微分几何第二章习题

黄鹏飞

习题 1 (P28,T1). 求下列曲线的弧长与曲率:

(1) $y = ax^2$;

(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$;

(3) $\boldsymbol{r}(t) = (a\cosh t, b\sinh t)$;

(4) $\boldsymbol{r}(t) = (t, a\cosh\frac{t}{a})(a > 0)$.

解. 对于平面参数曲线 $\boldsymbol{r}(t) = (x(t), y(t))$, 有弧长参数

$$s(t) = \int_0^t |\boldsymbol{r}'(\theta)|d\theta, \qquad |\boldsymbol{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2},$$

从而我们有

$$\boldsymbol{t}(s) = \boldsymbol{t}(s(t)) = \frac{d\boldsymbol{r}}{ds} = \frac{d\boldsymbol{r}}{dt}\Big/\frac{ds}{dt} = \frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|} = \frac{(x'(t), y'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

及

$$\boldsymbol{n}(s) = \boldsymbol{n}(s(t)) = \frac{(-y'(t), x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}},$$

求导有

$$\dot{\boldsymbol{t}}(s) = \frac{d\boldsymbol{t}}{ds} = \frac{d\boldsymbol{t}}{dt}\Big/\frac{ds}{dt} = \frac{\Big(y'(t)(x''(t)y'(t) - x'(t)y''(t)), x'(t)(x'(t)y''(t) - x''(t)y'(t))\Big)}{\{(x'(t))^2 + (y'(t))^2\}^2},$$

由曲率公式 $\dot{\boldsymbol{t}}(s(t)) = \kappa(t)\boldsymbol{n}(s(t))$ 便得平面曲线曲率公式为

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\{(x'(t))^2 + (y'(t))^2\}^{3/2}},$$

这便是第二题的结论。

(1) 曲线参数化 $\boldsymbol{r}(t) = (t, at^2)$, $\quad |\boldsymbol{r}'(t)| = \sqrt{1 + 4a^2t^2}$, 从而弧长参数为

$$\begin{aligned} s(t) = \int_0^t \sqrt{1 + 4a^2\theta^2}d\theta &= \frac{1}{2}\theta\sqrt{1 + 4a^2\theta^2} + \frac{1}{4a}\ln(2a\theta + \sqrt{1 + 4a^2\theta^2})\Big|_{\theta=0}^{\theta=t} \\ &= \frac{1}{2}t\sqrt{1 + 4a^2t^2} + \frac{1}{4a}\ln(2at + \sqrt{1 + 4a^2t^2}), \end{aligned}$$

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pfhwang@mail.ustc.edu.cn.

代入公式得曲率为

$$\kappa(t)=\frac{2a}{(1+4a^2t^2)^{3/2}}.$$

(2) 曲线参数化 $\boldsymbol{r}(t)=(a\cos t.b\sin t),\quad t\in[0,2\pi),\quad |\boldsymbol{r}'(t)|=\sqrt{a^2\sin^2 t+b^2\cos^2 t}$, 直接计算便有

$$s(t)=\int_0^t|\boldsymbol{r}'(\theta)|d\theta=\int_0^t\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}d\theta,$$

$$\kappa(t)=\frac{ab}{(a^2\sin^2 t+b^2\cos^2 t)^{3/2}}.$$

(3) $|\boldsymbol{r}'(t)|=\sqrt{a^2\sinh^2 t+b^2\cosh^2 t}$, 直接计算便有

$$s(t)=\int_0^t|\boldsymbol{r}'(\theta)|d\theta=\int_0^t\sqrt{a^2\sinh^2\theta+b^2\cosh^2\theta}d\theta,$$

$$\kappa(t)=-\frac{ab}{(a^2\sinh^2 t+b^2\cosh^2 t)^{3/2}}.$$

(4) $|\boldsymbol{r}'(t)|=\sqrt{1+\sinh^2\frac{t}{a}}$, 直接计算便有

$$s(t)=\int_0^t|\boldsymbol{r}'(\theta)|d\theta=\int_0^t\sqrt{1+\sinh^2\frac{\theta}{a}}d\theta,$$

$$\kappa(t)=\frac{\cosh\frac{t}{a}}{a(1+\sinh^2\frac{t}{a})^{3/2}}=\frac{1}{a\cosh^2\frac{t}{a}}.$$

习题 2 (P28,T3)**.** 设曲线 C 在极坐标 (r,θ) 下的表示为 $r=f(\theta)$, 证明曲线 C 的曲率表达式为

$$\kappa(\theta)=\frac{f^2(\theta)+2(f'(\theta))^2-f(\theta)f''(\theta)}{\left\{f^2(\theta)+(f'(\theta))^2\right\}^{\frac{3}{2}}}.$$

解. 曲线 C 参数化 $\boldsymbol{r}(\theta)=(x(\theta),y(\theta))=(r\cos\theta,r\sin\theta)=(f(\theta)\cos\theta,f(\theta)\sin\theta)$, 直接计算有

$$\begin{aligned}
&x'(\theta)=f'(\theta)\cos\theta-f(\theta)\sin\theta,\\
&x''(\theta)=f''(\theta)\cos\theta-2f'(\theta)\sin\theta-f(\theta)\cos\theta,\\
&y'(\theta)=f'(\theta)\sin\theta+f(\theta)\cos\theta,\\
&y''(\theta)=f''(\theta)\sin\theta+2f'(\theta)\cos\theta-f(\theta)\sin\theta,\\
&(x'(\theta))^2+(y'(\theta))^2=f^2(\theta)+(f'(\theta))^2,\\
&x'(\theta)y''(\theta)-x''(\theta)y'(\theta)=f^2(\theta)+2(f'(\theta))^2-f(\theta)f''(\theta),
\end{aligned}$$

代入第二题结果即可.

习题 3 (P28,T4)**.** 求下列曲线的曲率和挠率:

(1) $\boldsymbol{r}(t)=(a\cosh t,a\sinh t,bt)(a>0)$;

(2) $\boldsymbol{r}(t)=(3t-t^2,3t^2,3t+t^2)$;

(3) $\boldsymbol{r}(t)=(a(1-\sin t),a(1-\cos t),bt)(a>0)$;

(4) $\boldsymbol{r}(t)=(at,\sqrt{2}a\ln t,\frac{a}{t})(a>0)$.

解. 首先证明下述 $\mathbb{E}^3$ 中参数曲线 $\boldsymbol{r}(t)$ 的曲率和挠率公式:

$$\kappa(t)=\frac{\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3},\quad \tau(t)=\frac{\left(\boldsymbol{r}'(t),\boldsymbol{r}''(t),\boldsymbol{r}'''(t)\right)}{\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|^2}.$$

直接计算, 弧长参数为

$$s(t)=\int_0^t|\boldsymbol{r}'(\theta)|d\theta,$$

$$\boldsymbol{t}=\frac{d\boldsymbol{r}}{ds}=\frac{d\boldsymbol{r}}{dt}\Big/\frac{ds}{dt}=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|},$$

$$\begin{aligned}\dot{\boldsymbol{t}}&=\frac{1}{|\boldsymbol{r}'(t)|}\frac{d\boldsymbol{t}}{dt}=\frac{1}{|\boldsymbol{r}'(t)|^4}\Big(\langle\boldsymbol{r}'(t),\boldsymbol{r}'(t)\rangle\boldsymbol{r}''(t)-\langle\boldsymbol{r}'(t),\boldsymbol{r}''(t)\rangle\boldsymbol{r}'(t)\Big)\\&=\frac{\boldsymbol{r}'(t)\wedge\Big(\boldsymbol{r}''(t)\wedge\boldsymbol{r}'(t)\Big)}{\left|\boldsymbol{r}'(t)\right|^4},\end{aligned}$$

上述最后一个等式用到了向量外积公式

$$\boldsymbol{a}\wedge(\boldsymbol{b}\wedge\boldsymbol{c})=\langle\boldsymbol{a},\boldsymbol{c}\rangle\boldsymbol{b}-\langle\boldsymbol{a},\boldsymbol{b}\rangle\boldsymbol{c}.$$

注意到 $\boldsymbol{r}'(t)\perp\Big(\boldsymbol{r}''(t)\wedge\boldsymbol{r}'(t)\Big)$, 则有

$$\kappa(t)=|\dot{\boldsymbol{t}}|=\frac{\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3},$$

故而

$$\boldsymbol{n}=\frac{\dot{\boldsymbol{t}}}{\kappa}=\frac{\boldsymbol{r}'(t)\wedge\Big(\boldsymbol{r}''(t)\wedge\boldsymbol{r}'(t)\Big)}{\left|\boldsymbol{r}'(t)\right|\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|},$$

$$\begin{aligned}\boldsymbol{b}=\boldsymbol{t}\wedge\boldsymbol{n}&=\frac{\boldsymbol{r}'(t)\wedge\Big(\boldsymbol{r}'(t)\wedge\Big(\boldsymbol{r}''(t)\wedge\boldsymbol{r}'(t)\Big)\Big)}{\left|\boldsymbol{r}'(t)\right|^2\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|}\\&=\frac{\langle\boldsymbol{r}'(t),\boldsymbol{r}''(t)\wedge\boldsymbol{r}'(t)\rangle\boldsymbol{r}'(t)-\langle\boldsymbol{r}'(t),\boldsymbol{r}'(t)\rangle(\boldsymbol{r}''(t)\wedge\boldsymbol{r}'(t))}{\left|\boldsymbol{r}'(t)\right|^2\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|}\\&=\frac{\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)}{\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|},\end{aligned}$$

$$\begin{aligned}\dot{\boldsymbol{b}}&=\frac{1}{\left|\boldsymbol{r}'(t)\right|}\frac{d\boldsymbol{b}}{dt}\\&=\frac{\big(\boldsymbol{r}'(t)\wedge\boldsymbol{r}'''(t)\big)\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|^2-\big\langle\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t),\boldsymbol{r}'(t)\wedge\boldsymbol{r}'''(t)\big\rangle\big(\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\big)}{\left|\boldsymbol{r}'(t)\right|\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|^3},\end{aligned}$$

注意到

$$\boldsymbol{n}=\frac{\langle\boldsymbol{r}'(t),\boldsymbol{r}'(t)\rangle\boldsymbol{r}''(t)-\langle\boldsymbol{r}'(t),\boldsymbol{r}''(t)\rangle\boldsymbol{r}'(t)}{\left|\boldsymbol{r}'(t)\right|\left|\boldsymbol{r}'(t)\wedge\boldsymbol{r}''(t)\right|},$$

则得挠率

$$\begin{aligned}\tau(t) &= -\langle \dot{\boldsymbol{b}}(t), \boldsymbol{n}(t)\rangle \\ &= -\frac{\langle \boldsymbol{r}'(t)\wedge \boldsymbol{r}'''(t), \boldsymbol{r}''(t)\rangle}{\left|\boldsymbol{r}'(t)\wedge \boldsymbol{r}''(t)\right|^2} \\ &= \frac{\Big(\boldsymbol{r}'(t), \boldsymbol{r}''(t), \boldsymbol{r}'''(t)\Big)}{\left|\boldsymbol{r}'(t)\wedge \boldsymbol{r}''(t)\right|^2}.\end{aligned}$$

这样我们便得到了 $\mathbb{E}^3$ 中参数曲线 $\boldsymbol{r}(t)$ 的曲率和挠率公式:

$$\kappa(t) = \frac{\left|\boldsymbol{r}'(t)\wedge \boldsymbol{r}''(t)\right|}{\left|\boldsymbol{r}'(t)\right|^3}, \quad \tau(t) = \frac{\Big(\boldsymbol{r}'(t), \boldsymbol{r}''(t), \boldsymbol{r}'''(t)\Big)}{\left|\boldsymbol{r}'(t)\wedge \boldsymbol{r}''(t)\right|^2}.$$

(1) 求导得

$$\begin{aligned}\boldsymbol{r}'(t) &= (a\sinh t, a\cosh t, b), \\ \boldsymbol{r}''(t) &= (a\cosh t, a\sinh t, 0), \\ \boldsymbol{r}'''(t) &= (a\sinh t, a\cosh t, 0),\end{aligned}$$

直接代入上述公式便有

$$\kappa(t) = \frac{a\sqrt{a^2+b^2\cosh 2t}}{(a^2\cosh 2t+b^2)^{3/2}}, \quad \tau(t) = \frac{b}{a^2+b^2\cosh 2t}.$$

(2) 求导得

$$\begin{aligned}\boldsymbol{r}'(t) &= (3-2t, 6t, 3+2t), \\ \boldsymbol{r}''(t) &= (-2, 6, 2), \\ \boldsymbol{r}'''(t) &= (0, 0, 0),\end{aligned}$$

直接代入上述公式便有

$$\kappa(t) = \frac{3\sqrt{11}}{(22t^2+9)^{3/2}}, \quad \tau(t) \equiv 0.$$

(3) 求导得

$$\begin{aligned}\boldsymbol{r}'(t) &= (-a\cos t, a\sin t, b), \\ \boldsymbol{r}''(t) &= (a\sin t, a\cos t, 0), \\ \boldsymbol{r}'''(t) &= (a\cos t, -a\sin t, 0),\end{aligned}$$

直接代入上述公式便有

$$\kappa(t) = \frac{a}{a^2+b^2}, \quad \tau(t) = -\frac{b}{a^2+b^2}.$$

(4) 求导得

$$\begin{aligned} \boldsymbol{r}'(t) &= (a, \frac{\sqrt{2}a}{t}, -\frac{a}{t^2}), \\ \boldsymbol{r}''(t) &= (0, -\frac{\sqrt{2}a}{t^2}, \frac{2a}{t^3}), \\ \boldsymbol{r}'''(t) &= (0, \frac{2\sqrt{2}a}{t^3}, -\frac{6a}{t^4}), \end{aligned}$$

直接代入上述公式便有

$$\kappa(t) = \frac{\sqrt{2}t^2}{a(1+t^2)^2}, \quad \tau(t) = \frac{\sqrt{2}t^2}{a(1+t^2)^2}.$$

习题 4 (P28,T6)**.** 证明: 曲线

$$\boldsymbol{r}(s) = \Big(\frac{(1+s)^{\frac{3}{2}}}{3}, \frac{(1-s)^{\frac{3}{2}}}{3}, \frac{s}{\sqrt{2}}\Big) \quad (-1 < s < 1)$$

以 s 为弧长参数, 并求出它的曲率、挠率和 *Frenet* 标架.

证明.

$$|\mathbf{r}'(s)| = \Big|\Big(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}}\Big)\Big| = 1,$$

从而可知 s 是弧长参数.

$$\begin{aligned} \mathbf{t}(s) = \mathbf{r}'(s) &= \Big(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}}\Big), \\ \dot{\mathbf{t}}(s) &= \frac{1}{4}\Big(\frac{1}{\sqrt{1+s}}, \frac{1}{\sqrt{1-s}}, 0\Big), \end{aligned}$$

故得曲率和主法向量

$$\kappa(s) = |\dot{\mathbf{t}}(s)| = \frac{\sqrt{2}}{4\sqrt{1-s^2}}, \quad \mathbf{n}(s) = \frac{\dot{\mathbf{t}}(s)}{|\dot{\mathbf{t}}(s)|} = \frac{\sqrt{2}}{2}\Big(\sqrt{1-s}, \sqrt{1+s}, 0\Big).$$

则副法向量和挠率为

$$\mathbf{b}(s) = \mathbf{t}(s) \wedge \mathbf{n}(s) = \Big(-\frac{\sqrt{1+s}}{2}, \frac{\sqrt{1-s}}{2}, \frac{\sqrt{2}}{2}\Big), \quad \tau(s) = \langle \mathbf{n}'(s), \mathbf{b}(s) \rangle = \frac{\sqrt{2}}{4\sqrt{1-s^2}}.$$

Frenet 标架为 $\{\mathbf{r}(s); \mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$, 数据从上.

□

习题 5 (P29,T10)**.** 设 $\mathcal{T}(X) = X\boldsymbol{T} + P$ 是 $\mathbb{E}^3$ 的一个合同变换, $\det \boldsymbol{T} = -1$, $\boldsymbol{r}(t)$ 是 $\mathbb{E}^3$ 的正则曲线. 求曲线 $\tilde{\boldsymbol{r}} = \boldsymbol{T} \circ \boldsymbol{r}$ 与曲线 $\boldsymbol{r}$ 的弧长参数、曲率、挠率之间的关系.

解. 分别记 $\boldsymbol{r}(t)$ 和 $\tilde{\boldsymbol{r}}(t)$ 的弧长参数为 s 和 $\tilde{s}$, 则有

$$\tilde{s}(t) = \int_0^t |\boldsymbol{r}'(\theta)| d\theta = \int_0^t |\boldsymbol{r}'(\theta)| |\det \boldsymbol{T}| d\theta = \int_0^t |\boldsymbol{r}'(\theta)| d\theta = s(t),$$

故二者弧长参数一致, 依次记两条曲线的相关参数为 $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}, \kappa, \tau, \tilde{\boldsymbol{t}}, \tilde{\boldsymbol{n}}, \tilde{\boldsymbol{b}}, \tilde{\kappa}, \tilde{\tau}$. 则有

$$\tilde{\boldsymbol{t}} = \frac{d\tilde{\boldsymbol{r}}}{d\tilde{s}} = \frac{d(\boldsymbol{r}\boldsymbol{T} + P)}{ds} = \frac{d\boldsymbol{r}}{ds}\boldsymbol{T} = \boldsymbol{t}\boldsymbol{T},$$

继而

$$\tilde{\kappa}\tilde{\boldsymbol{n}} = \frac{d\tilde{\boldsymbol{t}}}{d\tilde{s}} = \frac{d\tilde{\boldsymbol{t}}}{ds} = \frac{d\boldsymbol{t}}{ds}\boldsymbol{T} = \kappa\boldsymbol{n}\boldsymbol{T},$$

这便表明

$$\tilde{\kappa} = \kappa, \quad \tilde{\boldsymbol{n}} = \boldsymbol{n}\boldsymbol{T}.$$

$\tilde{\boldsymbol{r}}(t)$ 副法向量

$$\tilde{\boldsymbol{b}} = \tilde{\boldsymbol{t}} \wedge \tilde{\boldsymbol{n}} = (\boldsymbol{t}\boldsymbol{T}) \wedge (\boldsymbol{n}\boldsymbol{T}) = \det \boldsymbol{T} \cdot (\boldsymbol{t} \wedge \boldsymbol{n})\boldsymbol{T} = -(\boldsymbol{t} \wedge \boldsymbol{n})\boldsymbol{T} = -\boldsymbol{b}\boldsymbol{T},$$

上述倒数第二个等式用到了第一章习题 5 的结果:

$$(\mathcal{T}\boldsymbol{v}) \wedge (\mathcal{T}\boldsymbol{w}) = \det \mathcal{T} \cdot \mathcal{T}(\boldsymbol{v} \wedge \boldsymbol{w}),$$

其中 $\mathcal{T}$ 是 $\mathbb{E}^3$ 的一个合同变换, $\tilde{\boldsymbol{r}}(t)$ 挠率为

$$\tilde{\tau} = \left\langle \frac{d\tilde{\boldsymbol{n}}}{d\tilde{s}} \right\rangle = \left\langle \frac{d\boldsymbol{n}}{ds}\boldsymbol{T}, -\boldsymbol{b}\boldsymbol{T} \right\rangle = -\frac{d\boldsymbol{n}}{ds}\boldsymbol{T} \cdot (\boldsymbol{b}\boldsymbol{T})^t = -\frac{d\boldsymbol{n}}{ds}\boldsymbol{T} \cdot \boldsymbol{T}^t\boldsymbol{b}^t = -\left\langle \frac{d\boldsymbol{n}}{ds}, \boldsymbol{b} \right\rangle = -\tau.$$

综上所述, $\tilde{\boldsymbol{r}}(t)$ 和 $\boldsymbol{r}(t)$ 满足关系:

$$\tilde{s}(t) = s(t), \quad \tilde{\kappa}(t) = \kappa(t), \quad \tilde{\tau}(t) = -\tau(t).$$

习题 6 (P29,T11)**.** 设弧长参数曲线 $C : \boldsymbol{r}(s)$ 的曲率 $\kappa > 0$, 挠率 $\tau > 0$, $\boldsymbol{b}(s)$ 是 C 的副法向量, 定义曲线 $\tilde{C}$:

$$\tilde{\boldsymbol{r}}(s) = \int_0^s \boldsymbol{b}(u)du,$$

(1) 证明 s 是曲线 C 的弧长参数并且 $\tilde{\kappa} = \tau, \tilde{\tau} = \kappa$;

(2) 求 C 的 *Frenet* 标架.

证明. (1) 由于 $|\tilde{\mathbf{r}}'(s)| = |\mathbf{b}(s)| = 1$, 故 s 是曲线 $\tilde{C}$ 的弧长参数.

$$\tilde{\mathbf{t}}(s) = \frac{d\tilde{\mathbf{r}}}{ds} = \mathbf{b}(s), \quad \dot{\tilde{\mathbf{t}}}(s) = \frac{d\tilde{\mathbf{t}}}{ds} = \frac{d\mathbf{b}}{ds} = -\tau(s)\mathbf{n}(s),$$

故有

$$\tilde{\kappa}(s) = \tau(s), \quad \tilde{\mathbf{n}}(s) = \frac{\dot{\tilde{\mathbf{t}}}(s)}{\tilde{\kappa}(s)} = -\mathbf{n}(s), \quad \tilde{\mathbf{b}}(s) = \tilde{\mathbf{t}}(s) \wedge \tilde{\mathbf{n}}(s) = -\mathbf{b}(s) \wedge \mathbf{n}(s) = \mathbf{t}(s),$$

$$\dot{\tilde{\mathbf{b}}}(s) = \frac{d\mathbf{t}}{ds} = \kappa(s)\mathbf{n}(s), \quad \tilde{\tau}(s) = -\langle \dot{\tilde{\mathbf{b}}}(s), \tilde{\mathbf{b}}(s) \rangle = \kappa(s).$$

(2) 由上述计算过程可知,曲线 $\tilde{C}$ 的 *Frenet* 标架为

$$\{\tilde{\mathbf{r}}(s); \tilde{\mathbf{t}}(s), \tilde{\mathbf{n}}(s), \tilde{\mathbf{b}}(s)\} = \{\tilde{\mathbf{r}}(s); \mathbf{b}(s), -\mathbf{n}(s), \mathbf{t}(s)\},$$

由 $\mathbf{r}(s)$ 易得曲线 $\tilde{C}$ 的 *Frenet* 方程. □

习题 7 (P29,T12)**.** 设曲线 $\boldsymbol{r}(s)$, 它的曲率和挠率分别是 κ, τ, $\boldsymbol{r}(s)$ 的单位切向量 $\boldsymbol{t}(s)$ 可视作单位球面 S^2 上的一条曲线, 称为曲线 $\boldsymbol{r}(s)$ 的切线像. 证明:曲线 $\tilde{\boldsymbol{r}}(s) = \boldsymbol{t}(s)$ 的曲率、挠率分别为

$$\tilde{\kappa} = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}, \quad \tilde{\tau} = \frac{\frac{d}{ds}\left(\frac{\tau}{\kappa}\right)}{\kappa[1 + (\frac{\tau}{\kappa})^2]}.$$

证明. 注意 s 不一定是曲线 $\tilde{r}(s)$ 的弧长参数,可利用第二章第 5 题的结论,直接计算有

$$\begin{aligned}
\tilde{r}'(s) &= \frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \quad |\tilde{r}'(s)| = \kappa, \\
\tilde{r}''(s) &= \frac{d}{ds}(\kappa\mathbf{n}) = \kappa'\mathbf{n} + \kappa\frac{d\mathbf{n}}{ds} = \kappa'\mathbf{n} + \kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) \\
&= -\kappa^2\mathbf{t} + \kappa'\mathbf{n} + \kappa\tau\mathbf{b}, \\
\tilde{r}'''(s) &= \kappa''\mathbf{n} + \kappa'\frac{d\mathbf{n}}{ds} - 2\kappa\kappa'\mathbf{t} - \kappa^2\frac{d\mathbf{t}}{ds} + \kappa'\tau\mathbf{b} + \kappa\tau'\mathbf{b} + \kappa\tau\frac{d\mathbf{b}}{ds} \\
&= \kappa''\mathbf{n} + \kappa'(-\kappa\mathbf{t} + \tau\mathbf{b}) - 2\kappa\kappa'\mathbf{t} - \kappa^2(\kappa\mathbf{n}) + \kappa'\tau\mathbf{b} + \kappa\tau'\mathbf{b} + \kappa\tau(-\tau\mathbf{n}) \\
&= -3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{n} + (2\kappa'\tau + \kappa\tau')\mathbf{b},
\end{aligned}$$

则有

$$\begin{aligned}
\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s) &= \kappa^2\tau\mathbf{t} + \kappa^3\mathbf{b}, \quad |\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)| = \kappa^2\sqrt{\kappa^2 + \tau^2}, \\
\left(\tilde{\mathbf{r}}'(s), \tilde{\mathbf{r}}''(s), \tilde{\mathbf{r}}'''(s)\right) &= \left\langle \tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s), \tilde{\mathbf{r}}'''(s) \right\rangle \\
&= \kappa^2\tau(-3\kappa\kappa') + \kappa^3(2\kappa'\tau + \kappa\tau') \\
&= \kappa^4\tau' - \kappa^3\kappa'\tau,
\end{aligned}$$

故得 $\tilde{\mathbf{r}}(s)$ 的曲率和挠率分别为

$$\begin{aligned}
\tilde{k} &= \frac{|\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)|}{|\tilde{\mathbf{r}}'(s)|^3} = \sqrt{1 + \left(\frac{\tau}{\kappa}\right)^2}, \\
\tilde{\tau} &= \frac{\left(\tilde{\mathbf{r}}'(s), \tilde{\mathbf{r}}''(s), \tilde{\mathbf{r}}'''(s)\right)}{|\tilde{\mathbf{r}}'(s) \wedge \tilde{\mathbf{r}}''(s)|^2} = \frac{\kappa^4\tau' - \kappa^3\kappa'\tau}{\kappa^4(\kappa^2 + \tau^2)} = \frac{\frac{d}{ds}(\frac{\tau}{\kappa})}{\kappa[1 + (\frac{\tau}{\kappa})^2]}.
\end{aligned}$$

□

习题 8 (P29,T13). (1) 求曲率 $\kappa(s) = \frac{a}{a^2+s^2}$($s$ 是弧长参数) 的平面曲线;

(2) 求曲率 $\kappa(s) = \frac{1}{\sqrt{a^2-s^2}}$($s$ 是弧长参数) 的平面曲线.

解. 对于弧长参数平面曲线 $\boldsymbol{r}(s) = (x(s), y(s))$, 假设已知曲率 $\kappa(s)$, 下面我们给出曲线方程的一般求解过程:

首先 s 作为弧长参数给出了约束条件:

$$(x'(s))^2 + (y'(s))^2 = 1 \tag{1}$$

接下来有

$$\begin{aligned}
\boldsymbol{t}(s) &= \boldsymbol{r}'(s) = (x'(s), y'(s)), \\
\boldsymbol{n}(s) &= (-y'(s), x'(s)), \\
\dot{\boldsymbol{t}}(s) &= \boldsymbol{r}''(s) = (x''(s), y''(s)) = \kappa(s)\boldsymbol{n}(s),
\end{aligned}$$

则应有方程:

$$\begin{cases} x''(s) &= -\kappa(s)y'(s), \\ y''(s) &= \quad \kappa(s)x'(s), \end{cases} \tag{2}$$

根据方程 (1) 我们可以假设

$$\begin{cases} x'(s) &= \cos(\theta(s)), \\ y''(s) &= \sin(\theta(s)), \end{cases} \tag{3}$$

其中 $\theta=\theta(s)$ 是关于 s 的函数, 结合方程 (2)、(3) 我们有关于 $\theta(s)$ 的一阶 ODE

$$\theta'(s)=\kappa(s), \tag{4}$$

则有

$$\theta(s)=\theta(0)+\int_0^s \kappa(t)dt=\int_0^s \kappa(t)dt, (\text{相差一个刚体运动}) \tag{5}$$

(1) 若 $\kappa(s)=\frac{a}{a^2+s^2}$, 则由 (4) 知 $\theta(s)=\int_0^s \frac{a}{a^2+s^2}ds=\arctan\frac{s}{a}$, 从而有

$$\begin{cases} x(s) &= x(0)+\int_0^s x'(t)dt = x(0)+\int_0^s \cos(\arctan\frac{t}{a})dt \\ &= x(0)+\int_0^s \frac{a}{\sqrt{a^2+t^2}}dt = x(0)+a\ln(s+\sqrt{a^2+s^2}), \\ y(s) &= y(0)+\int_0^s y'(t)dt = y(0)+\int_0^s \sin(\arctan\frac{t}{a})dt \\ &= y(0)+\int_0^s \frac{s}{\sqrt{a^2+t^2}}dt = y(0)+\sqrt{a^2+s^2}, \end{cases}$$

所以我们要求的平面曲线为 $\boldsymbol{r}(s)=\left(a\ln(s+\sqrt{a^2+s^2}),\sqrt{a^2+s^2}\right)$(相差一个刚体运动, 这是由于曲率的刚体不变性).
(2) 若 $\kappa(s)=\frac{1}{\sqrt{a^2-s^2}}$, 则由 (4) 知 $\theta(s)=\int_0^s \frac{1}{\sqrt{a^2-s^2}}ds=\arcsin\frac{s}{a}$, 从而有

$$\begin{cases} x(s) &= x(0)+\int_0^s x'(t)dt = x(0)+\int_0^s \cos\left(\arcsin\frac{t}{a}\right)dt \\ &= x(0)+\int_0^s \sqrt{1-(\frac{t}{a})^2}dt = x(0)+\frac{1}{2}\left(s\sqrt{1-(\frac{t}{a})^2}+a\arcsin\frac{s}{a}\right), \\ y(s) &= y(0)+\int_0^s y'(s)ds = y(0)+\int_0^s \sin\left(\arcsin\frac{t}{a}\right) \\ &= y(0)+\int_0^s \frac{t}{a}dt = y(0)+\frac{s^2}{2a}, \end{cases}$$

所以我们要求的平面曲线为 $\boldsymbol{r}(s)=\left(\frac{1}{2}\left(s\sqrt{1-(\frac{t}{a})^2}+a\arcsin\frac{s}{a}\right),\frac{s^2}{2a}\right)$ (相差一个刚体运动, 这是由于曲率的刚体不变性).

习题 9 (P29,T14). 证明: 对 $\mathbb{E}^3$ 的弧长参数曲线 $\boldsymbol{r}(s)$, 有

(1) $\left(\frac{d\boldsymbol{r}}{ds},\frac{d^2\boldsymbol{r}}{ds^2},\frac{d^3\boldsymbol{r}}{ds^3}\right)=\kappa^2\tau;$

(2) $\left(\frac{d\boldsymbol{t}}{ds},\frac{d^2\boldsymbol{t}}{ds^2},\frac{d^3\boldsymbol{t}}{ds^3}\right)=\kappa^3(\kappa\dot{\tau}-\dot{\kappa}\tau)=\kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right).$

证明. (1) 直接计算有

$$\frac{d\mathbf{r}}{ds}=\mathbf{t},\quad \frac{d^2\mathbf{r}}{ds^2}=\dot{\mathbf{t}}=\kappa\mathbf{n},\quad \frac{d^3\mathbf{r}}{ds^3}=\dot{\kappa}\mathbf{n}+\kappa\dot{\mathbf{n}}=\dot{\kappa}\mathbf{n}+\kappa(-\kappa\mathbf{t}+\tau\mathbf{b})=-\kappa^2\mathbf{t}+\dot{\kappa}\mathbf{n}+\kappa\tau\mathbf{b},$$

故有

$$\begin{aligned}\left(\frac{d\mathbf{r}}{ds},\frac{d^2\mathbf{r}}{ds^2},\frac{d^3\mathbf{r}}{ds^3}\right)=\left\langle\frac{d\mathbf{r}}{ds}\wedge\frac{d^2\mathbf{r}}{ds^2},\frac{d^3\mathbf{r}}{ds^3}\right\rangle &= \langle\mathbf{t}\wedge\kappa\mathbf{n},-\kappa^2\mathbf{t}+\dot{\kappa}\mathbf{n}+\kappa\tau\mathbf{b}\rangle \\ &= \langle\kappa\mathbf{b},-\kappa^2\mathbf{t}+\dot{\kappa}\mathbf{n}+\kappa\tau\mathbf{b}\rangle \\ &= \kappa^2\tau.\end{aligned}$$

(2) 利用 (1) 直接计算有

$$\frac{d\mathbf{t}}{ds}=\frac{d^2\mathbf{r}}{ds^2}=\kappa\mathbf{n},\quad \frac{d^2\mathbf{t}}{ds^2}=\frac{d^3\mathbf{r}}{ds^3}=-\kappa^2\mathbf{t}+\dot{\kappa}\mathbf{n}+\kappa\tau\mathbf{b},$$

$$\begin{aligned}\frac{d^3\mathbf{t}}{ds^3} &= -2\kappa\dot\kappa\mathbf{t} + \ddot\kappa\mathbf{n} + \dot\kappa\dot{\mathbf{n}} + \dot\kappa\tau\mathbf{b} + \kappa\dot\tau\mathbf{b} + \kappa\tau\dot{\mathbf{b}} \\ &= -2\kappa\dot\kappa\mathbf{t} + \ddot\kappa\mathbf{n} + \dot\kappa(-\kappa\mathbf{t} + \tau\mathbf{b}) + \dot\kappa\tau\mathbf{b} + \kappa\dot\tau\mathbf{b} + \kappa\tau(-\tau\mathbf{n}) \\ &= -3\kappa\dot\kappa\mathbf{t} + (\ddot\kappa - \kappa\tau^2)\mathbf{n} + (2\dot\kappa\tau + \kappa\dot\tau)\mathbf{b},\end{aligned}$$

故有

$$\begin{aligned}\left(\frac{d\mathbf{t}}{ds}, \frac{d^2\mathbf{t}}{ds^2}, \frac{d^3\mathbf{t}}{ds^3}\right) &= \left\langle \frac{d\mathbf{t}}{ds} \wedge \frac{d^2\mathbf{t}}{ds^2}, \frac{d^3\mathbf{t}}{ds^3}\right\rangle \\ &= \left\langle \kappa\mathbf{n} \wedge (-\kappa^2\mathbf{t} + \dot\kappa\mathbf{n} + \kappa\tau\mathbf{b}), -3\kappa\dot\kappa\mathbf{t} + (\ddot\kappa - \kappa\tau^2)\mathbf{n} + (2\dot\kappa\tau + \kappa\dot\tau)\mathbf{b}\right\rangle \\ &= \left\langle \kappa^2\tau\mathbf{t} + \kappa^3\mathbf{b}, -3\kappa\dot\kappa\mathbf{t} + (\ddot\kappa - \kappa\tau^2)\mathbf{n} + (2\dot\kappa\tau + \kappa\dot\tau)\mathbf{b}\right\rangle \\ &= \kappa^4\dot\tau - \kappa^3\dot\kappa\tau \\ &= \kappa^5\frac{d}{ds}\left(\frac{\tau}{\kappa}\right).\end{aligned}$$

□

习题 10 (P29,T15)**.** 证明*: 满足条件*

$$\left(\frac{1}{\kappa}\right)^2 + \left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{\kappa}\right)\right]^2 = \text{常数} \tag{6}$$

的曲线, 或者是球面曲线, 或者 κ 是常数.

证明. 显然 κ 是常数满足条件,下设 κ 不是常数. 实际上若 κ 不为常数,$\mathbf{r}(s)$ 是球面曲线当且仅当上述方程成立,不妨设 s 为弧长参数.

先证必要性:

若 $\mathbf{r}(s)$ 是球面曲线,则存在某常数 C 和固定点 $\mathbf{r}(s_0)$ 使得

$$|\mathbf{r}(s) - \mathbf{r}(s_0)| = C, \tag{7}$$

求导有

$$\langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{t}(s)\rangle = 0, \tag{8}$$

$$1 + \langle \mathbf{r}(s) - \mathbf{r}(s_0), \kappa(s)\mathbf{n}(s)\rangle = 0, \tag{9}$$

$$\langle \mathbf{t}(s), \kappa(s)\mathbf{n}(s)\rangle + \langle \mathbf{r}(s) - \mathbf{r}(s_0), \kappa'(s)\mathbf{n}(s) + \kappa(s)(-\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s))\rangle = 0, \tag{10}$$

由 (8)、(9)、(10) 可得 $\mathbf{r}(s) - \mathbf{r}(s_0) \in Span\{\mathbf{n}(s), \mathbf{b}(s)\}$,并且

$$\langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{n}(s)\rangle = -\frac{1}{\kappa(s)}, \tag{11}$$

$$\langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{b}(s)\rangle = \frac{1}{\tau(s)}\frac{\kappa'(s)}{\kappa^2(s)} = -\frac{1}{\tau(s)}\frac{d}{ds}\left(\frac{1}{\kappa(s)}\right), \tag{12}$$

由于 $\mathbf{r}(s) - \mathbf{r}(s_0) = \langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{n}(s)\rangle\mathbf{n}(s) + \langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{b}(s)\rangle\mathbf{b}(s)$, 再结合 (7)、(11)、(12) 便得方程 (6).

再证充分性:

构造参数曲线

$$\tilde{\mathbf{r}}(s) = \mathbf{r}(s) + \frac{1}{\kappa(s)}\mathbf{n}(s) + \frac{1}{\tau(s)}\frac{d}{ds}\left(\frac{1}{\kappa(s)}\right)\mathbf{b}(s),$$

下面证如此构造的参数曲线 $\tilde{r}(s)$ 是常数曲线:

$$\begin{aligned}\tilde{\mathbf{r}}'(s)=&\mathbf{t}(s)+\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)\mathbf{n}(s)+\frac{1}{\kappa(s)}\big(-\kappa(s)\mathbf{t}(s)+\tau(s)\mathbf{b}(s)\big)\\&+\frac{d}{ds}\Big(\frac{1}{\tau(s)}\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)\Big)\mathbf{b}(s)+\frac{1}{\tau(s)}\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)\big(-\tau(s)\mathbf{n}(s)\big)\\=&\tau(s)\Big\{\frac{1}{\kappa(s)}+\frac{1}{\tau(s)}\frac{d}{ds}\Big[\frac{1}{\tau(s)}\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)\Big]\Big\}\mathbf{b}(s),\end{aligned}\tag{13}$$

由原方程 (6) 有

$$2\frac{1}{\kappa(s)}\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)+2\frac{1}{\tau(s)}\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)\frac{d}{ds}\Big(\frac{1}{\tau(s)}\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)\Big)=0,\tag{14}$$

这表明

$$\frac{1}{\kappa(s)}+\frac{1}{\tau(s)}\frac{d}{ds}\Big(\frac{1}{\tau(s)}\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)\Big)=0,$$

结合 (13) 我们可知

$$\tilde{\mathbf{r}}'(s)\equiv 0,$$

从而

$$\mathbf{r}(s_0)=\tilde{\mathbf{r}}(s)=\tilde{\mathbf{r}}(s)=\mathbf{r}(s)+\frac{1}{\kappa(s)}\mathbf{n}(s)+\frac{1}{\tau(s)}\frac{d}{ds}\Big(\frac{1}{\kappa(s)}\Big)\mathbf{b}(s),$$

结合 (6) 我们最终得到

$$|\mathbf{r}(s)-\mathbf{r}(s_0)|=\text{常数},$$

这样我们便证明了 $\mathbf{r}(s)$ 是球面曲线.

□

习题 11 (P30,T20)**.** *证明: 曲线 $\boldsymbol{r}(t)=\big(t+\sqrt{3}\sin t,2\cos t,\sqrt{3}t-\sin t\big)$ 与曲线 $\tilde{r}(t)=\big(2\cos\frac{t}{2},2\sin\frac{t}{2},-t\big)$ 是合同的.*

证明. 要证明 $\mathbf{r}(t)$ 和 $\tilde{\mathbf{r}}(t)$ 的合同性,我们需要下面的定理:

定理 (P25, 定理 4.2)**.** *设 $\boldsymbol{r}_1(s)$ 和 $\boldsymbol{r}_2(s)$ 是 $\mathbb{E}^3$ 的两条弧长参数曲线, 定义在同一个参数区间 (a,b) 上. 设 $\kappa_1(s)=\kappa_2(s)>0,\quad \tau_1(s)=\tau_2(s),\forall s\in(a,b)$. 则存在 $\mathbb{E}^3$ 的一个刚体运动 $\mathcal{T}$ 把曲线 $\boldsymbol{r}_2$ 变成 $\boldsymbol{r}_1$, 即 $\boldsymbol{r}_1=\mathcal{T}\circ\boldsymbol{r}_2$, 也即曲线 $\boldsymbol{r}_1$ 和 $\boldsymbol{r}_2$ 是合同的.*

直接计算有

$$\begin{aligned}\mathbf{r}'(t)&=(1+\sqrt{3}\cos t,-2\sin t,\sqrt{3}-\cos t), &\tilde{\mathbf{r}}'(t)&=(-\sin\frac{t}{2},\cos\frac{t}{2},-1);\\ \mathbf{r}''(t)&=(-\sqrt{3}\sin t,-2\cos t,\sin t), &\tilde{\mathbf{r}}''(t)&=(-\frac{1}{2}\cos\frac{t}{2},-\frac{1}{2}\sin\frac{t}{2},0);\\ \mathbf{r}'''(t)&=(-\sqrt{3}\cos t,2\sin t,\cos t), &\tilde{\mathbf{r}}'''(t)&=(\frac{1}{4}\sin\frac{t}{2},-\frac{1}{4}\cos\frac{t}{2},0);\end{aligned}$$

从而便有

$$\begin{aligned}\kappa(t)&=\frac{|\mathbf{r}'(t)\wedge\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}=\frac{1}{4}, &\tilde{\kappa}(t)&=\frac{|\tilde{\mathbf{r}}'(t)\wedge\tilde{\mathbf{r}}''(t)|}{|\tilde{\mathbf{r}}'(t)|^3}=\frac{1}{4};\\ \tau(t)&=\frac{(\mathbf{r}'(t),\mathbf{r}''(t),\mathbf{r}'''(t))}{|\mathbf{r}'(t)\wedge\mathbf{r}''(t)|^2}=-\frac{1}{4}, &\tilde{\tau}(t)&=\frac{(\tilde{\mathbf{r}}'(t),\tilde{\mathbf{r}}''(t),\tilde{\mathbf{r}}'''(t))}{|\tilde{\mathbf{r}}'(t)\wedge\tilde{\mathbf{r}}''(t)|^2}=-\frac{1}{4};\end{aligned}$$

由此可知曲线 $\mathbf{r}(t)$ 与 $\tilde{\mathbf{r}}(t)$ 是合同的. □

2.2 习题解析

2.2.1

注 弧长指曲线长度，若曲线长度无限，求一段的长度。

(1)提示 $\sqrt{ax^2+c}$ 的原函数为

$$\frac{x}{2}\sqrt{ax^2+c}+\frac{c}{2\sqrt{a}}\log(x\sqrt{a}+\sqrt{ax^2+c})$$

$$k=\frac{2|a|}{(4a^2x^2+1)^{\frac{3}{2}}}$$

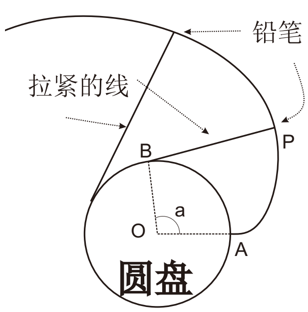
(2)

$$k=\frac{1}{\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)^{\frac{3}{2}}a^2b^2}$$

$$\text{周长}=2a\pi\left[1-(\frac{1}{2})^2e^2-(\frac{1\cdot3}{2\cdot4})^2\frac{e^4}{3}-(\frac{1\cdot3\cdot5}{2\cdot4\cdot6})^2\frac{e^6}{5}-...\right]$$

$$(3)\quad k=\frac{|ab|}{(a^2sh^2t+b^2ch^2t)^{\frac{3}{2}}},\quad (4)k=\frac{1}{ach^2\frac{t}{a}}$$

☺补充习题.



如左图所示，在圆盘上绕上一根不会伸缩的细线，线端栓一支铅笔，将线端拉紧，并逐渐拉开，铅笔尖在纸上画出的曲线就是圆的渐开线。求

(1)渐开线的参数表达式，

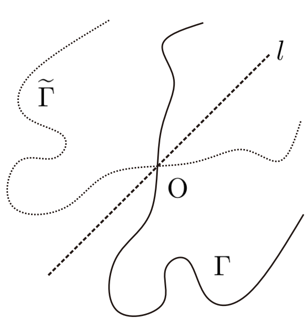
(2)渐开线的曲率，

(3) AP 段的弧长，已知 OA 与 OB 的夹角为 a。

2.2.2

$$\vec{t}=\frac{(x',y')}{\sqrt{x'^2+y'^2}},\ \frac{ds}{dt}=\left|\frac{d\vec{r}}{dt}\right|=\sqrt{x'^2+y'^2}$$

若记θ为曲线切向与x轴的夹角，如果在一点处$x'\neq0$ 则在这一点的邻域内 $x'\neq0$，$\theta=\arctan\frac{y'}{x'}$



$$k=\frac{d\theta}{ds}=\frac{d\theta}{dt}\frac{dt}{ds}=\frac{\frac{-y'x''}{x'^2}+\frac{y''}{x'}}{1+\left(\frac{y'}{x'}\right)^2}\frac{1}{\sqrt{x'^2+y'^2}}$$

若在某点P处，$x' = 0$ 则 $y' \neq 0$，不妨设这点为原点。 可将曲线沿直线l翻转(这里$l = \{x = y\}$)。翻转得到的曲线记为$\widetilde{\Gamma}$, $\widetilde{\Gamma}$的参数表达式为$\widetilde{r}(t) = (y(t), x(t))$。$\widetilde{\Gamma}$的曲率$\widetilde{k}(t) = -k(t)$,再利用前面结果(交换$x, y$)，得:

$$\widetilde{k}(t) = -k(t) = -\frac{-y''x' + y'x''}{(x'^2 + y'^2)^{\frac{3}{2}}}$$

2.2.3　略

2.2.4

$$(1)\ \ k = \frac{a\sqrt{a^2 + b^2 ch2t}}{(a^2 ch2t + b^2)^{\frac{3}{2}}},\ \ \tau = \frac{b}{b^2 ch2t + a^2} \qquad (2)\ \ \tau = 0,\ \ k = \frac{3\sqrt{11}}{(22t^2 + 9)^{\frac{3}{2}}}$$

$$(3)\ \ k = \frac{a}{a^2 + b^2},\ \ \tau = \frac{-b}{a^2 + b^2} \quad (4)\quad k = \frac{\sqrt{2}t^2}{a(1 + t^2)^2},\ \ \tau = \frac{\sqrt{2}t^2}{a(1 + t^2)}$$

2.2.5

$$\vec{r}'(t) = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{t}\left|\frac{d\vec{r}}{dt}\right|$$

$$\vec{r}''(t) = \frac{d\vec{t}}{ds}\left(\frac{ds}{dt}\right)^2 + \vec{t}\frac{d}{dt}\left|\frac{d\vec{r}}{dt}\right| = k\vec{n}\left|\frac{d\vec{r}}{dt}\right|^2 + \vec{t}\frac{d}{dt}\left|\frac{d\vec{r}}{dt}\right|$$

$$\vec{r}'''(t) = k\tau\vec{b}\left|\frac{d\vec{r}}{dt}\right|^2 + (*)\vec{n} + (*)\vec{t}$$

将以上表达式代入，并利用$(\vec{t}, \vec{n}, \vec{b}) = 1$即可。

☺补充习题. 判断如下两等式是否正确：(1) $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$,(2)$\frac{d^2 s}{dt^2} = \left|\frac{d^2\vec{r}}{dt^2}\right|$.

2.2.6

$$\vec{t} = (\frac{\sqrt{1+s}}{2}, \frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}})$$

$$\vec{n} = (\frac{\sqrt{1-s}}{\sqrt{2}}, -\frac{\sqrt{1+s}}{\sqrt{2}}, 0)$$

$$\vec{b} = (-\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}})$$

$$b = k = \frac{1}{2\sqrt{2}\sqrt{1-s^2}}$$

☺补充习题. 证明该曲线落在一个圆锥面上。

2.2.7

(1)由于$e^{-\frac{1}{t^2}}$的各阶导数$=0$.
(2)

$$t\to 0^-,\ \{\vec{t},\vec{n},\vec{b}\}\to\{(0,1,0),(1,0,0),(0,0,-1)\},$$
$$t\to 0^+,\ \{\vec{t},\vec{n},\vec{b}\}\to\{(0,1,0),(0,0,1),(1,0,0)\},$$

☺补充习题．是否存在满足以下要求的定义在$(0,1]$上的光滑空间曲线$\vec{r}(t)$, 若存在请具体构造，若不存在请说明理由

1. 当$t\to 0$时， $k,\tau\to\infty$, $\vec{r}(t)\to$ 原点, 且曲线长度有限，即

$$\int_\epsilon^1 |\vec{r}(t)|dt\le 1,\ \text{对所有}\ \epsilon>0$$

2. 当$t\to 0$时， $k\to 0,\tau\to\infty$(且要求存在常数$\delta>0$, s.t $k>\delta t$), $\vec{r}(t)\to$ 原点. 仍然要求曲线长度有限。
3. 当$t\to 0$时， $k\to\infty,\ \tau\to 0$(且要求 $\tau>0$), $\vec{r}(t)\to$ 原点. 仍然要求曲线长度有限。
4. 当$t\to 0$时， $k\to 0,\ \tau\to 0$, $\vec{r}(t)\to$ 原点, 但要求曲线长度无限，即

$$\int_\epsilon^1 |\vec{r}(t)|dt\to\infty,\ \text{as}\ \epsilon\to 0$$

2.2.8

若存在$\epsilon>0$, 使得$\vec{r}(t)$在$\{t_0-\epsilon,t_0+\epsilon\}$上有定义，则

$$0=\frac{d|\vec{r}-P_0|^2}{dt}(t_0)=\frac{d}{dt}<\vec{r}-P_0,\vec{r}-P_0>(t_0)=2<\frac{d}{dt}\vec{r}(t_0),\vec{r}(t_0)-P_0>=2<\vec{t}(t_0)\frac{ds}{dt},\vec{r}(t_0)-P_0>$$

2.2.9

(1)设存在λ,使得$\vec{r}+\lambda\vec{t}=0$, 求导得 $\vec{t}+\lambda k\vec{n}+\lambda'\vec{t}=0\ \Rightarrow\ \lambda'=-1,\ \lambda k=0$. 故在$\lambda\neq 0$处$k=0$, 由于$k$是连续函数，且$\lambda$只能在孤立点处取零，所以$k\equiv 0$.
(2)设存在λ,使得$\vec{r}+\lambda\vec{n}=0$, 求导得 $\vec{t}+\lambda(-k\vec{t}+\tau\vec{b})+\lambda'\vec{n}=0\ \Rightarrow\ \lambda k=1,\ \lambda\tau=0,\ \lambda'=0$ 。故$\lambda\equiv Constant$, 且 $\lambda\neq 0$(因为 $\lambda k=1$). 由此$\tau\equiv 0, k=\frac{1}{\lambda}\equiv Constant$
☺补充习题．我们称空间曲线一点处的主法向$\vec{n}$ 和副法向$\vec{b}$张成的平面为曲线在该点处的法平面，证明： 若一条曲线($k\neq 0$) 在任意点处的法平面过定点，则该曲面是球面曲线。
☺补充习题．是否存在一条光滑的空间曲线, 副法线过定点?
☺补充习题． $\vec{r}(t)$为R^3中正则曲线，存在常数$\alpha_1,\alpha_2,\alpha_3$, 使得直线族$\{\vec{r}(t)+\lambda(\alpha_1\vec{t}(t)+\alpha_2\vec{n}(t)+\alpha_3\vec{b}(t))|\lambda\in R\}$过定点，问给定$\alpha_1,\alpha_2,\alpha_3$,如果曲线足够长，我们是否可以确定这条曲线的几何形状?
☺补充习题．判断如下命题是否正确，如正确请证明，不正确请举例说明。
若R^3中一条曲线主法线过给定直线，则该曲线为圆柱螺线或圆。

2.2.10

弧长参数不变，$\widetilde{n}(t)=n(t),\ \widetilde{\tau}(t)=-\tau(t)$

2.2.11

先求出$\widetilde{r}(s)$的Frenet标架，由于$|\vec{b}|=1$ 故s 也是 $\widetilde{r}$ 的弧长参数。

$$\widetilde{t}(s)=\frac{d\widetilde{r}(s)}{ds}=\vec{b}(s)$$

$$\widetilde{n}(s)=\frac{d\widetilde{t}(s)}{ds}/\left|\frac{d\widetilde{t}(s)}{ds}\right|=\frac{d\vec{b}(s)}{ds}/\left|\frac{d\vec{b}(s)}{ds}\right|=\frac{-k\vec{n}}{k}=-\vec{n}$$

$$\widetilde{b}(s)=\widetilde{t}\wedge\widetilde{n}=\widetilde{b}\wedge(-\vec{n})=\vec{t}$$

$$\Rightarrow\ \widetilde{k}=\left|\frac{d\widetilde{t}(s)}{ds}\right|=\left|\frac{d\vec{b}(s)}{ds}\right|=\tau,\ \widetilde{\tau}=<\frac{d\widetilde{n}}{ds},\widetilde{b}(s)>=<k\vec{t}-\tau\vec{b},\vec{t}>=k$$

2.2.12

首先我们计算$\widetilde{r}(s)=\vec{t}(s)$的Frenet标架(记$\widetilde{r}$的弧长参数为$\sigma$)

$$\widetilde{t}(s)=\frac{d\widetilde{r}(\sigma)}{d\sigma}=\frac{d\widetilde{r}(s)}{ds}\frac{ds}{d\sigma}=\frac{d\vec{t}(s)}{ds}\frac{ds}{d\sigma}=k\vec{n}(s)\frac{ds}{d\sigma}$$

由于$\widetilde{t}$长度为1，故$\frac{d\sigma}{ds}=k$, 且$\widetilde{t}=\vec{n}$

$$\widetilde{n}(s)=\frac{d\widetilde{t}(\sigma)}{d\sigma}/\left|\frac{d\widetilde{t}(\sigma)}{d\sigma}\right|=\frac{d\vec{n}(s)}{ds}\frac{ds}{d\sigma}/\left|\frac{d\widetilde{t}(\sigma)}{d\sigma}\right|=\frac{-k\vec{t}(s)+\tau\vec{b}(s)}{k\left|\frac{d\widetilde{t}(\sigma)}{d\sigma}\right|}$$

$$\text{由于}|\widetilde{n}|=1,\ \text{故}k\left|\frac{d\widetilde{t}(\sigma)}{d\sigma}\right|=k\widetilde{k}=\sqrt{k^2+\tau^2}\Rightarrow\widetilde{k}=\sqrt{1+(\frac{\tau}{k})^2},\ \widetilde{n}=\frac{-k\vec{t}+\tau\vec{b}}{\sqrt{k^2+\tau^2}}$$

$$\text{由}\vec{b}=\vec{t}\wedge\vec{n},\ \text{得}\vec{b}\wedge\vec{t}=\vec{n},\ \vec{n}\wedge\vec{b}=\vec{t}\ \Rightarrow\ \widetilde{b}=\widetilde{t}\wedge\widetilde{n}=\frac{k\vec{b}+\tau\vec{t}}{\sqrt{k^2+\tau^2}}$$

$$\begin{aligned}\widetilde{\tau}&=<\frac{d\widetilde{n}(\sigma)}{d\sigma},\ \widetilde{b}(\sigma)>\\&=<\frac{ds}{d\sigma}\frac{d}{ds}\left(\frac{-k\vec{t}+\tau\vec{b}}{\sqrt{k^2+\tau^2}}\right),\ \frac{k\vec{b}+\tau\vec{t}}{\sqrt{k^2+\tau^2}}>\\&=\left[\frac{d}{ds}\left(\frac{-k}{\sqrt{k^2+\tau^2}}\right)\right]\frac{\tau}{k\sqrt{k^2+\tau^2}}+\left[\frac{d}{ds}\left(\frac{\tau}{\sqrt{k^2+\tau^2}}\right)\right]\frac{1}{\sqrt{k^2+\tau^2}}\\&=\frac{1}{k}\frac{\frac{d}{ds}\left(\frac{\tau}{k}\right)}{1+\left(\frac{\tau}{k}\right)^2}\end{aligned}$$

☺补充习题. 设 $k\neq$, $\tau\neq 0$ 对 $\hat{r}=\vec{b}$ 计算曲率挠率。

2.2.13

设曲线在(x,y)平面上，曲线切向与x轴的夹角为θ。
(1)由曲率定义，本题应有$a>0$

$$k=\frac{d\theta}{ds}=\frac{a}{a^2+s^2}\quad \text{积分得}\quad \theta=\arctan\frac{s}{a}+C,\ a\tan(\theta-C)=s$$

不同的C意味着将曲线旋转，故不失一般性我们可以取$C\equiv 0$.

$$\text{这时}\ \vec{t}=\frac{(a,s)}{\sqrt{a^2+s^2}},\quad \text{积分得}\ \vec{r}=\left(\log(\sqrt{\frac{s^2}{a^2}+1}+\frac{s}{a}),\sqrt{a^2+s^2}\right)$$

(2)

$$k=\frac{d\theta}{ds}=\frac{1}{\sqrt{a^2-s^2}},\quad \text{积分得}\ \theta=\arcsin\frac{s}{a}+C,\ \sin(\theta-C)=\frac{s}{a}$$

$$\text{同上可以取}\ C\equiv 0,\ \text{这时}\ \vec{t}=\left(\frac{\sqrt{a^2-s^2}}{a},\frac{s}{a}\right),\ (\text{这里假设}a>0)$$

$$\text{积分得}\ \vec{r}=\left(\frac{a}{2}(\frac{s}{a}\sqrt{1-\frac{s^2}{a^2}}+\arcsin\frac{s}{a}),\ \frac{s^2}{2a}\right)$$

☺补充习题. 试求出以上两曲线更简洁的(非弧长参数)表达式。
☺补充习题. 有一条平面曲线 $\vec{r}(t)$,

$$|\frac{d\vec{r}}{dt}|=\sqrt{1+t^2},\ k=\frac{t^2+2}{(t^2+1)^{\frac{3}{2}}}$$

求 $\vec{r}(t)$ 表达式。
☺补充习题. 有一条平面弧长参数曲线 $\vec{r}(s)$,

$$k=\frac{1}{\sqrt{2s}},$$

求 $\vec{r}(s)$ 参数表达式.

2.2.14

方法一: 利用$(\vec{t},\vec{n},\vec{b})=<\vec{t}\wedge\vec{n},\vec{b}>=1$，及Frenet标架的运动方程。

$$\frac{d^4\vec{r}}{ds^4}=(-3\dot{k}k)\vec{t}+(\ddot{k}-k^3-k\tau^2)\vec{n}+(2\dot{k}\tau+k\dot{\tau})\vec{b}$$

方法二: 对于第二问，将本节12题的$\widetilde{k},\widetilde{\tau}$, 代入本题第一问，并注意参数的选取。
☺补充习题. 计算:

$$\left(\frac{d\vec{b}}{ds},\frac{d^2\vec{b}}{ds^2},\frac{d^3\vec{b}}{ds^3}\right)$$

2.2.15

如$\vec{r}$为球面曲线，则$\vec{r}$的法平面过定点(参考本节第9题补充习题)，可设存在λ, μ, 使得

$$\vec{r}+\lambda\vec{n}+\mu\vec{b}\equiv 0 \text{ 求导得 } \vec{t}+\lambda'\vec{n}+\lambda(-k\vec{t}+\tau\vec{b})+\mu'\vec{b}+\mu(-\tau\vec{n})=0$$

$$\Rightarrow\ \lambda k=1,\ \lambda'=\mu\tau,\ \lambda\tau+\mu'=0$$

$$\Rightarrow\ \vec{r}+\frac{1}{k}\vec{n}+\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{k}\right)\vec{b} \text{ 为圆心}$$

由此我们希望由条件$\left(\frac{1}{k}\right)^2+\left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{k}\right)\right]^2$ 可以推出$\vec{r}+\frac{1}{k}\vec{n}+\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{k}\right)\vec{b}$为常值

$$\begin{aligned}0=&\frac{d}{ds}\left[\left(\frac{1}{k}\right)^2+\left(\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{k}\right)\right)^2\right]\\=&\frac{2}{k}\left(\frac{1}{k}\right)'+\frac{2}{\tau}\left(\frac{1}{k}\right)'\left(\frac{1}{\tau}\left(\frac{1}{k}\right)'\right)'\\=&\frac{2}{\tau}\left(\frac{1}{k}\right)'\left[\frac{\tau}{k}+\left(\frac{1}{\tau}\left(\frac{1}{k}\right)'\right)'\right]\end{aligned}$$

$$\frac{d}{ds}\left[\vec{r}+\frac{1}{k}\vec{n}+\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{k}\right)\vec{b}\right]=\left[\left(\frac{1}{\tau}\left(\frac{1}{k}\right)'\right)'+\frac{\tau}{k}\right]\vec{b}$$

所以如果k不是常值，则$\vec{r}+\frac{1}{k}\vec{n}+\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{k}\right)\vec{b}$为常值。

2.2.16

设P_0为原点，l为z轴，则在P_0点附近，可设曲线为:

$$\vec{r}(t)=(x(t),y(t),t),\ x(0)=y(0)=x'(0)=y'(0)=0$$

$$x(t)=x''(0)\frac{t^2}{2}+O(t^3),\ y(t)=y''(0)\frac{t^2}{2}+O(t^3)$$

计算得

$$k(0)=\left|\frac{d^2\vec{r}}{dt^2}\right|(0)=\left|\frac{(x'',y'',0)}{\sqrt{x'^2+y'^2+1}}\right|(0)=\sqrt{x''(0)^2+y''(0)^2}$$

设$P=\vec{r}(t)$,则

$$\begin{aligned}\lim_{P\to P_0}\frac{2d(P,l)}{d^2(P_0,P)}&=\lim_{t\to 0}\frac{2\sqrt{x^2(t)+y^2(t)}}{x^2(t)+y^2(t)+t^2}=\lim_{t\to 0}\frac{\sqrt{x''(0)^2t^4+y''(0)^2t^4+O(t^5)}}{t^2+O(t^3)}\\&=\sqrt{x''(0)^2+y''(0)^2}=k(0)\end{aligned}$$

☺补充习题. 仍用本题的记号，设L是P_0点处曲线C的切向与主法向张成的平面。证明:

$$\lim_{P\to P_0}\frac{3d(P,L)}{d(P,l)d(P,P_0)}=\tau(P_0)$$

☺补充习题. 证明:通过平移旋转, 一条正则曲线可局部表示为

$$\vec{r}(t)=(t,\frac{k}{2}t^2+O(t^3),\frac{k\tau}{6}t^3+O(t^4))$$

2.2.17

方法一：将$\tau=c\kappa(\kappa\neq 0)$代入本节12题，得 $\widetilde{k}=\sqrt{1+c^2}$，$\widetilde{\tau}=0$, 故$\vec{t}(s)$是一个圆，$\vec{b}(s)$是常向量(但注意此时$s$不是$\vec{r}(s)=\vec{t}(s)$的弧长参数)。
由此任给某圆的一个参数化$\vec{a}(t)$,

$$\int_0^s \vec{a}(t)dt \triangleq \vec{r}(s)$$

为满足条件的曲线。
方法✌:

$$\frac{d}{ds}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}=\begin{pmatrix} & k & \\ -k & & \tau\\ & -\tau & \end{pmatrix}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}=k\begin{pmatrix} & 1 & \\ -1 & & c\\ & -c & \end{pmatrix}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}$$

$$\begin{pmatrix} & 1 & \\ -1 & & c\\ & -c & \end{pmatrix}=O^T\begin{pmatrix} & & \\ & & \sqrt{c^2+1}\\ & -\sqrt{c^2+1} & \end{pmatrix}O$$

$$\text{上式中，}\quad O=\begin{pmatrix}\frac{c}{\sqrt{c^2+1}} & & \frac{1}{\sqrt{c^2+1}}\\ & 1 & \\ -\frac{1}{\sqrt{c^2+1}} & & \frac{c}{\sqrt{c^2+1}}\end{pmatrix},\ O^TO=OO^T=\mathrm{I}$$

$$\text{故}\frac{d}{ds}O\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}=k\begin{pmatrix} & & \\ & & \sqrt{c^2+1}\\ & -\sqrt{c^2+1} & \end{pmatrix}O\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}$$

$$\text{由此得}\frac{d}{ds}\left(\frac{c\vec{t}+\vec{b}}{\sqrt{c^2+1}}\right)=0,$$

$$\frac{d}{ds}\vec{n}=k\sqrt{c^2+1}\left(\frac{-c\vec{t}+\vec{b}}{\sqrt{c^2+1}}\right),$$

$$\frac{d}{ds}\left(\frac{-c\vec{t}+\vec{b}}{\sqrt{c^2+1}}\right)=-k\sqrt{c^2+1}\vec{n}$$

$$c\vec{t}+\vec{b}\equiv c\vec{t}(0)+\vec{b}(0)$$

$$\frac{-c\vec{t}+\vec{b}}{\sqrt{c^2+1}}=-\sin(\sqrt{c^2+1}\int k)\vec{n}(0)+\cos(\sqrt{c^2+1}\int k)\left(\frac{-c\vec{t}(0)+\vec{b}(0)}{\sqrt{c^2+1}}\right)$$

$$\Rightarrow \vec{t}=\frac{1}{2c}\left(c\vec{t}(0)+\vec{b}(0)+\sqrt{c^2+1}\sin(\sqrt{c^2+1}\int_0^s k)\vec{n}(0)-\cos(\sqrt{c^2+1}\int_0^s k)(-c\vec{t}(0)+\vec{b}(0))\right)$$

对$\vec{t}$积分可得$\vec{r}$。
☺补充习题．注意到在本节15题中，若

$$\left(\frac{1}{k}\right)^2+\left[\frac{1}{\tau}\frac{d}{ds}\left(\frac{1}{k}\right)\right]^2=1,\text{ 则 }\ \tau=\frac{\frac{d}{ds}\left(\frac{1}{k}\right)}{\sqrt{1-\left(\frac{1}{k}\right)^2}}$$

$$\text{取 }k=\frac{1}{\sqrt{1-a^2}},\ \text{得到}\tau=\frac{-a'}{\sqrt{1-a^2}},\ \frac{\tau}{k}=-a'$$

所以若$a'=const$则$\frac{\tau}{k}$为常数。如果取$a=s$，则

$$k=\frac{1}{\sqrt{1-s^2}},\ \tau=\frac{-1}{\sqrt{1-s^2}}$$

请具体写出该曲线。
☺补充习题．证明，一条空间曲线若切向与固定方向成定角，则该曲线满足$\tau=ck$。证明，反之亦然，即若一条曲线满足满足$\tau=ck$，则该曲线的切向与某一固定方向成定角。提示利用上面的方法✌.
☃注 参考本节第六题。

2.2.18

(1)本书定义曲率时要求选定$\vec{n}$使得$\{\vec{t},\vec{n}\}$ 与$\{\partial_x,\partial_y\}$定向相同，即可通过旋转平移使$\{\vec{t},\vec{n}\}$ 与$\{\partial_x,\partial_y\}$重合。

$$\widetilde{k}(t)=-k(-t)$$

(2)

$$\widetilde{k}(t)=k(-t),\ \widetilde{\tau}(t)=\tau(-t)$$

2.2.19

$$\vec{v}=k\vec{b}+\tau\vec{t}$$

2.2.20

$\vec{r}(\frac{t}{2})$沿方向$(\frac{1}{2},0,\frac{\sqrt{3}}{2})$跑向无穷远处， $\widetilde{r}(t)$沿方向$(0,0,-1)$跑向无穷远处。 为找到使两曲线重合的刚体运动，我们先将 $\vec{r}(\frac{t}{2})$ 绕 y 轴旋转使 $\vec{r}(\frac{t}{2})$, $\widetilde{r}(t)$ 以相同速度跑向无穷远处。

$$\vec{r}(\frac{t}{2})\begin{pmatrix}-\frac{\sqrt{3}}{2} & & -\frac{1}{2}\\ & 1 & \\ \frac{1}{2} & & -\frac{\sqrt{3}}{2}\end{pmatrix}=(-2\sin\frac{t}{2},\ 2\cos\frac{t}{2},\ -t)$$

然后再将 $(-2\sin\frac{t}{2},\ 2\cos\frac{t}{2},\ -t)$ 绕 z轴旋转，可使其与$\widetilde{r}(t)$重合。

2.2.21

☺补充习题．给定k,τ为x,y,z,s 的光滑函数。证明，存在空间曲线$\vec{r}(s):R\to R^3=\{(x,y,z)\}$, 使得 $k=k(\vec{r}(s),s),\tau=\tau(\vec{r}(s),s)$