Complex Transfinite Barycentric Mappings with Similarity Kernels

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Abstract

Transfinite barycentric kernels are the continuous version of traditional barycentric coordinates and are used to define interpolants of values given on a smooth planar contour. When the data is two-dimensional, i.e. the boundary of a planar map, these kernels may be conveniently expressed using complex number algebra, simplifying much of the notation and results. In this paper we develop some of the basic complex-valued algebra needed to describe these planar maps, and use it to define **similarity** kernels, a natural alternative to the usual barycentric kernels.

We develop the theory behind similarity kernels, explore their properties, and show that the transfinite versions of the popular three-point barycentric coordinates (Laplace, mean value and Wachspress) have surprisingly simple similarity kernels.

We furthermore show how similarity kernels may be used to invert injective transfinite barycentric mappings using an iterative algorithm which converges quite rapidly. This is useful for rendering images deformed by planar barycentric mappings.

Categories and Subject Descriptors (according to ACM CCS): I.3.5 [Computer Graphics]: Computer Gemetry and Object Modeling—Boundary representations; G.1.1 [Numerical Analysis]: Interpolation—Interpolation formulas

1. Introduction

1.1 Polygon barycentric coordinates

Barycentric coordinates are typically used to interpolate a real function given on the boundary of a two-dimensional polygon, where the values of the function are specified on the polygon vertices and assumed to vary linearly along the edges. The objective is to generate for any interior point in the polygon a real value which is some natural combination of the values at the vertices.

More concretely, let *P* be a planar polygon with vertices $p_j = (x_j, y_j), j = 1, ..., n$. Given real values f_j at p_j , what should the value f(p) at a point p = (x, y) interior to *P* be ? One way to achieve this is to associate with the *j*'th vertex a *barycentric coordinate function* $B_j(p)$ which satisfies a number of natural conditions, and then define

$$f(p) = \sum_{j=1}^{n} B_j(p) f_j$$
 (1)

In the field of computer graphics and image deformation, this interpolation method has been used to generate mappings between two 2D polygonal regions by associating a 2D vector $f_j = (u_j, v_j)$ with each vertex p_i of P instead of the usual scalar value. This implies that the edges of the source polygon $P = (p_1, ..., p_n)$ are linearly mapped to the edges of the target polygon $F = (f_1, ..., f_n)$ and, through (1), the barycentric coordinate functions $B_i(p)$ define the 2D image f(p) of an interior point $p \in P$.

© 2016 The Author(s) Computer Graphics Forum © 2016 The Eurographics Association and John Wiley & Sons Ltd. Published by Joh Wiley & Sons Ltd. In recent years, many formulae for B_j have been proposed, the most prominent being the Laplace (also called *discrete harmonic* or *cotangent*) [PP93], mean value [Flo03; HF06], Wachspress [Wac75] and harmonic [JMD*07]. The first three are given in closed form for any interior point, while the harmonic coordinates must be computed numerically, typically using a Finite-Element Method (FEM) resulting in a discrete Laplace equation with appropriate Dirichlet boundary conditions on *P*. The reader may consult the recent survey by Floater [Flo15] for a comprehensive overview of many barycentric recipes.

1.2 Transfinite barycentric kernels

The concept of barycentric coordinates of a polygon *P* may be generalized to the case of a closed simple planar contour *C*, which may be treated as the limit of a polygon with an increasing number of vertices. Thus the discrete index *j* becomes a continuum (represented by the parameter *c*), the value of *f* is given at all points of *C* (the so-called *boundary values*), the discrete sum is replaced by a boundary integral, the finite set of barycentric coordinate functions $B_j(p)$ is replaced by a *kernel* function K(c, p), and (1) becomes:

$$f(p) = \oint_C K(c, p) f(c) ds$$
 (2)

namely, the value of f at an interior point p is defined as some weighted average of the continuum of values of f on the boundary C. The quantity ds is the usual arc-length differential of C.

In general, it seems to be difficult to obtain closed formulae for the continuous kernels of most discrete barycentric recipes, even those for which closed forms are available in the discrete case. Belyaev [Bel06] and others (e.g. [SJW07, DF09]) were able to express the kernels in some limited cases, showing that one of the most basic kernels is the mean value kernel, and other members of the so-called *three-point* family are generalizations of this basic kernel. However, their formulation is complicated somewhat by the fact that the integral (2) is computed using the density implied by a unit circle centered at *p*:

$$f(p) = \oint_{C} K(c, p) f(c) d\theta$$
(3)

where the differential $d\theta$ is the projection of ds onto this unit circle and quite difficult to work with.

1.3 Contribution

Our contribution in this paper is mostly theoretical. Representing 2D vectors using complex numbers, we develop in Theorem 1 the complex algebra needed to conveniently and compactly describe planar transfinite barycentric kernels. This is achieved by introducing the key quantity h(w, z) related to the geometry of the source contour. We then introduce in Section 4 the concept of similarity kernel, which allows an alternative description of tranfinite barycentric mappings using the complex dw and df differentials (on the source and target contours, repectively). After exploring the properties of similarity kernels, we use the quantities of Theorem 1 to compactly express the similarity kernels of the popular three-point barycentric coordinates in Theorem 10. Despite the theoretical focus of this paper, we make a practical contribution in Section 7 by showing how to use the developed theory to efficiently invert 2D mappings based on transfinite barycentric coordinates, in particular the mean value mapping. This inversion is important in texture mapping and image deformation applications. Proofs of all Theorems and Corollaries are provided in the Appendix.

2. Complex Barycentric Coordinates

In the planar scenario, it is convenient to represent 2D vectors $(x, y) \in \mathbb{R}^2$ as complex numbers $z \in \mathbb{C}$. Thus, analogously to (1), the mapping of the interior of a source polygon *W* whose vertices are $(w_1, ..., w_n)$ to the interior of a target polygon whose vertices are $(f_1, ..., f_n)$ is:

$$f(z) = \sum_{j=1}^{n} B_j(z) f_j$$
 (4)

where z is a point in the interior of W, and the B_j should satisfy: C1. Constant precision:

C2. Linear precision:

$$\sum_{j=1}^{n} B_j(z) = 1, \quad \forall z \in int(W)$$

$$\sum_{j=1}^{n} z_j B_j(z) = z, \quad \forall z \in int(W)$$
C3. Lagrange property:

$$B_j(z_k) = \delta_{jk}, \quad j, k = 1, \dots, n$$

Weber et al. [WBG09] showed that it is possible to express any barycentric coordinate function using a set of complex-valued functions γ_j associated with the *j*'th edge (j = 1, ..., n):

$$\hat{B}_{j}(z) = \gamma_{j}(z) \frac{r_{j+1}(z)}{e_{j}} - \gamma_{j-1}(z) \frac{r_{j-1}(z)}{e_{j-1}},$$
$$B_{j}(z) = \frac{\hat{B}_{j}(z)}{\sum_{j=1}^{n} \hat{B}_{j}(z)}$$
(5)

 $r_j(z)$ is the difference $w_j - z$ and e_j is the edge vector $w_{j+1} - w_j$. See Fig. 1. Weber et al. [WBGH11] show that the so-called "three-point coordinates" - Wachspress, mean value and Laplace - may be obtained for

$$\gamma_j(z) = \frac{e_j}{\operatorname{Im}(\bar{r}_j(z)r_{j+1}(z))} \left(\frac{|r_{j+1}(z)|^{\nu}}{r_{j+1}(z)} - \frac{|r_j(z)|^{\nu}}{r_j(z)} \right)$$
(6)

with p = 0, 1, 2, respectively.

The advantage of using the complex formulation is that many of the formulae become very simple. For example, Weber et al. [WBG09] showed that the so-called *Green coordinates* introduced by Lipman et al. [LLC08] to generate conformal mappings may be expressed very simply in the complex formulation, essentially by integrating the very simple kernel $K(w, z) = \frac{1}{w-z}$ featuring in Cauchy's integral formula [Ahl79] over each polygon edge (with the complex differential dw).



Figure 1: The terminology of discrete complex barycentric coordinates.

Interpreting (4) and (5) in a different way, Weber et al. [WBGH11] observed that there is a more natural way of expressing the planar barycentric mapping. Consider a source edge vector $e_j = w_{j+1} - w_j$ and its corresponding target edge vector $\hat{e}_j = f_{j+1} - f_j$. This pair of edges defines a unique similarity mapping (translation, rotation and scale) of the plane

$$S_j(z) = \frac{r_{j+1}(z)f_j - r_j(z)f_{j+1}}{e_j} = f_j + \frac{\widehat{e_j}}{e_j}(z - w_j)$$

and the barycentric mapping can be defined as a weighted average of these *edge-to-edge similarities*. It turns out that the equivalent to (4) with the $B_i(z)$ as defined in (5) is:

$$f(z) = \frac{\sum_{j=1}^{n} \gamma_j(z) S_j(z)}{\sum_{j=1}^{n} \gamma_j(z)}$$
(7)

so (after normalization), the $\gamma_j(z)$ are the correct way to weight the edge-to-edge similarities. Note that, as opposed to the barycentric coordinate functions $B_j(z)$, which are real-valued functions, the $\gamma_j(z)$ functions will typically be complex-valued. We will return to this interpretation in Section 4, when we generalize it to the transfinite case using *similarity kernels*.

3. Complex Transfinite Barycentric Coordinates

When the discrete polygon W of Section 2 is replaced with a continuous contour C, the finite set of *barycentric coordinate functions* $B_j(z)$ is replaced by a bivariate *transfinite barycentric kernel* K(w, z). The kernel should satisfy conditions analogous to conditions C1-C2 above:

C1. Constant precision:

$$\oint_{C} K(w, z) dw = 1, \forall z \in int(W)$$
C2. Linear precision:

$$\oint_{C} zK(w, z) dw = z, \forall z \in int(W)$$

In an attempt to express the transfinite version of a number of popular barycentric coordinate schemes, Belyaev [Bel06] showed that the easiest way to do this was to introduce a new differential $d\theta$, the projection of the *ds* differential on the unit circle centered at *z* (see Fig. 2). Defining r = w - z, Belyaev showed that, for a convex curve *C*, the (un-normalized) kernel of the mean value coordinates is:

$$K_{MV}(w,z)ds = \frac{1}{|r|}d\theta \tag{8}$$

so the mean value interpolant is

$$f(z) = \frac{\int_{0}^{2\pi} \frac{1}{|r|} f(w) d\theta}{\int_{0}^{2\pi} \frac{1}{|r|} d\theta}$$
(9)

as the interval $[0,2\pi]$ represents the extent of the θ variable, which, in the convex case, is positive and monotone as the boundary is traversed. While this is elegant, it does not generalize well to the non-convex case. Belvaev overcomes this in the same way that Hormann and Floater [HF06] overcome it in the discrete mean value case (and this is also applicable to the non-convex case for the Gordon-Wixom coordinates [GW74]), by considering all intersections of a ray originating at z with the contour. Dyken and Floater [DF09] express the transfinite mean value kernel using vector algebra, but have the same problem as Belyaev for non-convex contours. It turns out that it is much more natural, and simple, to address the non-convex case using true boundary integrals, in particular contour integrals of complex analysis. This was done, albeit for the discrete case, by Weber et al. [WBGH11], who showed that using complex-valued expressions, it was possible to simply capture also the non-convex case. In the sequel, we extend Weber et al's approach to the transfinite case, using complex contour integrals with the complex differential dw.

3.1 The basic quantities

To facilitate the complex approach, we introduce the *height* function, $h(w, z): C \times int(C) \rightarrow C$, the difference between z and its projection onto the tangent line to C at w, which will play a central role in everything we will do from now on. See Fig. 2 for an illustration.

The following key Theorem summarizes the properties of h and relates it to the other known quantities and differentials expressed in the algebra of complex numbers. Many of the quantities are functions of variables, but from now on, and throughout the paper, for compactness sake we will drop some of the variables where they are obviously implied. For example r(w, z) = w - z is a function of w and z, but we will just write r. Similarly h(w, z) is also a function of w and z, but we will mostly write just h. Note that we also use the $d\theta$ differential, which is always a real-valued quantity, but in the general non-convex case may be negative.





Figure 2: *The terminology of complex transfinite barycentric mappings.*

Theorem 1: Let *C* be a simple closed planar contour in counterclockwise orientation, and *z* a point within *C*. For any point $w \in C$, denote r = w - z, $\theta = \arg(r)$, namely $r = |r| \exp(i\theta)$, A =the (signed) area swept by *r*, and *h* = the difference between *z* and its projection onto the tangent line to *C* at *w*. Then

(a)
$$h(w, z) = \frac{1}{2} \left(r - \frac{dw}{dw} \bar{r} \right)$$

(b)
$$hd\bar{w} = -\bar{h}dw = i \text{Im}(rd\bar{w}),$$

$$(r - h)d\bar{w} = \text{Re}(rd\bar{w})$$

(c)
$$dA = \frac{1}{2} |r|^2 d\theta = \frac{1}{2i} \bar{h}dw = \frac{1}{2} \text{Im}(\bar{r}dw)$$

(d)
$$d\theta = -i\frac{\bar{h}}{|r|^2} dw = \text{Im}\left(\frac{dr}{r}\right) = \text{Im}\left(\frac{dw}{r}\right),$$

$$d\log|r| = \text{Re}\left(\frac{dw}{r}\right)$$

(e)
$$\oint_{\mathbb{C}} \bar{h} dw = 2i\text{Area}(C) = \oint_{\mathbb{C}} \bar{w} dw$$

(f)
$$\text{Re}\left(\frac{dw}{h}\right) = 0$$

(g)
$$\text{Re}\left(\frac{r}{h}\right) = 1$$

(h)
$$d|r| = |r|\text{Re}\left(\frac{dw}{r}\right), \quad \frac{d\theta}{|r|} = -\frac{i}{r}d\left(\frac{r}{|r|}\right)$$

(i)
$$\frac{\partial h}{\partial z\partial \bar{z}} = \frac{\partial \bar{h}}{\partial z\partial \bar{z}} = 0$$

3.2 The polar dual

To illustrate some of the concepts introduced in the previous section, we look at some special cases. First we observe that the *polar dual* used by Schaefer et al. [SJW07] in their investigation of transfinite barycentric coordinates may be expressed very simply using *h*. The polar dual p(w, z) of a point *w* on a given contour *C* relative to an interior point *z* is defined as the vector orthogonal to *dw* having unit scalar product with w - z.

Corollary 1: The polar dual of a contour relative to a point *z* is $1/\overline{h}$.

3.3 The unit circle

We now examine the simplest possible contour – the (unit) circle. First we derive a simple expression for h in this case:

Corollary 2: If *C* is the unit circle, then $h(w,z) = w \operatorname{Re}(\bar{r}w) = w \operatorname{Re}(r\bar{w}) = w \operatorname{Re}\left(\frac{r}{w}\right)$ $= w \left(1 - \operatorname{Re}(w\bar{z})\right)$

The interested reader is also referred to the end of Section 5.2 for an alternative expression for h on the unit circle.

It is well known that the harmonic barycentric kernel on the unit circle is the so-called *Poisson kernel* [Bel06], given in complex form as:

$$K_P^N(w,z)ds = \frac{1}{2\pi} \operatorname{Re}\left(\frac{w+z}{w-z}\right) ds = \frac{1}{2\pi} \frac{1-|z|^2}{|r|^2} ds \qquad (10)$$

The superscript N means that this kernel is normalized: $\oint_C K_P^N(w, z) ds = 1$ for all z. It is possible to superficially simplify this formula by noticing that some of it is a function of z only, thus may be omitted:

$$K_P(w,z)ds = \frac{1}{|r|^2}ds$$
 (11)

and recovered as part of a standard normalization procedure, i.e. given boundary values f(w), the Dirichlet problem on the unit disk may be solved as

$$H(z) = \frac{\oint_C K_P(w, z) f(w) \, ds}{\oint_C K_P(w, z) \, ds}$$

If a given transfinite barycentric coordinate scheme reduces to the Poisson kernel for the special case that the contour is the unit circle, we say that the coordinates are *pseudo-harmonic*. Belyaev [Bel06] has shown that the three-point coordinates, in the continuous limit, are not pseudo-harmonic, but a number of others, most notably those introduced by Gordon and Wixom [GW74], are. Recently, Chen and Gotsman [CG15] have shown that the Moving Least Squares (MLS) coordinates introduced by Manson and Schaefer [MS10] are also pseudo-harmonic.

As an exercise, we may use Corollary 2 to prove Belyaev's [Bel06] observation that the Poisson kernel K_P may be expressed very simply using h:

Corollary 3: On the unit circle, the (un-normalized) Poisson kernel is

$$K_P ds = \frac{1}{|h|} d\theta$$

4. Similarity and Anti-Similarity Kernels

4.1 Motivation

When constructing barycentric coordinates, a relevant and natural transformation of the plane is the similarity transformation (i.e. scale, rotation and translation), which, in complex-valued algebra, is just a simple linear transformation. Similarities are also the fundamental building block of conformal maps, which locally are just similarities. The guiding principle will be then, given a boundary mapping $f: C \rightarrow F$ and z, to identify and blend all similarities relevant to z and the given mapping, resulting in f(z).

The main question is how to derive relevant similarities from the boundary mapping, i.e. the source contour and its target image. Recall that a planar similarity is defined by the images of two points, namely two corresponding line segments. Fig. 3 illustrates two possibilities, given a point $z \in int(C)$, to glean similarities from the data related to an aribitrary point w on C: The first method is to consider the line segment through w and some other point $u \in C$. Both w and u have images on the target contour -f(w) and f(u) respectively. Thus the line segment [f(w), f(u)] relative to the line segment [w, u] defines a unique similarity transformation $S_f^1(z)$, called a *boundary similarity*, which may be applied to z. An interesting special case is when w and u are *antipodes* relative to z, namely $z \in [w, u]$. This will force $f(z) \in$

[f(w), f(u)]. We call S_f^1 in this case an *antipodal similarity*. The second method is to consider a small portion of *C* at *w*, and the corresponding image of this portion at f(w). Essentially these are the differentials dw and df, defining a unique similarity transformation $S_f^2(z)$, which may be applied to *z*. We call S_f^2 a *differential similarity*. The precise formulae for these two similarities are:

$$S_{f}^{1}(z) = f(w) - (w - z)\frac{f(w) - f(u)}{w - u} = f - r\frac{f(w) - f(u)}{w - u}$$
$$S_{f}^{2}(z) = f(w) - (w - z)\frac{df}{dw} = f - r\frac{df}{dw}$$

where df is the differential of f along the contour, which can be expressed mathematically as $df = \frac{\partial f}{\partial w} dw + \frac{\partial f}{\partial \overline{w}} d\overline{w}$.

When constructing transfinite barycentric coordinates using similarities, the key questions are, given z, which similarities to use, and how to blend them in order to satisfy properties C1-C2 of Section 1.2. Probably one of the earliest attempts to do this was by Gordon and Wixom [GW74], who considered all possible antipodal similarities of z. These were blended (i.e. integrated) using the angular differential $d\theta$ to generate f(z). In the sequel we will work exclusively with differential similarities, showing that these are a quite natural way to describe transfinite barycentric mappings.



Figure 3: *(left)* Boundary similarity defined by [u, w] and [f(u), f(w)], *(right)* differential similarity defined by dw and df.

4.2 Similarity kernels

We now show how to use differential similarities to construct transfinite barycentric coordinates, analogously to what Weber et al. [WBGH11] did with edge-to-edge similarities in the discrete case (as described in Section 2). Assume that, given z, we want to blend all possible differential similarities using a complex *dw* integral with weighting function $\sigma(w, z)$, which we call a *similarity kernel* relating a boundary point w with an interior point z. Dropping the superscript of S_f^2 , f(z) is defined as a normalized complex contour integral of the form:

$$f(z) = \frac{\oint_{\mathbb{C}} \sigma(w, z) S_f(w, z) dw}{\oint_{\mathbb{C}} \sigma(w, z) dw}$$
(12)

Note that, since z can be considered a constant, dw = d(w - z) = dr,

$$S_f dw = f dr - r df = -r^2 d\left(\frac{f}{r}\right) \tag{13}$$

Thus (12) becomes

$$f(z) \oint_{C} \sigma(w, z) dw = \oint_{C} \sigma(w, z) (f dw - r df)$$

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implying, since f(z) can be considered a constant:

$$0 = \oint_{\mathsf{C}} \sigma(r_f dr - r dr_f) = -\oint_{\mathsf{C}} \sigma r^2 d\left(\frac{r_f}{r}\right)$$

where $r_f = f(w) - f(z)$. By the derivative product rule (d(fg) = fdg + gdf), this is equivalent to:

$$\oint_{\mathbb{C}} \frac{r_f}{r} d(\sigma r^2) = 0$$

This means that blending differential similarities with density σdw to obtain f(z) (as in (12)) is equivalent to defining f(z) such that:

$$f(z) = \frac{\oint_C \frac{f}{r} d(\sigma r^2)}{\oint_C \frac{1}{r} d(\sigma r^2)}$$
(14)

Example: As we shall see later, the transfinite mean value barycentric coordinates can be generated using differential similarities with the similarity kernel $\sigma = \frac{1}{|r|r}$. So (14) implies that the mean value coordinates satisfy:

$$f(z) = \frac{\oint_{\mathbb{C}} \frac{f}{r} d\left(\frac{r}{|r|}\right)}{\oint_{\mathbb{C}} \frac{1}{r} d\left(\frac{r}{|r|}\right)} = \frac{\oint_{\mathbb{C}} \frac{f}{r} \frac{r}{i|r|} d\theta}{\oint_{\mathbb{C}} \frac{1}{r} \frac{r}{i|r|} d\theta} = \frac{\oint_{\mathbb{C}} \frac{f(w)}{|r|} d\theta}{\oint_{\mathbb{C}} \frac{1}{|r|} d\theta}$$

where the second equality is due to Theorem 1(h). This is consistent with Belyaev's [Bel06] formula (9) for the transfinite mean-value coordinates.

In general, (14) can be considered a formulation analogous to that of Schaefer et al. [SJW07]. Whereas they express general barycentric coordinates as a Shepard-type integral (using the 1/|r| kernel and *ds* differential) over an auxiliary contour, we express the coordinates as a Cauchy-type integral (using the 1/r kernel and *dw* differential) over the auxiliary contour σr^2 .

4.3 Anti-similarity kernels

Analogously to the use of differential similarities, we could use differential *anti-similarities*:

$$f(z) = \frac{\oint_{\mathbb{C}} \alpha(w, z) A_f(z) d\overline{w}}{\oint_{\mathbb{C}} \alpha(w, z) d\overline{w}}$$
(15)

where $A_f(z)$ is the differential *anti-similarity* function

$$A_f(w,z) = f(w) - \bar{r}\frac{df}{d\bar{w}}$$

and $\alpha(w, z)$ is an *anti-similarity kernel*. Analogously to (13),

$$A_f d\overline{w} = f d\overline{r} - \overline{r} df = -\overline{r}^2 d\left(\frac{J}{\overline{r}}\right) \tag{16}$$

We say that the similarity kernel σ and the anti-similarity kernel α *correspond* if they generate identical mappings, i.e. for all contours *C* and all boundary mappings *f*:

$$\frac{\oint_{\mathbb{C}} \sigma(w,z) S_f(w,z) dw}{\oint_{\mathbb{C}} \sigma(w,z) dw} = \frac{\oint_{\mathbb{C}} \alpha(w,z) A_f(z) d\overline{w}}{\oint_{\mathbb{C}} \alpha(w,z) d\overline{w}}$$

4.4 Boundary interpolation

In many applications, barycentric coordinates are used for interpolating a given boundary mapping. Thus it is essential that applying the similarity kernel in (12) or anti-similarity kernel in (15) reproduce the given boundary conditions, which is not at all obvious. The following theorem characterizes when boundary mapping reproduction occurs.

Theorem 2: A similarity kernel σ reproduces a given boundary mapping f(w) of ∂C iff the following conditions are satisfied for every $u \in \partial C$: © 2016 The Author(s)

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(a)	$\lim_{z \to u} \oint_C \sigma(w, z) dw = \infty$
(b)	$\lim_{z \to u} \oint_C r\sigma(w, z) dw < \infty$
(c)	$\forall x \neq u \lim \sigma(x,z) < \infty$

4.5 Affine Reproduction

A fundamental requirement from the similarity and anti-similarity kernels is that it they have the correct precision, namely reproduce constant, similarity and affine functions. Constant precision is immediate if the kernel is normalized to unity. It is also easy to show (in analogy to the discrete case treated by Weber et al. [WBGH11]) that any similarity kernel automatically reproduces similarity functions f(z) = az + b and any anti-similarity kernel automatically reproduces anti-similarity functions $f(z) = a\bar{z} + b$:

Theorem 3:

- (1) Every similarity kernel σ reproduces similarity transformations.
- (2) Every anti-similarity kernel α reproduces anti-similarity transformations.

Reproduction of affine functions, namely reproduction of *both* f(z) = z and $f(z) = \overline{z}$ is not guaranteed for all similarity or anti-similarity kernels. The following theorem characterizes when this happens.

Theorem 4: A similarity kernel σ reproduces affine transformations iff (for every $z \in C$)

$$\oint_C \sigma(w,z)h(w,z)d\overline{w}=0$$

The following theorem provides an alternative characterization of affine reproduction based on the relationship between the similarity kernel σ and its conjugate $\bar{\sigma}$.

Theorem 5: σ is an affine reproducing similarity kernel iff $\overline{\sigma}$ is an affine reproducing similarity kernel.

The following theorem provides a family of similarity kernels which are affine-reproducing.

Theorem 6: Any similarity kernel of the forms:

$$\sigma_1(w, z) = \frac{1}{h\bar{r}^k}, \text{ integer } k \neq 1$$

$$\sigma_2(w, z) = \frac{1}{\bar{h}r^k}, \text{ integer } k \neq 1$$

$$\sigma_3(w, k) = \frac{1}{|r|r}$$

$$\sigma_4(w, k) = \frac{1}{|r|\bar{r}}$$
is affine-reproducing

is affine-reproducing.

4.6 Conjugate Reproduction

Similarly to affine reproduction, it is natural to ask under which conditions is the conjugate function \bar{f} reproduced by a similarity kernel σ if we know that σ reproduces f.

First we characterize the conjugation relationship between a similarity kernel σ and its corresponding anti-similarity counterpart α.

Theorem 7: If a similarity kernel σ satisfies $\operatorname{Re}(\sigma dr) + d\operatorname{Re}(\sigma r) = 0$ then a corresponding anti-similarity kernel is $\alpha = \overline{\sigma}$.

Now we can see when a similarity kernel reproduces conjugates:

Theorem 8: Under the conditions of Theorem 7, if σ reproduces f, then σ also reproduces \bar{f} .

4.7 Uniqueness

The following Theorem shows that a similarity kernel of a barycentric mapping is not unique:

Theorem 9: If σ is a similarity kernel of a barycentric mapping, then so is $\sigma' = a\sigma + \frac{b}{r^2}$, for any complex functions a(z) and b(z) that depend only on z.

5. **Transfinite three-point coordinates**

We now ready to state a central theoretical result of this paper, namely that using similarity kernels allows us to express the transfinite versions of the popular "three-point" barycentric coordinate schemes in a surprisingly simple form. Some of them involve the h(w, z) function introduced in Section 3.1.

5.1 Discrete three-point coordinates

Given a simple planar polygon W having vertices (in complex form) w_i , j = 1, ..., n, the discrete three-point schemes mentioned in Section 2 express $B_i(z)$ – the un-normalized barycentric coordinate function associated with w_i as a function of just w_{j-1}, w_j, w_{j+1} (and of course z). This is reflected also in (6). For example, the Laplace coordinate function is just the sum of the cotangents of the two angles formed by (w_i, w_{i-1}, z) and (w_j, w_{j+1}, z) : α_{j-1} and β_j in Fig. 1. Similarly, the mean-value and Wachspress coordinates involve only angles and edge lengths in these two triangles.

The mean value coordinates are particularly interesting as they are derived by trying to mimic the mean value property of harmonic functions [Flo03]. Yet, as Belyaev [Bel06] notes, they are not pseudo-harmonic, and neither are the Wachspress or Laplace coordinates. In contrast, when the polygon is a square, both the Wachspress and Laplace coordinates are exactly the harmonic bilinear coordinates. For the square whose vertices are (1, i, -1, -i), these are:

$$B_k(z) = \frac{1}{4}((1+zi^{1-k})^2), \quad k = 1,..,4$$

This was noted by Floater et al. [FHK06], who also noted that these two coordinates coincide for any circular polygon, i.e. a polygon whose vertices all lie on a circle, including the regular polygons (those having equal edge lengths).

5.2 Similarity kernels of three-point coordinates

We now show that the transfinite versions of the three-point coordinates may be expressed very simply using similarity kernels.

Theorem 10: The similarity kernels of the three-point barycentric coordinates are:

• Laplace:
$$\sigma_L = \frac{1}{h}$$

• Wachspress:
$$\sigma_W = \frac{1}{hr}$$

Mean value: $\sigma_{MV} = \frac{1}{|r|r}$

Note that these kernels are affine-reproducing by Theorem 6, Furthermore, the conditions of Theorems 7 and 8 also hold:

Laplace: $\operatorname{Re}(\sigma_L dr) + d\operatorname{Re}(\sigma_L r) = \operatorname{Re}\left(\frac{dw}{h}\right) + d\operatorname{Re}\left(\frac{r}{h}\right) = 0 + d1 = 0$ by Theorem 1(f) and Theorem 1(g).

Mean-Value: $\operatorname{Re}(\sigma_{MV}dr) + d\operatorname{Re}(\sigma_{MV}r) = \operatorname{Re}\left(\frac{dr}{r|r|}\right) + d\operatorname{Re}\left(\frac{r}{r|r|}\right) = \frac{1}{|r|}\operatorname{Re}\left(\frac{dr}{r}\right) + d\frac{1}{|r|} = \frac{1}{|r|^2}d|r| - \frac{1}{|r|^2}d|r| = 0$ by Theorem 1(h). $\operatorname{Re}(\sigma_W dr) + d\operatorname{Re}(\sigma_W r) = \operatorname{Re}\left(\frac{dr}{\bar{h}r^2}\right) +$ Wachspress: $d\operatorname{Re}\left(\frac{1}{\overline{h}r}\right) = \operatorname{Re}\left(\frac{dr}{\overline{h}r^2} + \frac{1}{r}d\left(\frac{1}{\overline{h}}\right) + \frac{1}{\overline{h}}d\left(\frac{1}{r}\right)\right) = \operatorname{Re}\left(-\frac{1}{\overline{h}}d\left(\frac{1}{r}\right)\right) + \frac{1}{\overline{h}}d\left(\frac{1}{\overline{h}}\right) = \operatorname{Re}\left(-\frac{1}{\overline{h}}d\left(\frac{1}{\overline{h}}\right) + \frac{1}{\overline{h}}d\left(\frac{1}{\overline{h}}\right)\right) = \operatorname{Re}\left(-\frac{1}{\overline{h}}d\left(\frac{1}{\overline{h}}\right)\right) = \operatorname{Re}\left(-\frac{1}{\overline{h}}d\left(\frac{1}{\overline{h}}d\left(\frac{1}{\overline{h}}\right)\right) = \operatorname{Re}\left(-\frac{1}{\overline{h}}d\left(\frac{1}{\overline{h}}d\left$ $\frac{1}{r}d\left(\frac{1}{\bar{h}}\right) + \frac{1}{\bar{h}}d\left(\frac{1}{r}\right) = \operatorname{Re}\left(\frac{1}{r}d\left(\frac{1}{\bar{h}}\right)\right) = \frac{1}{|r|^2}\operatorname{Re}\left(\bar{r}d\left(\frac{1}{\bar{h}}\right)\right) =$ $\frac{1}{|r|^2} \operatorname{Re}\left(d\left(\frac{\bar{r}}{\bar{h}}\right) - \frac{d\bar{r}}{\bar{h}}\right) = \frac{1}{|r|^2} \left(d\operatorname{Re}\left(\frac{\bar{r}}{\bar{h}}\right) - \operatorname{Re}\left(\frac{d\bar{r}}{\bar{h}}\right)\right) = \frac{1}{|r|^2} \left(d1 - 0\right) = 0 \text{ by Theorem 1(f) and Theorem 1(g).}$

To show one simple consequence of Theorem 10, we now prove the continuous equivalent of the result of Floater et al [FHK06] that the Wachspress and Laplace mappings are equivalent on a circular polygon.

Theorem 11: The similarity kernels of the Wachspress mapping and the Laplace mapping (as expressed in Theorem 10) are equivalent on the unit circle.

It is worth mentioning that the normalization integrals for the Laplace and Wachspress similarity kernels on the unit circle may be computed using Cauchy's Residue Theorem [Ahl79] based on the observation:

$$h = -\frac{\bar{z}}{2}(w-a)(w-b) \quad where \ a,b = \frac{1 \pm \sqrt{1-|z|^2}}{\bar{z}}, a\bar{b}$$
$$= 1, |a| > 1, |b| < 1$$

 $2\pi i$

which leads to:

Wachspress:

Laplace:

dw

$$\oint_{C} h = \frac{1}{(1-|z|^{2})^{\frac{1}{2}}}$$
$$\oint_{C} \frac{dw}{\bar{h}r^{2}} = \frac{2\pi i}{(1-|z|^{2})^{\frac{3}{2}}}$$

5.3 The Cauchy kernel

It is worth comparing the situation for the three-point barycentric coordinates to that of the "Cauchy coordinates" of Weber et al. [WBG09], inspired by the celebrated Cauchy integral theorem of complex analysis, which makes use of the so-called "Cauchy ker-nel" $K_C dw = \frac{1}{2\pi i} \frac{dw}{r}$. Defining

$$g(z) = \frac{1}{2\pi i} \oint_{\mathbb{C}} \frac{f(w)}{r} dw$$

yields a g which is always holomorphic. If f is the boundary mapping of a holomorphic function, then g = f, otherwise g is the *projection* of f onto the linear space of holomorphic functions.

When used in the discrete barycentric setting, Cauchy coordinates always generate a conformal map which does not always interpolate the given target boundary. Nonetheless, it is interesting to observe that for these special coordinates, the similarity kernel is identical to the Cauchy kernel:

$$\frac{1}{2\pi i} \oint_{\mathsf{C}} \frac{(fdr - rdf)}{r} = \frac{1}{2\pi i} \oint_{\mathsf{C}} \frac{f}{r} dw - \frac{1}{2\pi i} \oint_{\mathsf{C}} df = g(z) - 0$$
$$= g(z)$$

6. Relating the barycentric kernel and the similarity kernel

In the previous two sections we introduced the concept of a similarity kernel as a simpler alternative to the conventional barycentric kernel. In this section we relate between the two.

6.1 The differential equation

Theorem 12: If *B* is a barycentric kernel (generating *f* using the complex dw contour integral), and σ satisfies $Bdw = 2\sigma dw + rd\sigma$ then σ is a similarity kernel corresponding to *B*.

To illustrate, it is easy to use Theorem 12 to verify that for the Cauchy kernel $B_C = \frac{1}{2\pi i r}$, we have $B_C = \sigma_C$. It is also easy to verify Theorem 9, namely that both σ and $a(z)\sigma + \frac{b(z)}{r^2}$ correspond to the same *B*.

Now let us examine a little more closely the mean value similarity kernel: $\sigma_{MV} = \frac{1}{r|r|} = r^{-\frac{3}{2}} \bar{r}^{-\frac{1}{2}}$. Observing that $d\sigma_{MV} = -\frac{3}{2} r^{-\frac{5}{2}} \bar{r}^{-\frac{1}{2}} dw - \frac{1}{2} r^{-\frac{3}{2}} \bar{r}^{-\frac{3}{2}} d\overline{w} = -\frac{1}{2} \left(\frac{3dw}{r^2|r|} + \frac{d\overline{w}}{|r|^3} \right)$, The-

orem 12 implies $1 (dw d\overline{w})$

$$B_{MV}dw = 2\sigma_{MV}dw + rd\sigma_{MV} = \frac{1}{2}\left(\frac{dw}{|r|} - \frac{dw}{\bar{r}|r|}\right)$$
$$= \frac{i}{|r|} \operatorname{Im}\left(\frac{dw}{r}\right)$$
by Theorem 1(d).

So that, by Theorem 1(d),

$$\oint_{C} fB_{MV}dw = i \oint_{C} f \frac{1}{|r|} \operatorname{Im}\left(\frac{dw}{r}\right) = i \oint_{C} f \frac{1}{|r|} d\theta$$

which is equivalent (up to normalization) to Belyaev's [Bel06] mean-value integral (9).

7. **Inverting Barycentric Maps**

We now present an application of similarity kernels, which arises when using barycentric mappings in a real-world application. Suppose we wish to render a deformation of a digital image given within a source contour, where the deformation is described using a barycentric mapping of the interior of the source contour to the interior of a target contour. This rendering is typically done using *texture mapping*, which involves generating the resulting image one pixel at a time. For each such pixel in the resulting image, the so-called pre-image of the pixel in the source image is found and

this area is filtered. In order to determine the pre-image, it is necessary to invert the mapping.

Inverting a barycentric mapping is not straightforward. The simplest version of the problem - inverting the bilinear map of a unit square - has been treated since these mappings are frequently used in computer graphics. The exact solution reduces to a quadratic equation, which involves computing a square root [Qui16]. While this in principle should not be difficult, it is known to suffer from numerical imprecision in single-precision shaders. Thus, in practice, the bilinear map is inverted using an iterative algorithm. The interested reader is referred to details in [S14].

We are not aware of any existing algorithm to invert a general barycentric mapping. We now describe an iterative Newton algorithm which achieves this, taking advantage of the similarity kernel concept. Given u = f(z), and assuming f is injective, we would like to compute z, namely solve the equation u - f(z) =0 for z. The Newton method over the field of complex numbers implies the following iteration scheme, given an initial guess z_0 :

$$z_{n+1} = z_n + \frac{(u - f(z_n))\bar{a} - (u - f(z_n))b}{|a|^2 - |b|^2}$$
(17)

where the key quantities are the gradients of *f*:

$$a = \frac{\partial f}{\partial z}(z_n), \quad b = \frac{\partial f}{\partial \bar{z}}(z_n)$$

The observant reader will note that the expression $|a|^2 - |b|^2$ in (17) is the Jacobian of f at z_n . Computing the gradients of f as a linear combination of gradients of the traditional barycentric coordinates is not that easy, and the formulae for the mean value case, when using the usual barycentric formulae, as derived by Thiery et al. [TTB14] for the discrete case and by Dyken and Floater [DF09] for the transfinite case, are quite complicated. While numerical computation of the gradients is a possibility, it is always better to use analytic formulae, especially if they are simple. Indeed, the main advantage of using our similarity and anti-similarity kernels is that the resulting formulae are quite simple. This is thanks to the fact that the mappings involve integrals using only the dw and df differentials, which are independent of z, thus permit differentiation under the integral sign. Contrast this to Belyaev's formula (9) for the mean value mapping, which although simple, involves the $d\theta$ differential, which depends on both w and z, thus cannot be differentiated under the integral sign.

We proceed by exploiting the similarity and anti-similarity kernel formulations (12) and (15), differentiating under the integral sign:

$$a = \frac{\partial f}{\partial z}(z) = \frac{\partial}{\partial z} \left(\frac{\oint_{c} \sigma(f \, dw - r \, df)}{\oint_{c} \sigma \, dw} \right)$$

$$= \frac{\oint_{c} \frac{\partial \sigma}{\partial z} (f \, dw - r \, df) + \oint_{c} \sigma \, df}{\oint_{c} \sigma \, dw} - \frac{\oint_{c} \sigma(f \, dw - r \, df) \oint_{c} \frac{\partial \sigma}{\partial z} \, dw}{(\oint_{c} \sigma \, dw)^{2}}$$

$$= \frac{\oint_{c} \frac{\partial \sigma}{\partial z} (f \, dw - r \, df) + \oint_{c} \sigma \, df - f(z) \oint_{c} \frac{\partial \sigma}{\partial z} \, dw}{\oint_{c} \sigma \, dw}$$

$$= \frac{\oint_{c} (f - f(z)) \frac{\partial \sigma}{\partial z} \, dw + \oint_{c} \left(\sigma - \frac{\partial \sigma}{\partial z} r\right) \, df}{\oint_{c} \sigma \, dw}$$

$$b = \frac{\partial f}{\partial \overline{z}}(z) = \frac{\partial}{\partial \overline{z}} \left(\frac{\oint_{c} \alpha(f \, d\overline{w} - \overline{r} \, df)}{\oint_{c} \alpha \, d\overline{w}} \right)$$

$$= \frac{\oint_{c} (f - f(z)) \frac{\partial \alpha}{\partial \overline{z}} \, d\overline{w} + \oint_{c} \left(\alpha - \frac{\partial \alpha}{\partial \overline{z}} \overline{r}\right) \, df}{\oint_{c} \alpha \, d\overline{w}}$$
(19)

These expressions may be significantly simplified for the threepoint coordinates, as follows:

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Laplace: Recall that $\sigma = \frac{1}{h}$. Thus

$$\frac{\partial \sigma}{\partial z} = -\frac{1}{h^2} \left(-\frac{1}{2} \right) = \frac{1}{2h^2} = \frac{\sigma^2}{2}$$

Mean value: Recall that $\sigma = \frac{1}{r|r|} = r^{-\frac{3}{2}} \bar{r}^{-\frac{1}{2}}$. Thus $\frac{\partial \sigma}{\partial z} = \frac{3}{2}r^{-\frac{5}{2}} \bar{r}^{-\frac{1}{2}} = \frac{3}{2}r^{-2}|r|^{-1} = \frac{3}{2}\frac{\sigma}{r}$

Wachspress: Recall that
$$\sigma = \frac{1}{\bar{h}r^2}$$
. From
 $\bar{h} = \frac{1}{2} \left(\bar{r} - \frac{d\bar{w}}{dw} r \right)$ we conclude that $r^2 \frac{d\bar{w}}{dw} = |r|^2 - 2r\bar{h}$, so
 $\frac{\partial \sigma}{\partial z} = -\sigma^2 \left(-2r\bar{h} + r^2 \frac{1}{2} \frac{d\bar{w}}{dw} \right) = \sigma^2 \left(2r\bar{h} - \frac{1}{2} |r|^2 + r\bar{h} \right)$
 $= \sigma^2 \left(3r\bar{h} - \frac{1}{2} |r|^2 \right) = \sigma^2 \left(\frac{3}{\sigma r} - \frac{1}{2} |r|^2 \right)$
 $= \frac{3\sigma}{r} - \frac{\sigma^2 |r|^2}{2}$
In all of the above, $\alpha = \bar{\sigma}$, thus
 $\frac{\partial \alpha}{\partial \bar{z}} = \frac{\partial \bar{\sigma}}{\partial \bar{z}} = \overline{\left(\frac{\partial \sigma}{\partial z} \right)}$

The same method applies to the discrete (polygon) case using the γ_i functions of (5): Denoting

$$r_{j} = w_{j} - z, e_{j} = w_{j+1} - w_{j}, e_{j} = f_{j+1} - f_{j}$$

$$a = \frac{\partial f(z)}{\partial z} = \frac{\partial}{\partial z} \frac{\sum_{j} S_{j} \gamma_{j}}{\sum_{j} \gamma_{j}} = \frac{\sum_{j} \frac{\hat{e}_{j}}{e_{j}} \gamma_{j} + \sum_{j} S_{j} \frac{\partial \gamma_{j}}{\partial z} - f \sum_{j} \frac{\partial \gamma_{j}}{\partial z}}{\sum_{j} \gamma_{j}}$$

$$= \frac{\sum_{j} \left(\frac{\hat{e}_{j}}{e_{j}} \gamma_{j} + (S_{j} - f) \frac{\partial \gamma_{j}}{\partial z}\right)}{\sum_{j} \gamma_{j}}$$

$$\sum_{j} \left(\frac{\hat{e}_{j}}{e_{j}} \overline{y}_{j} + A \frac{\partial \overline{\gamma}_{j}}{\partial z} - f \frac{\partial \overline{\gamma}_{j}}{\partial z}\right)$$

$$b = \frac{\partial f(z)}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}} \frac{\sum_{j} A_{j} \bar{\gamma}_{j}}{\sum_{j} \bar{\gamma}_{j}} = \frac{\sum_{j} \left(\overline{e_{j}} \gamma_{j} + A_{j} \overline{\partial \bar{z}} - J \overline{\partial \bar{z}}\right)}{\sum_{j} \bar{\gamma}_{j}}$$
$$= \frac{\sum_{j} \left(\frac{\hat{e}_{j}}{\bar{e}_{j}} \bar{\gamma}_{j} + (A_{j} - f) \overline{\left(\frac{\partial \gamma_{j}}{\partial z}\right)}\right)}{\sum_{j} \bar{\gamma}_{j}}$$

For the three-point coordinates, (6) implies

$$\gamma_j = \frac{i}{\overline{h}_j} \left(\frac{\left| r_{j+1} \right|^p}{r_{j+1}} - \frac{\left| r_j \right|^p}{r_j} \right)$$

resulting in

$$\frac{\partial \gamma_j}{\partial z} = \left(1 - \frac{p}{2}\right) \frac{i}{\bar{h}_j} \left(\frac{\left|r_{j+1}\right|^p}{r_{j+1}^2} - \frac{\left|r_j\right|^p}{r_j^2}\right) + \frac{i}{2\bar{h}_j^2} \frac{\bar{e}_j}{e_j} \left(\frac{\left|r_{j+1}\right|^p}{r_{j+1}} - \frac{\left|r_j\right|^p}{r_j}\right)$$

In particular, for the Laplace coordinates (p = 2), this simplifies to

$$\gamma_j = i \frac{e_j}{h_j}, \quad \frac{\partial \gamma_j}{\partial z} = \frac{e_j}{2h_j^2}$$

and for the Wachspress coordinates (p = 0):

$$\gamma_j = \frac{e_j}{\bar{h}_j r_j r_{j+1}}$$

7.1 Experimental results

We implemented the Newton-Raphson scheme described above for inverting injective transfinite mean value mappings. The iteration was initialized using the reverse mean value mapping, i.e. the mapping obtained when the roles of the source and target contours are reversed. Of course, this mapping can be arbitrary far from the desired inverse, and will typically not even be injective. Nonetheless, it serves as a good initialization. We did experience some numerical instability, especially in regions close to the boundary (where the kernels are singular), which may be somewhat mitigated by making smaller steps (e.g. by making just half the step size implied by (17)). Figs. 4 and 5 show the convergence of the iteration on two example inputs. In both cases, we show the source and the target contours with the mean value mapping between their interiors computed on a triangulation of the source domain (a constrained Delaunay triangulation generated by the Triangle library [She05] based on a polygonal sample of the source contour), meaning that the mapping was computed on the vertices of the triangulation and then linearly interpolated within each triangle. We then applied the inversion algorithm on the vertices of target triangulation in an attempt to reproduce the vertices of the source triangulation. The contour integrals required for the computation of a and b in (18) and (19) were computed using a simple sum, sampling the contours at 10,000 points. The iterations are numbered, with 0 meaning the initialization to the reverse mean value mapping. The source was texture-mapped using a color code reflecting distance from a central point to illustrate the mapping. We measured the approximation of the inverse by the average distance between the source vertex and the inverse computed on the target vertex. As is evident in the figures, the iteration converges in less than 6 iterations, with the approximation measure dropping more than 4 orders of magnitude. This is consistent with the theory that Newton-Raphson converges at a quadratic rate under very mild conditions on f [SM03].

8. Conclusion

We have showed how complex-valued algebra simplifies the expression of barycentric kernels in the plane, and even more so when we use similarity kernels instead of the traditional kernels. This allows us to express the transfinite versions of the popular three-point coordinates very simply and invert the resulting mappings efficiently.

It would be interesting to determine which other transfinite barycentric coordinates can be expressed simply using similarity kernels, and if this is related to them (not) being pseudo-harmonic.

Although complex-valued algebra is not natural in 3D space, there are still analogous expressions using 3D vector calculus, and we wonder whether a similar formulation to that presented here could be extended also to higher dimensional spaces.

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Figure 4: Inverting the transfinite mean value map. (top left and middle) Map between two corresponding planar contours (the numbers on the contours are a sample of the correspondence) computed at the vertices of a triangulation of 446 vertices including a boundary of 110 vertices. (0-6) Iterations of the Newton-Raphson inversion algorithm applied to the target, initialized to the reverse mean value map. The numbers are the number of iterations and the average distance of the vertices from those of the source. The contours diameter is 1.15.



Figure 5: Inverting the transfinite mean value map. (top left and middle) Map between two corresponding planar contours (the numbers are a sample of the correspondence) computed on the vertices of a triangulation of 386 vertices including a boundary of 125 vertices. (0-6) Iterations of the Newton-Raphson inversion algorithm applied to the target, initialized to the reverse mean value map. The numbers are the number of iterations and the average distance of the vertices from those of the source. The diameter of the contours is 1.37. Note the inverted ("flipped") triangles during the first few iterations.

Appendix: Proofs of Theorems

Theorem 1: Let *C* be a planar curve in counter-clockwise orientation, and z a point within C. For any point $w \in C$, denote $r = w - z, \theta = \arg(r)$, namely $r = |r| \exp(i\theta)$, A =the (signed) area swept by r, and h = the difference between z and the tangent to C at w. Then

(a)
$$h(w, z) = \frac{1}{2} \left(r - \frac{dw}{d\overline{w}} \overline{r} \right)$$

(b)
$$hd\overline{w} = -\overline{h}dw = i\mathrm{Im}(rd\overline{w}),$$

$$(r - h)d\overline{w} = \mathrm{Re}(rd\overline{w})$$

(c)
$$dA = \frac{1}{2} |r|^2 d\theta = \frac{1}{2i} \overline{h}dw = \frac{1}{2} \mathrm{Im}(\overline{r}dw)$$

(d)
$$d\theta = -i\frac{\overline{h}}{|r|^2} dw = \mathrm{Im}\left(\frac{dr}{r}\right) = \mathrm{Im}\left(\frac{dw}{r}\right),$$

$$d\log|r| = \mathrm{Re}\left(\frac{dw}{r}\right)$$

(e)
$$\oint_{\mathbb{C}} \overline{h} dw = 2i\mathrm{Area}(C) = \oint_{\mathbb{C}} \overline{w} dw$$

(f)
$$\mathrm{Re}\left(\frac{dw}{h}\right) = 0$$

(g)
$$\mathrm{Re}\left(\frac{r}{h}\right) = 1$$

(h)
$$d|r| = |r|\mathrm{Re}\left(\frac{dw}{r}\right), \frac{d\theta}{|r|} = -\frac{i}{r}d\left(\frac{r}{|r|}\right)$$

(i)
$$\frac{\partial h}{\partial z\partial \overline{z}} = \frac{\partial \overline{h}}{\partial z\partial \overline{z}} = 0$$

Proof:

(a) The unit vector tangent to C at w is $t = \frac{dw}{|dw|}$, thus $t^2 = \frac{dw}{d\overline{w}}$. Now since the point u on the tangent closest to w satisfies $\frac{w-u}{|w-u|} = t$ and $\operatorname{also} \frac{z-u}{|z-u|} = it$, this implies $w - u = t^2(\overline{w-u})$ and $h = u - z = t^2(\overline{z - u})$. Subtracting these two identities results in (a).

(b) Since h and dw are orthogonal:

 $\frac{h}{|h|} = i \frac{dw}{|dw|}$ Thus, by squaring both sides of the equation: $hd\overline{w} = -\overline{h}dw$

But also, using the expression (1) for h

$$hd\overline{w} = \frac{1}{2}(rd\overline{w} - dw\overline{r}) = i\mathrm{Im}(rd\overline{w})$$

and then

$$(r-h)d\overline{w} = rd\overline{w} - i\mathrm{Im}(rd\overline{w}) = \mathrm{Re}(rd\overline{w}).$$

(c) The connection to dA is by definition:

$$dA = \frac{1}{2}|r|^2 d\theta$$

or, since *h* is the height of the base *dw*:
$$dA = \frac{1}{\bar{h}} dw$$

$$dA = \frac{1}{2i}\bar{h}dw$$

which, by (b), gives also:

$$dA = \frac{1}{2} \operatorname{Im}(\bar{r}dw)$$

(d) An expression for $d\theta$ is obtained by comparing the first expression in (c) to the second:

 $d\theta = -i\frac{\bar{h}}{|r|^2}dw$ Moreover, observe that $r = |r| \exp(i\theta)$, thus $\log r = \log|r| + i\theta$

So

$$\frac{dr}{r} = d\log r = d\log|r| + id\theta$$

Implying:

$$d \log |r| = \operatorname{Re}\left(\frac{dr}{r}\right), \ d\theta = \operatorname{Im}\left(\frac{dr}{r}\right)$$
(e) Applying (b) and (c):

$$\oint_{C} \overline{h} \ dw = i \oint_{C} \operatorname{Im}(\overline{r} \ dw) = 2i \oint_{C} dA = 2iArea(C).$$
The area enclosed by the contour *C* is also well known to tained as $\frac{1}{2i} \oint_{C} \overline{w} \ dw.$

(f) Applying (b), namely, that h and dw are orthogonal:

$$\operatorname{Re}\left(\frac{dw}{h}\right) = \frac{\operatorname{Re}(hdw)}{|h|^2} = 0$$

be ob-

(g) Since, by (a):

thus

$$|h|^{2} = \frac{1}{2} \left(|r|^{2} - \operatorname{Re}\left(r^{2} \frac{d\overline{w}}{dw}\right) \right) = \operatorname{Re}(\overline{r}h) = \operatorname{Re}(r\overline{h})$$

we conclude that
$$\operatorname{Re}\left(\frac{r}{h}\right) = \frac{\operatorname{Re}(r\overline{h})}{|h|^{2}} = \frac{|h|^{2}}{|h|^{2}} = 1$$

(h) An easy consequence of (d) is:

$$d|r| = |r|d\log|r| = |r|\operatorname{Re}\left(\frac{dw}{r}\right),$$

$$\frac{i}{r}d\left(\frac{r}{|r|}\right) = \frac{i}{r}\frac{|r|dr - rd|r|}{|r|^2} = \frac{i}{|r|}\left(\frac{dr}{r} - \frac{d|r|}{|r|}\right)$$
$$= \frac{i}{|r|}\left(\frac{dw}{r} - \operatorname{Re}\left(\frac{dw}{r}\right)\right) = -\frac{1}{|r|}\operatorname{Im}\left(\frac{dw}{r}\right)$$
$$= -\frac{d\theta}{|r|}$$

(i) For any w, h(w, z) is an affine function of z, thus harmonic in z. 🔳

Corollary 1: The polar dual of a contour relative to a point z is $\frac{1}{\overline{h}}$

Proof: The polar dual relative to z at contour point w is defined as the vector p orthogonal to dw having unit scalar product with r = w - z, namely $\operatorname{Re}(\bar{p}dw) = 0$ and $\operatorname{Re}(\bar{p}r) = 1$. Comparing this to Theorem 1(f) and Theorem 1(g) implies $p = \frac{1}{b}$.

Corollary 2: If *C* is the unit circle, then

$$h(w,z) = w \operatorname{Re}(\bar{r}w) = w \operatorname{Re}(r\bar{w}) = w \operatorname{Re}\left(\frac{r}{w}\right)$$
$$= w (1 - \operatorname{Re}(w\bar{z}))$$

Proof:

On the unit circle $\frac{1}{w} = \overline{w}$, hence $d\overline{w} = -\frac{dw}{w^2}$ and $\frac{dw}{d\overline{w}} = -w^2$, implying: $h = \frac{1}{2} \left(r - \frac{dw}{d\overline{w}} \overline{r} \right) = \frac{1}{2} \left(r + w^2 \overline{r} \right) = w \frac{1}{2} \left(r \overline{w} + w \overline{r} \right)$ (7) $= w \operatorname{Re}(\bar{r}w) = w \operatorname{Re}(r\bar{w}) = w \operatorname{Re}\left(\frac{r}{w}\right)$ $= w \operatorname{Re}\left((w - z)\bar{w}\right) = w(1 - \operatorname{Re}(w\bar{z}))$

Corollary 3: On the unit circle, the (un-normalized) Poisson kernel is

$$K_P ds = \frac{1}{|h|} d\theta$$

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Proof: By Corollary 2, on the unit circle

 $|h| = |\operatorname{Re}(\bar{r}w)| = \operatorname{Re}(\bar{r}w).$

The last equality is because the angle between the vectors r and w is always acute. Using Theorem 1(d), the fact that dw = iwds, and Corollary 2 again:

$$d\theta = -i\frac{h}{|r|^2}dw = \frac{wh}{|r|^2}ds = \frac{\operatorname{Re}(\bar{r}w)}{|r|^2}ds$$

So, combining the two:

 $\frac{1}{|h|}d\theta = \frac{1}{|r|^2}ds$

which is the (un-normalized) Poisson kernel (11). \blacksquare

Theorem 2: A similarity kernel σ reproduces a given boundary mapping f(w) of ∂C iff the following conditions are satisfied for every $u \in \partial C$:

(a)
$$\lim_{z \to u} |\oint_C \sigma(w, z) dw| = \infty$$

(b)
$$\lim_{z \to u} |\oint_C r\sigma(w, z) dw| < \infty$$

(c)
$$\forall x \neq u \quad \lim_{z \to u} |\sigma(x, z)| < \infty$$

Proof: Consider any $u \in \partial C$

$$\lim_{z \to u} \oint_C S_f(w, z) \sigma(w, z) dw = \lim_{z \to u} \oint_C \left(f(w) - r \frac{df}{dw} \right) \sigma(w, z) dw$$
$$= \lim_{z \to u} \oint_C f(w) \sigma(w, z) dw$$
$$- \lim_{z \to u} \oint_C r \sigma(w, z) df$$

We may neglect the second term relative to the first because of (a) and (b):

$$\lim_{z \to u} \frac{\oint_C S_f(w, z) \sigma(w, z) dw}{\oint_C \sigma(w, z) dw} = \lim_{z \to u} \frac{\oint_C f(w) \sigma(w, z) dw}{\oint_C \sigma(w, z) dw}$$

Define $\delta(x)$ on ∂C by:

$$\delta(x) = \lim_{z \to u} \left(\frac{\sigma(x, z)}{\oint_C \sigma(w, z) dw} \right) = \frac{\lim_{z \to u} \sigma(x, z)}{\lim_{z \to u} \oint_C \sigma(w, z) dw}$$

Now (a) and (c) and the fact that $\oint_C \delta(x) dx = 1$ imply that δ behaves like a delta function centered at u, thus

$$\lim_{z \to u} \frac{\oint_C S_w(z)\sigma(w, z)dw}{\oint_C \sigma(w, z)dw} = \oint_C f(x)\delta(x)dx = f(u)$$

Theorem 3:

- (3) Every similarity kernel σ reproduces similarity transformations.
- (4) Every anti-similarity kernel α reproduces anti-similarity transformations.

Proof:

(1) It suffices to prove for
$$f(z) = z$$
:

$$\oint_C \sigma(w, z) S_f(w, z) dw = \oint_C \sigma(f dr - r df)$$

$$= \oint_C \sigma(w dw - r dw) = \oint_C \sigma z dw = z \oint_C \sigma dw$$

Hence

$$\frac{\oint_C \sigma S_f dw}{\oint_C \sigma dw} = z$$
(2) It suffices to prove for $f(z) = \bar{z}$:

$$\oint_C \alpha(w, z) A_f(w, z) d\bar{w} = \oint_C \alpha (f d\bar{r} - \bar{r} df)$$

$$= \oint_C \alpha (\bar{w} d\bar{w} - \bar{r} d\bar{w}) = \oint_C \alpha \bar{z} d\bar{w} = \bar{z} \oint_C \alpha d\bar{w}$$
Hence

Hence

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$$\frac{\oint_C \alpha A_f d\overline{w}}{\oint_C \alpha d\overline{w}} = \overline{z}$$

Theorem 4: A similarity kernel σ reproduces affine transformations iff (for every $z \in C$)

$$\oint_C \sigma(w,z)h(w,z)d\overline{w}=0$$

Proof: It suffices to prove reproduction for $f(z) = \overline{z}$. Define $g(z) = \oint_C \sigma S_f dw = \oint_C \sigma \left(f - r \frac{df}{dw} \right) dw = \oint_C \sigma \left(\overline{w} dw - r d\overline{w} \right)$ Now, from Theorem 1(a) we have: $2hd\overline{w} = rd\overline{w} - \overline{r}dw$

thus

So

$$g(z) = \bar{z} \oint_C \sigma \, dw - 2 \oint_C \sigma \, h d\bar{w}$$

$$g(z) = \bar{z} \oint_C \sigma \, dw$$

$$\oint_C \sigma \, h d\overline{w} = 0$$

implying the Theorem.

Theorem 5: σ is an affine reproducing similarity kernel iff $\overline{\sigma}$ is an affine reproducing similarity kernel.

Proof: We use the fact that c = 0 iff $\bar{c} = 0$ and that $hd\bar{w} = -\bar{h}dw$ (as shown in Theorem 1(b)):

$$\oint_C \sigma h d\overline{w} = 0 \quad \text{iff} \quad \oint_C \overline{\sigma} \overline{h} dw = 0 \quad \text{iff} \quad -\oint_C \overline{\sigma} h d\overline{w} = 0$$

Theorem 6: Any similarity kernel of the forms:

$$\sigma_{1}(w,z) = \frac{1}{h\bar{r}^{k}}, \text{ integer } k \neq 1$$

$$\sigma_{2}(w,z) = \frac{1}{\bar{h}r^{k}}, \text{ integer } k \neq 1$$

$$\sigma_{3}(w,k) = \frac{1}{|r|r}$$

$$\sigma_{4}(w,k) = \frac{1}{|r|\bar{r}}$$

is affine-reproducing.

Proof: Since σ_1 and σ_2 are conjugates, and σ_3 and σ_4 are conjugates, it suffices to prove the theorem for σ_1 and σ_3 and the theorem for σ_2 and σ_4 will follow by Theorem 5. Let us check the condition of Theorem 4:

$$\oint_C \sigma_1 h d\overline{w} = \oint_C \frac{1}{h\overline{r}^k} h d\overline{w} = \oint_C \frac{1}{r^k} dw$$

which, by Cauchy's integral formula [Ahl79], vanishes iff $k \neq 1$. By Theorem 1(d) and Theorem 1(h):

$$\oint_C \sigma_3 h d\overline{w} = \oint_C \frac{1}{|r|r} h d\overline{w} = -i \oint_C \frac{1}{|r|r} |r|^2 d\theta = i \oint_C \frac{|r|}{r} d\theta$$
$$= -\oint_C \frac{|r|^2}{r^2} d\left(\frac{r}{|r|}\right) = \oint_C d\left(\frac{|r|}{r}\right) = 0$$

Theorem 7: If a similarity kernel
$$\sigma$$
 satisfies
 $\operatorname{Re}(\sigma dr) + d\operatorname{Re}(\sigma r) = 0$
then a corresponding anti-similarity kernel is $\alpha = \overline{\sigma}$.

iff

Proof: Define

$$f(z) = \frac{\oint_{C} \sigma S_{f} dw}{\oint_{C} \sigma dw}$$
$$g(z) = \frac{\oint_{C} \bar{\sigma} A_{f} d\bar{w}}{\oint_{C} \bar{\sigma} d_{f} d\bar{w}}$$

We have to prove that f(z) = g(z). Indeed, denote $D = \oint_C \sigma dw$, then, by the premise on σ :

 $\operatorname{Re}(D) = \operatorname{Re}\left(\oint_{C} \sigma dw\right) = -\operatorname{Re}\left(\oint_{C} d(\sigma r)\right) = 0$ so $\overline{D} = -D$. Then

$$f = \frac{\oint_C \sigma(fdr - rdf)}{D} = \frac{\oint_C f\sigma dr - \oint_C \sigma rdf}{D}$$

and

$$g = \frac{\oint_C \bar{\sigma}(f d\bar{r} - \bar{r} df)}{-D} = \frac{\oint_C f \bar{\sigma} d\bar{r} - \oint_C \bar{\sigma} \bar{r} df}{-D}$$

So

$$\frac{D}{2}(f-g) = \oint_{C} f \operatorname{Re}(\sigma dr) - \oint_{C} df \operatorname{Re}(\sigma r)$$
$$= -\oint_{C} f d \operatorname{Re}(\sigma r) - \oint_{C} df \operatorname{Re}(\sigma r)$$
$$= -\oint_{C} d \left(f \operatorname{Re}(\sigma r) \right) = 0 \quad \blacksquare$$

Theorem 8: Under the conditions of Theorem 7, if σ reproduces f, then σ reproduces \bar{f} .

Proof: Since Theorem 7 implies

$$f = \oint_C \bar{\sigma} (f d\bar{r} - \bar{r} df)$$

gation)

we have (by conju

$$\bar{f} = \oint_{C} \sigma(\bar{f}dr - rd\bar{f})$$

namely, that σ reproduces \overline{f} .

Theorem 9: If σ is a similarity kernel of a barycentric mapping, then so is $\sigma' = a\sigma + \frac{b}{r^2}$, for any complex functions a(z) and b(z) that depend only on z.

Proof: If σ is a similarity kernel of f, then $\oint_{\mathsf{C}} \sigma(fdr - rdf) = f \oint_{\mathsf{C}} \sigma dw$

For σ'

or
$$\sigma$$
:

$$\oint_{C} \sigma'(fdr - rdf) = \oint_{C} \left(a\sigma + \frac{b}{r^{2}} \right) (fdr - rdf)$$

$$= -\oint_{C} \left(a\sigma + \frac{b}{r^{2}} \right) r^{2} d\left(\frac{f}{r} \right)$$

$$= -a \oint_{C} \sigma r^{2} d\left(\frac{f}{r} \right) - b \oint_{C} d\left(\frac{f}{r} \right)$$

$$= -a \oint_{C} \sigma r^{2} d\left(\frac{f}{r} \right) = a \oint_{C} \sigma (fdr - rdf)$$

$$= af \oint_{C} \sigma dw$$

But

$$f \oint_{\mathsf{C}} \sigma' dw = f \oint_{\mathsf{C}} \left(a\sigma + \frac{b}{r^2} \right) dw = af \oint_{\mathsf{C}} \sigma dw + bf \oint_{\mathsf{C}} \frac{dw}{r^2}$$
$$= af \oint_{\mathsf{C}} \sigma dw$$

The last equality is by the Cauchy Theorem [Ahl79]. ■

Theorem 10: The similarity kernels of the three-point barycentric coordinates are:

- Laplace: $\sigma_L = \frac{1}{h}$
- Wachspress: $\sigma_W = \frac{1}{\bar{h}r^2}$ Mean value: $\sigma_{MV} = \frac{1}{|r|r|}$

Proof: The transfinite three-point similarity kernels are continuous, and their analogous discrete γ_i functions, as specified in (5), are defined on the edges of a polygonal contour $C = [w_1, \dots, w_n]$. These γ_j functions are obtained by integration of σ over the *j*'th edge $[w_j, w_{j+1}]$. Note the key fact that, for a given z, the height function h(w, z) is a constant h(z) over the entire edge, and also perpendicular to that edge $e_j = w_{j+1} - w_j$. We also use the notation $r_j = w_j - z$. Note that $e_j = r_{j+1} - r_j$. Thus the area of the triangle formed by z, w_j and w_{j+1} is $A_j = \frac{1}{2} |h| |e_j| = \frac{i}{2} h \bar{e}_j =$ $\begin{aligned} -\frac{i}{2}\bar{h}e_{j} &= \operatorname{Im}(r_{j+1}\bar{r}_{j}) = \frac{1}{2i}(\bar{r}_{j+1}r_{j} - r_{j+1}\bar{r}_{j}) = \operatorname{Im}(\bar{r}_{j}e_{j}) = \\ \frac{1}{2i}(\bar{r}_{j}e_{j} - r_{j}\bar{e}_{j}) &= \operatorname{Im}(\bar{r}_{j+1}e_{j}) = \frac{1}{2i}(\bar{r}_{j+1}e_{j} - r_{j+1}\bar{e}_{j}). \\ \text{So, for the Laplace kernel:} \\ \gamma_{j} &= \int_{w_{j}}^{w_{j+1}} \frac{dw}{h} = \frac{w_{j+1} - w_{j}}{h} = \frac{e_{j}}{h} = \frac{e_{j}\bar{e}_{j}}{h\bar{e}_{j}} \end{aligned}$ $= -i \frac{e_j}{2A_j} \left(\frac{|r_{j+1}|^2}{r_{j+1}} - \frac{|r_j|^2}{r_j} \right)$

For the Wachspress kernel:

$$\begin{split} \gamma_{j} &= \int_{w_{j}}^{w_{j+1}} \frac{dw}{\bar{h}r^{2}} = \frac{1}{\bar{h}} \int_{w_{j}}^{w_{j+1}} \frac{dw}{r^{2}} = \frac{1}{\bar{h}} \left(\frac{1}{r_{j+1}} - \frac{1}{r_{j}} \right) \\ &= \frac{e_{j}}{\bar{h}e_{j}} \left(\frac{1}{r_{j+1}} - \frac{1}{r_{j}} \right) = i \frac{e_{j}}{2A_{j}} \left(\frac{1}{r_{j+1}} - \frac{1}{r_{j}} \right) \end{split}$$

For the mean value kernel:

$$\gamma_j = \int_{w_j}^{w_{j+1}} \frac{dw}{|r|r} = \int_{w_j}^{w_{j+1}} \frac{dw}{(\overline{z-w})^{\frac{1}{2}}(z-w)^{\frac{3}{2}}}$$

We use the fact that $\overline{w} = aw + b$ on the line segment to solve the integral: ¬W:...

$$= -\frac{2(\overline{z-w})^{\frac{1}{2}}}{(z-w)^{\frac{1}{2}}(az+b-\overline{z})} \int_{w_{j}}^{w_{j+1}} = -\frac{2}{(az+b-\overline{z})} \frac{|z-w|}{(z-w)} \Big|_{w_{j}}^{w_{j+1}} = \frac{-2}{(az+b-\overline{z})} \left(\frac{|r_{j+1}|}{(r_{j+1})} - \frac{|r_{j}|}{r_{j}}\right)$$

Since $a = \frac{\overline{e}_{j}}{e_{j}}$ and $b = \overline{w}_{j} - \frac{w_{j}\overline{e}_{j}}{e_{j}}$, we have
 $\gamma_{j} = \frac{-2e_{j}}{i(\overline{e}_{j}z+\overline{w}_{j}e_{j}-w_{j}\overline{e}_{j}-e_{j}\overline{z})} \left(\frac{|r_{j+1}|}{r_{j+1}} - \frac{|r_{j}|}{r_{j}}\right)$
 $= \frac{-2e_{j}}{i(\overline{r}_{j}e_{j}-\overline{e}_{j}r_{j})} \left(\frac{|r_{j+1}|}{r_{j+1}} - \frac{|r_{j}|}{r_{j}}\right) = 2\frac{e_{j}}{2A_{j}} \left(\frac{|r_{j+1}|}{r_{j+1}} - \frac{|r_{j}|}{r_{j}}\right)$
This shows consistency with (5), up to normalization constants.

Theorem 11: The similarity kernels of the Wachspress mapping and the Laplace mapping are equivalent on the unit circle.

Proof: For the unit circle we have $\overline{w} = \frac{1}{w}$ and |w| = 1. By Corollary 1: vRe(*r̄w*)

$$h = w \operatorname{Re}(r)$$

therefore

$$\overline{h} = \overline{w} \operatorname{Re}(\overline{r}w)$$

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so $\frac{h}{\bar{h}} = w^2$. Also by Corollary 1, $h = w \left(1 - \frac{w\bar{z} + \bar{w}z}{2}\right) = \frac{1}{2}$

$$\dot{w} = w \left(1 - \frac{wz + wz}{2} \right) = \frac{1}{2} (2w - \bar{z}w^2 - z|w|^2)$$
$$= \frac{1}{2} (2w - \bar{z}w^2 - z).$$

So:

$$\sigma_{W} = \frac{1}{\bar{h}r^{2}} = \frac{w^{2}}{hr^{2}} = \frac{(w-z)^{2} + z(2w-\bar{z}w^{2}-z)}{1-|z|^{2}} \frac{1}{hr^{2}}$$
$$= \frac{1}{1-|z|^{2}} \left(\frac{r^{2}+2zh}{hr^{2}}\right)$$
$$= \frac{1}{1-|z|^{2}} \left(\frac{1}{h} + \frac{2z}{r^{2}}\right)$$

which, by Theorem 9, is equivalent to the Laplace similarity kernel $\sigma_L = \frac{1}{h}$.

Theorem 12: If *B* is a regular barycentric kernel (generating *f* using the complex *dw* contour integral), and σ satisfies $Bdw = 2\sigma dw + rd\sigma$ then σ is a similarity kernel corresponding to *B*.

Proof: First we show that:

$$\oint_{\mathsf{C}} Bdw = \oint_{\mathsf{C}} \sigma dw$$

Indeed:

$$\oint_{c} Bdw = \oint_{c} (2\sigma dw + rd\sigma) = \oint_{c} 2\sigma dw + \oint_{c} rd\sigma$$
$$= \oint_{c} \sigma dr + \oint_{c} d(\sigma r) = \oint_{c} \sigma dw$$

$$f(z) \oint_{C} \sigma dw = f(z) \oint_{C} Bdw = \oint_{C} Bfdw = \oint_{C} f(2\sigma dw + rd\sigma)$$
$$= \oint_{C} \sigma f dr + \oint_{C} (\sigma f dr + rf d\sigma)$$
$$= \oint_{C} \sigma f dr + \oint_{C} fd(r\sigma) = \oint_{C} \sigma f dr + \oint_{C} d(r\sigma f) - \oint_{C} r\sigma df$$
$$= \oint_{C} \sigma f dr + 0 - \oint_{C} r\sigma df$$
$$= \oint_{C} \sigma (f dr - rdf) = \oint_{C} \sigma S_{f}$$