPSD, so ΔL is SPSD iff $b \geqslant 0$ and also $\Delta L_{11} \geqslant 0$ iff $b \geqslant 0$.

Denote by ΔL_{11} the top left entry of ΔL. Direct computation of the Dirichlet energy leads to

$$2\Delta L_{11} = \cot\delta_1 - \cot\beta_1 + \cot\delta_2 - \cot\alpha_1 + \cot\alpha + \cot\beta \tag{3}$$

using the notation $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$.

Simple trigonometry shows that:

$$\cos\alpha + \sin\alpha\cot\delta_1 = \frac{\sin(\delta_1+\alpha)}{\sin\delta_1} = \frac{\|P_4 - P_3\|}{\|P_4 - P_1\|} = \frac{\sin\beta_2/\sin(\alpha_2+\beta_2)}{\sin\beta_1/\sin(\alpha_1+\beta_1)} = \frac{\sin\beta_2\sin(\alpha_1+\beta_1)}{\sin\beta_1\sin(\alpha_2+\beta_2)}$$
$$\cos\beta + \sin\beta\cot\delta_2 = \frac{\sin(\delta_2+\beta)}{\sin\delta_2} = \frac{\|P_2 - P_3\|}{\|P_2 - P_1\|} = \frac{\sin\alpha_2/\sin(\alpha_2+\beta_2)}{\sin\alpha_1/\sin(\alpha_1+\beta_1)} = \frac{\sin\alpha_2\sin(\alpha_1+\beta_1)}{\sin\alpha_1\sin(\alpha_2+\beta_2)}$$

Leading to

$$\cot\alpha + \cot\delta_1 = \frac{\sin\beta_2\sin(\alpha_1+\beta_1)}{\sin\beta_1\sin(\alpha_2+\beta_2)\sin\alpha} \tag{4}$$
$$\cot\beta + \cot\delta_2 = \frac{\sin\alpha_2\sin(\alpha_1+\beta_1)}{\sin\alpha_1\sin(\alpha_2+\beta_2)\sin\beta} \tag{5}$$

Substituting (4) and (5) into (3) eliminates δ_1 and δ_2:

$$2\Delta L_{11} = \frac{\sin\beta_2\sin(\alpha_1+\beta_1)}{\sin\beta_1\sin(\alpha_2+\beta_2)\sin\alpha} + \frac{\sin\alpha_2\sin(\alpha_1+\beta_1)}{\sin\alpha_1\sin(\alpha_2+\beta_2)\sin\beta} - \frac{\sin(\alpha_1+\beta_1)}{\sin\alpha_1\sin\beta_1}$$

$$= \frac{\sin(\alpha_1+\beta_1)}{\sin\beta_1 \sin(\alpha_2+\beta_2)\sin\alpha_1}\left[\frac{\sin\alpha_1 \sin\beta_2}{\sin\alpha} + \frac{\sin\beta_1 \sin\alpha_2}{\sin\beta} - \sin(\alpha_2+\beta_2)\right]$$

Using the trigonometric identity:

$$\sin x \sin(y-z) - \sin(x+z)\sin y = -\sin(x+y)\sin z$$

and noting that any angle or sum of any two angles inside a triangle always has positive sine, we have

$$\begin{aligned}
\operatorname{sign}(\Delta L_{11}) &= \operatorname{sign}\left[\frac{\sin\alpha_1 \sin\beta_2}{\sin\alpha} + \frac{\sin\beta_1 \sin\alpha_2}{\sin\beta} - \sin(\alpha_2+\beta_2)\right] \\
&= \operatorname{sign}\left[\frac{\sin\alpha_1 \sin\beta_2}{\sin\alpha} + \frac{\sin\alpha_2 \sin\beta_1 - \sin(\alpha_2+\beta_2)\sin\beta}{\sin\beta}\right] \\
&= \operatorname{sign}\left[\frac{\sin\alpha_1 \sin\beta_2}{\sin\alpha} + \frac{-\sin(\alpha_2+\beta)\sin\beta_2}{\sin\beta}\right] \\
&= \operatorname{sign}\left[\sin\alpha_1 \sin\beta - \sin\alpha \sin(\alpha_2+\beta)\right] \\
&= \operatorname{sign}\left[-\sin\alpha_2 \sin(\alpha+\beta)\right] = \operatorname{sign}\left[-\sin(\alpha+\beta)\right]
\end{aligned}$$

Leading to

$$\operatorname{sign}(b) = \operatorname{sign}(\Delta L_{11}) = \operatorname{sign}\left[-\sin(\alpha+\beta)\right]$$

Thus

$$T_2 \text{ is Delaunay} \quad\Leftrightarrow\quad \alpha+\beta \geqslant \pi \quad\Leftrightarrow\quad b \geqslant 0 \quad\Leftrightarrow\quad \Delta L \text{ is SPSD} \qquad \square$$

We are now in the position to provide a very simple proof of Rippa's theorem.

Theorem. *(See Rippa, 1990.) The Dirichlet energy of any piecewise-linear scalar function f on a triangulation obtains its minimum on the Delaunay triangulation.*

Proof. By Lawson's result (Lawson, 1972), it suffices to prove that the Dirichlet energy of a triangulation never increases following a Delaunay flip. Using the same notation as in the Key Lemma, where E_1 and E_2 are the Dirichlet energies of f on T_1 and T_2, respectively:

$$\text{Delaunay flip} \quad\Leftrightarrow\quad \Delta L \text{ is SPSD} \quad\Leftrightarrow\quad \Delta E = E_1 - E_2 = f^t(\Delta L)f \geqslant 0 \quad\Leftrightarrow\quad E_1 \geqslant E_2 \qquad \square$$

3. The Delaunay spectral theorem

Since the geometric Laplacian L depends on the particular triangulation, it is natural to ask whether the Laplacian associated with the Delaunay triangulation is special in any way. Note that the eigenvalue corresponding to an eigenvector of the geometric Laplacian is the Dirichlet energy of that eigenvector when viewed as a function on the graph vertices. Also note that the Delaunay triangulation is invariant to any similarity transformation in the plane, thus so is its Laplacian (which is not surprising, since it is based exclusively on angles, which are preserved by similarities).

The following theorem characterizes the Delaunay triangulation in terms of the spectrum of its geometric Laplacian:

Delaunay spectral theorem. *The spectrum of the geometric Laplacian obtains its minimum on a Delaunay triangulation. Namely if $\{\lambda_1 = 0, \lambda_2, \ldots, \lambda_n\}$ and $\{\mu_1 = 0, \mu_2, \ldots, \mu_n\}$ are the sequences of non-decreasing eigenvalues of the geometric Laplacian of a Delaunay triangulation and of any other triangulation of the same set of points, respectively, then $\lambda_i \leqslant \mu_i$ for $i = 1, \ldots, n$.*

Proof. Let T and T_D be a non-Delaunay triangulation and a Delaunay triangulation of the same set of points respectively, and L and L_D the geometric Laplacians of T and T_D, respectively. By Rippa's theorem, for any function f defined on the point set, $f^t L f \geqslant f^t L_D f$, therefore $\Delta L = L - L_D$ is SPSD. The spectral theorem follows immediately by Weyl's inequality from matrix perturbation theory (Wilkinson, 1965). $\square$

Corollary. *If the set of points is not in general position, resulting in many Delaunay triangulations, all these have identical spectra. This follows from the Delaunay spectral theorem.*

3.1. Musin's theorem

The Delaunay spectral theorem allows us to obtain an alternative proof of another related theorem. Musin (1997) studied a number of alternative optimality properties of the Delaunay triangulation. Among others, he defined the *harmonic index* of a triangulation:

$$H(T) = \sum_{t \in T} \frac{a_t^2 + b_t^2 + c_t^2}{A_t}$$

where t is a triangle in the triangulation T with edge lengths a_t, b_t, c_t, and area A_t. A small harmonic index means the triangle is "fatter".

Theorem. *(See Musin, 1997.) The harmonic index obtains its minimum on the Delaunay triangulation.*

Proof. Following Bobenko and Springborn (2007), elementary trigonometry shows that the harmonic index is equivalent to:

$$H(T) = 4 \sum_{(i,j,k) \in T} (\cot \alpha_{ij} + \cot \alpha_{jk} + \cot \alpha_{ki})$$

This implies:

$$H(T) = 4 \cdot \mathrm{trace}(L) = 4 \sum_{i=1}^{n} \lambda_i(L)$$

where $\lambda_i(L)$ are the eigenvalues of the geometric Laplacian of T. Applying the Delaunay spectral theorem completes the proof. □

4. Discussion

We have examined the spectral characteristics of the Delaunay triangulation using its geometric Laplacian matrix, and proved what seems to be quite a fundamental theorem: The spectrum of this matrix is minimal among all triangulations. Experimental results show that the combinatorial Laplacian does not have this property. Right now the Delaunay spectral theorem seems to be mostly of theoretical interest, and it would be intriguing to discover whether it has any algorithmic applications.

The spectral theorem leads to alternative proofs of existing theorems. In particular the trace of the Laplacian has a geometric interpretation as the harmonic index of the triangulation, related to its isoperimetric quantities. It would be interesting to see if there is any geometric interpretation of other functions of the spectrum. A strong candidate would be the second eigenvalue, i.e. the smallest non-zero eigenvalue, which, for unweighted graphs, Fiedler (1975) called the "algebraic connectivity". Numerical experiments have shown that if the vertices of the triangulation are distributed uniformly in the plane, then the ratio between the second and third eigenvalues of the geometric Laplacian reflects the "aspect ratio" of the triangulation, namely the ratio between the lengths of the two principal directions of the convex region of the plane that it occupies (the second eigenvalue corresponds to the shorter axis). Geometric interpretations of the Laplacian eigenvectors would also be interesting. The eigenvectors corresponding to the second and third eigenvalues, when used as embedding coordinates, produce a new "drawing" of the triangulation which is similar to the original and "normalized" to be axis-aligned.

There is an interesting analogy between graphs equipped with positive edge weights, random walks on graphs (i.e. Markov chains) and resistor networks (Doyle and Snell, 1984). In these scenarios, the edge weights may be considered the transition probabilities or "conductances" of the edges (namely, the edge "resistances" are $1/w_{ij}$). This value describes how easy it is for a random walker or electrical current to traverse the edge. It has been shown that the weighted Laplacian may be used to characterize the properties of the resulting circuits. In the random walk scenario we may define the "commute time" between any two vertices – the expected time it takes a random walker to travel from the first vertex to the second, and back. It is a weighted average of the traversal times of all possible paths between the two vertices. The equivalent for an electrical circuit is the effective resistance between the two vertices. The sum of the effective resistances between all pairs of vertices may be shown to be (Klein and Randic, 1993):

$$R(L) = \mathrm{trace}(L^+) = \sum_{i=2}^{n} \frac{1}{\lambda_i}$$

where L^+ is the Moore–Penrose pseudo-inverse of L. This is the so-called *total resistance* or *Kirchhoff index* of the weighted graph. The Delaunay spectral theorem implies that the Delaunay triangulation has the maximal total resistance among all triangulations (although the presence of negative weights in non-Delaunay triangulations damages this analogy somewhat).

Future work will extend this study to the case of mappings between two planar triangulations.

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