期中考试题 短评

"取乎其上，得乎其中"

第1题：做过之后，你就会算任一平面和给定曲面相截所得曲线的曲率了。

第2题：这是作业题

(1) $\vec{a}(t)$ 方向不变 $\Leftrightarrow$ $\vec{a}(t)\wedge\vec{a}'(t)=0$

(2) 若 $\vec{a}(t)\wedge\vec{a}'(t)\neq 0$，则

$\vec{a}(t)$ 和一个固定方向垂直 $\Leftrightarrow$ $(\vec{a}(t),\vec{a}'(t),\vec{a}''(t))=0$.

的一个具体应用。

更详细地 (1) 曲线是一般螺线 $\Leftrightarrow$ $\exists\,\vec{a}$ s.t. $\langle \dot{c}(s),\vec{a}\rangle = \text{const}$.

$\Leftrightarrow$ $\langle \ddot{c}(s),\vec{a}\rangle = k(s)\langle n(s),\vec{a}\rangle = 0$

故一条曲线为曲率处处非0的正则参数曲线 $\Leftrightarrow$ $\langle n(s),\vec{a}\rangle=0$

又注意到 $n(s)\wedge\dot{n}(s) = n(s)\wedge(-k t(s)+\tau b(s)) = -k\, n(s)\wedge t(s) + \tau\, n(s)\wedge b(s)$

$= kb+\tau t \neq 0$ （因为 b,t 线性无关，且 $k\neq 0$）

故而此时有

$$\begin{aligned}
\langle n(s),\vec{a}\rangle=0 \Leftrightarrow 0&=(n,\dot{n},\ddot{n}) = (n,-kt+\tau b,-\dot{k}t-k\dot{t}+\dot{\tau}b+\tau\dot{b})\\
&=(n,-kt+\tau b,-\dot{k}t+\dot{\tau}b-k^2n-\tau^2n)\\
&=(n,-kt+\tau b,-\dot{k}t+\dot{\tau}b)\\
&=(n,-kt,\dot{\tau}b)+(n,\tau b,-\dot{k}t)\\
&=(-k\dot{\tau})(n,t,b)+(-\tau\dot{k})(n,b,t)\\
&=(-k\dot{\tau}+\tau\dot{k})(n,t,b)
\end{aligned}$$

$\Leftrightarrow -k\dot{\tau}+\tau\dot{k}=0 \Leftrightarrow \left(\frac{\tau}{k}\right)^{\cdot}=0 \Leftrightarrow \frac{\tau}{k}=\text{const}.$

(2) $\sigma(s)=\tau(s)t(s)+k(s)b(s)$ 有固定的方向

$$\Leftrightarrow 0=\sigma(s)\wedge\dot\sigma(s)=(\tau(s)t(s)+k(s)b(s))\wedge(\dot\tau t+\tau\dot t+\dot k b+k\dot b)$$
$$=(\tau t+kb)\wedge(\dot\tau t+\dot k b+\tau k n-k\tau n)$$
$$=(\tau t+kb)\wedge(\dot\tau t+\dot k b)$$
$$=(\tau t)\wedge(\dot k b)+(kb)\wedge(\dot\tau t)$$
$$=(\tau\dot k-k\dot\tau)\,t\wedge b$$

$\Leftrightarrow \tau\dot k-k\dot\tau=0$

$\Leftrightarrow \left(\frac{\tau}{k}\right)^{\cdot}=0 \Leftrightarrow \frac{\tau}{k}=const \Leftrightarrow$ 曲线C是一般螺线

第3题：做过后，你了解到所有三种可能的曲面的基本形式
$\langle dr, dr\rangle$, $-\langle dr, dn\rangle$, $\langle dn, dn\rangle$

第4题：做过之后，你了解到高斯绝妙定理的"反过来"是不对的。

第5题：做过之后，你了解了一条空间曲线（$k(s)$处处非零）的管状邻域的表面的几何。

关于指标运算的一个说明

$$b_\alpha^\beta = b_{\alpha\gamma} g^{\gamma\beta}$$

意味着 b_α^β 是 $\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}$ 的(α,β)-项

同时又可写成

$$b_\alpha^\beta = g^{\gamma\beta} b_{\alpha\gamma} \overset{\text{对称性}}{=} g^{\beta\gamma} b_{\gamma\alpha}$$

意味着 b_α^β 是 $\begin{pmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ 的 (β,α)-项

$$A^t = A,\ B^t = B \Rightarrow (BA)^t = AB$$

所以,上次方程 $b_\beta^\xi b_{\xi\alpha} - b_\alpha^\xi b_{\xi\beta} = 0$ 的平凡性解释的有问题

解释: $b_\beta^\xi b_{\xi\alpha} = b_{\beta\eta} g^{\eta\xi} b_{\xi\alpha} = b_{\beta\eta} b_{\alpha\xi} g^{\xi\eta} = b_{\beta\eta} b_\alpha^\eta$

$$\cancel{= b_{\alpha\eta} b_{\beta\eta} = b_{\alpha\eta} b_{\beta\xi} g^{\xi\eta} = b_{\alpha\eta} g^{\eta\xi} b_{\xi\beta}}$$

$$= b_\alpha^\xi b_{\xi\beta} \qquad \square$$

方程(C)也称 Codazzi - Mainardi 方程，是由两位意大利数学家 Gaspare Mainardi (1856) 和 Delfino Codazzi (1868-1869) 各自独立推出。实际上这个方程的最早版本由俄国数学家 Karl Mikhailovich Peterson 在其1853年的毕业论文中（后来发表）中给出。

我们第一次遇到这么复杂一些的指标很多的公式，可能会问这种公式如何去学呢？

第一境界，就是观察它，各个指标是否具有对称性？熟悉它，或许试图记住它。

第二境界，知道它们从何处来，从什么基本原理来？也就是若要用到，可以临时去推导它。

第三境界，就是通过不断地琢磨，理解它背后的几何意义。这一层没有止境，要不断地重复，不断地品味，如切如磋，如琢如磨。

我们下面就一起朝这个方向讨论这两个结构方程。（这实际上是两组方程）。我们先来看 Gauss 方程（组）：

$$\boxed{\frac{\partial \Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma} - \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta} = b_{\alpha\beta} b^{\xi}_{\gamma} - b_{\alpha\gamma} b^{\xi}_{\beta} \qquad (G)}$$

我们先把上式右端上指标"拉下来"：(G) 等价于将两边同乘 $g_{\delta\xi}$ 再对 ξ 加和以后得到的方程

$$g_{\delta\xi}\left(\frac{\partial \Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma} - \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta}\right) = b_{\alpha\beta} b_{\gamma\delta} - b_{\alpha\gamma} b_{\beta\delta}$$

我们引入 Riemann 记号：

$$R_{\delta\alpha\beta\gamma} := -g_{\delta\xi}\left(\frac{\partial \Gamma^{\xi}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\xi}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\eta}_{\alpha\beta}\Gamma^{\xi}_{\eta\gamma} - \Gamma^{\eta}_{\alpha\gamma}\Gamma^{\xi}_{\eta\beta}\right) \qquad (*)$$

则 (G) $\Leftrightarrow$ $\boxed{R_{\delta\alpha\beta\gamma} = -b_{\alpha\beta} b_{\gamma\delta} + b_{\alpha\gamma} b_{\beta\delta}}$

从上式右端我们容易看到下面的对称性：

$$\boxed{R_{\delta\alpha\beta\gamma} = R_{\beta\gamma\delta\alpha} = -R_{\alpha\delta\beta\gamma} = -R_{\delta\alpha\gamma\beta}}$$

原则上讲，我应该可以从 $R_{\delta\alpha\beta\gamma}$ 的定义式 (*) 直接得到上述对称性： $R_{\delta\alpha\beta\gamma} = -R_{\delta\alpha\gamma\beta}$ 可以从 (*) 直接看出。

为看出其它对称性，我们推导：

$$R_{\delta\alpha\beta\gamma} = -\frac{\partial \Gamma_{\delta\alpha\beta}}{\partial u^\gamma} + \frac{\partial \Gamma_{\delta\alpha\gamma}}{\partial u^\beta} + \frac{\partial g_{\delta\xi}}{\partial u^\gamma}\Gamma^\xi_{\alpha\beta} - \frac{\partial g_{\delta\xi}}{\partial u^\beta}\Gamma^\xi_{\alpha\gamma}$$

$$- \Gamma^\eta_{\alpha\beta}\Gamma_{\delta\eta\gamma} + \Gamma^\eta_{\alpha\gamma}\Gamma_{\delta\eta\beta}$$

$$= -\frac{\partial \Gamma_{\delta\alpha\beta}}{\partial u^\gamma} + \frac{\partial \Gamma_{\delta\alpha\gamma}}{\partial u^\beta} - \Gamma^\eta_{\alpha\beta}\left(\Gamma_{\delta\eta\gamma} - \frac{\partial g_{\delta\eta}}{\partial u^\gamma}\right)$$

$$+ \Gamma^\eta_{\alpha\gamma}\left(\Gamma_{\delta\eta\beta} - \frac{\partial g_{\delta\eta}}{\partial u^\beta}\right)$$

$$\cancel{= \frac{1}{2}\frac{\partial}{\partial u^\gamma}\left(\frac{\partial g_{\delta\alpha}}{\partial u^\delta} + \frac{\partial g_{\delta\beta}}{\partial u^\alpha} - \frac{\partial g_{\delta\alpha}}{\partial u^\beta}\right) - \frac{1}{2}\frac{\partial}{\partial u^\beta}\left(\partial g\right.}$$

$$= -\frac{1}{2}\frac{\partial}{\partial u^\gamma}\left(\frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\delta\beta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta}\right) + \frac{1}{2}\frac{\partial}{\partial u^\beta}\left(\frac{\partial g_{\alpha\delta}}{\partial u^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial u^\alpha} - \frac{\partial g_{\alpha\gamma}}{\partial u^\delta}\right)$$

$$- \Gamma^\eta_{\alpha\beta}\left[\left(\frac{1}{2}\right)\cdot\left(\frac{\partial g_{\eta\delta}}{\partial u^\gamma} + \frac{\partial g_{\delta\gamma}}{\partial u^\eta} - \frac{\partial g_{\eta\gamma}}{\partial u^\delta}\right) - \frac{\partial g_{\delta\eta}}{\partial u^\gamma}\right]$$

$$+ \Gamma^\eta_{\alpha\gamma}\left[\frac{1}{2}\left(\frac{\partial g_{\eta\delta}}{\partial u^\beta} + \frac{\partial g_{\delta\beta}}{\partial u^\eta} - \frac{\partial g_{\eta\beta}}{\partial u^\delta}\right) - \frac{\partial g_{\delta\eta}}{\partial u^\beta}\right]$$

$\Longleftrightarrow$

$$\boxed{R_{\delta\alpha\beta\gamma} = \frac{1}{2}\left(-\frac{\partial^2 g_{\delta\beta}}{\partial u^\gamma \partial u^\alpha} + \frac{\partial^2 g_{\alpha\beta}}{\partial u^\gamma \partial u^\delta} + \frac{\partial^2 g_{\delta\gamma}}{\partial u^\beta \partial u^\alpha} - \frac{\partial^2 g_{\alpha\gamma}}{\partial u^\beta \partial u^\delta}\right) + \Gamma^\eta_{\alpha\beta}\underbrace{\Gamma_{\eta\delta\gamma}}_{= g_{\eta\xi}\Gamma^\xi_{\delta\gamma}} - \Gamma^\eta_{\alpha\gamma}\underbrace{\Gamma_{\eta\delta\beta}}_{= g_{\eta\xi}\Gamma^\xi_{\delta\beta}}}$$

另一方面 $\Gamma^\eta_{\alpha\beta} = g^{\eta\xi}\Gamma_{\xi\alpha\beta}$

也即

$$R_{\delta\alpha\beta\gamma} = \frac{1}{2}\left(-\frac{\partial^2 g_{\delta\beta}}{\partial u^\gamma \partial u^\alpha} + \frac{\partial^2 g_{\alpha\beta}}{\partial u^\gamma \partial u^\delta} + \frac{\partial^2 g_{\delta\gamma}}{\partial u^\beta \partial u^\alpha} - \frac{\partial^2 g_{\alpha\gamma}}{\partial u^\beta \partial u^\delta}\right)$$
$$+ g^{\eta\xi}\left(\Gamma_{\xi\alpha\beta}\Gamma_{\eta\delta\gamma} - \Gamma_{\xi\alpha\gamma}\Gamma_{\eta\delta\beta}\right)$$

从此式容易看出其它二对称性：$R_{\delta\alpha\beta\gamma} = -R_{\alpha\delta\beta\gamma} = R_{\beta\gamma\delta\alpha}$

由这些对称性我们有 $R_{\delta\alpha\beta\gamma} = 0$ whenever $\delta = \alpha$ or $\beta = \gamma$.

从而 Gauss 方程组只有一个独立方程

$$\boxed{R_{1212} = b_{11} b_{22} - b_{12}^2}$$

其中
$$\boxed{\begin{aligned} R_{1212} = {} & \frac{1}{2}\left(-\frac{\partial^2 g_{11}}{\partial u^2 \partial u^2} + \frac{\partial^2 g_{21}}{\partial u^2 \partial u^1} + \frac{\partial^2 g_{12}}{\partial u^1 \partial u^2} - \frac{\partial^2 g_{22}}{\partial u^1 \partial u^1}\right) \\ & + g^{\eta\xi}\left(\Gamma_{\xi 21}\Gamma_{\eta 12} - \Gamma_{\xi 22}\Gamma_{\eta 11}\right). \end{aligned}}$$

我们换回原来的记号 (E, F, G)，就能看清 R_{1212} 的含义了。回忆

$$g_{11} = E, \quad g_{12} = g_{21} = F, \quad g_{22} = G$$
$$g^{11} = \frac{G}{EG - F^2}, \quad g^{12} = g^{21} = \frac{-F}{EG - F^2}, \quad g^{22} = \frac{E}{EG - F^2}.$$

从而
$$\begin{aligned} R_{1212} = {} & \frac{1}{2}\left(-E_{vv} + 2F_{uv} - G_{uu}\right) \\ & + g^{11}\left(\Gamma_{121}\Gamma_{112} - \Gamma_{122}\Gamma_{111}\right) \\ & + g^{12}\left(\Gamma_{221}\Gamma_{112} - \Gamma_{222}\Gamma_{111} + \Gamma_{121}\Gamma_{212} - \Gamma_{122}\Gamma_{211}\right) \\ & + g^{22}\left(\Gamma_{221}\Gamma_{212} - \Gamma_{222}\Gamma_{211}\right) \end{aligned}$$

其中 $\Gamma_{121}\Gamma_{112} - \Gamma_{122}\Gamma_{111}$

$$= \frac{1}{4}\left(\frac{\partial g_{11}}{\partial u^2} + \cancel{\frac{\partial g_{21}}{\partial u^1}} - \cancel{\frac{\partial g_{21}}{\partial u^1}}\right)\left(\cancel{\frac{\partial g_{12}}{\partial u^1}} + \frac{\partial g_{11}}{\partial u^2} - \cancel{\frac{\partial g_{12}}{\partial u^1}}\right)$$
$$- \frac{1}{4}\left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1}\right)\left(\frac{\partial g_{11}}{\partial u^1} + \cancel{\frac{\partial g_{11}}{\partial u^1}} - \cancel{\frac{\partial g_{11}}{\partial u^1}}\right)$$
$$= \frac{1}{4}\left(E_v \cdot E_v - (2F_v - G_u)E_u\right) = \frac{1}{4}\left((E_v)^2 - 2E_uF_v + E_uG_u\right)$$

$$4\left(\Gamma_{221}\Gamma_{112}-\Gamma_{222}\Gamma_{111}+\Gamma_{121}\Gamma_{212}-\Gamma_{122}\Gamma_{211}\right)$$

$$=\left(\cancel{\frac{\partial g_{21}}{\partial u^2}}+\frac{\partial g_{22}}{\partial u^1}-\cancel{\frac{\partial g_{21}}{\partial u^2}}\right)\left(\cancel{\frac{\partial g_{12}}{\partial u^1}}+\frac{\partial g_{11}}{\partial u^2}-\cancel{\frac{\partial g_{12}}{\partial u^1}}\right)$$

$$-\left(\frac{\partial g_{22}}{\partial u^2}+\frac{\partial g_{22}}{\partial u^2}-\frac{\partial g_{22}}{\partial u^2}\right)\left(\frac{\partial g_{11}}{\partial u^1}+\frac{\partial g_{11}}{\partial u^1}-\frac{\partial g_{11}}{\partial u^1}\right)$$

$$+\left(\frac{\partial g_{11}}{\partial u^2}+\frac{\partial g_{21}}{\partial u^1}-\frac{\partial g_{21}}{\partial u^1}\right)\left(\frac{\partial g_{22}}{\partial u^1}+\frac{\partial g_{12}}{\partial u^2}-\frac{\partial g_{12}}{\partial u^2}\right)$$

$$-\left(\frac{\partial g_{12}}{\partial u^2}+\frac{\partial g_{21}}{\partial u^2}-\frac{\partial g_{22}}{\partial u^1}\right)\left(\frac{\partial g_{21}}{\partial u^1}+\frac{\partial g_{12}}{\partial u^1}-\frac{\partial g_{11}}{\partial u^2}\right)$$

$$= G_uE_v - G_vE_u + E_vG_u - (2F_v - G_u)(2F_u - E_v)$$

$$= 2E_vG_u - E_uG_v - 4F_uF_v + 2E_vF_v + 2F_uG_u - G_uE_v$$

$$= -E_uG_v + E_vG_u + \cancel{2E_vG_u} + 2E_vF_v - 4F_uF_v + 2F_uG_u$$

$$4\left(\Gamma_{221}\Gamma_{212}-\Gamma_{222}\Gamma_{211}\right)$$

$$=\left(\frac{\partial g_{21}}{\partial u^2}+\frac{\partial g_{22}}{\partial u^1}-\frac{\partial g_{21}}{\partial u^2}\right)\left(\frac{\partial g_{22}}{\partial u^1}+\frac{\partial g_{12}}{\partial u^2}-\frac{\partial g_{12}}{\partial u^2}\right)$$

$$-\left(\frac{\partial g_{22}}{\partial u^2}+\frac{\partial g_{22}}{\partial u^2}-\frac{\partial g_{22}}{\partial u^2}\right)\left(\frac{\partial g_{21}}{\partial u^1}+\frac{\partial g_{12}}{\partial u^1}-\frac{\partial g_{11}}{\partial u^2}\right)$$

$$= (G_u)^2 - G_v\cdot(2F_u - E_v)$$

$$= (G_u)^2 - 2F_uG_v + E_vG_v$$

综合起来，我们有

$$4(EG-F^2)R_{1212} = E\left(E_vG_v - 2F_uG_v + (G_u)^2\right)$$
$$+F\left(E_uG_v - E_vG_u - 2E_vF_v + 4F_uF_v - 2F_uG_u\right)$$
$$+G\left(E_uG_u - 2E_uF_v + (E_v)^2\right)$$
$$-2(EG-F^2)(E_{vv} - 2F_{uv} + G_{uu})$$

回想(p.104，定理9.1及Theorema Egregium)，上式右端正是

$$4(EG-F^2)^2K, \quad K\text{ 为 Gauss 曲率}$$

故得 $\boxed{R_{1212} = (EG-F^2)K}$

从而，我们得到 Gauss 方程

$$(G) \Leftrightarrow \quad R_{1212} = b_{11}b_{22} - (b_{12})^2$$

$$\Leftrightarrow \quad \underbrace{\frac{R_{1212}}{EG-F^2}}_{K} = \frac{LM-N^2}{EG-F^2}.$$

也即 Gauss 方程等价于高斯曲率 K 的计算式 $K = \frac{LM-N^2}{EG-F^2}$.

注记．这里我们定义的 $R_{\delta\alpha\beta\gamma}$ 和教材 [PC] 中相差一个符号 □

实际上，我们有

$$K = \frac{LM-N^2}{EG-F^2} \overset{\text{Gauss eq.}}{=} \frac{R_{1212}}{g_{11}g_{22}-g_{12}^2}.$$

到此为止，我们对 Gauss 方程 (G) 有了一个比较好的理解

我们再来看 Codazzi 方程（组）：

$$\boxed{\frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} + \Gamma^\xi_{\alpha\beta} b_{\xi\gamma} - \Gamma^\xi_{\alpha\gamma} b_{\xi\beta} = 0 \quad (C)}$$

前面已经提到过，(C) 的左端关于指标 β 和 γ 反称。故当 β=γ 时，(C) 为平凡等式 (0=0). 由此我们知方程组 (C) 只有两个独立方程

$$\begin{cases} \dfrac{\partial b_{11}}{\partial u^2} - \dfrac{\partial b_{12}}{\partial u^1} = \Gamma^\xi_{12} b_{\xi 1} - \Gamma^\xi_{11} b_{\xi 2} \\ \dfrac{\partial b_{21}}{\partial u^2} - \dfrac{\partial b_{22}}{\partial u^1} = \Gamma^\xi_{22} b_{\xi 1} - \Gamma^\xi_{21} b_{\xi 2} \end{cases} \quad (*)$$

下面我们换回原来符号来理解 Codazzi 方程．(*) 可重写为

$$\begin{cases} L_v - M_u = \Gamma^1_{12} L + \Gamma^2_{12} M - \Gamma^1_{11} M - \Gamma^2_{11} N & (C1) \\ M_v - N_u = \Gamma^1_{22} L + \Gamma^2_{22} M - \Gamma^1_{21} M - \Gamma^2_{21} N & (C2) \end{cases}$$

所以我们需要写出 $\Gamma^1_{11}, \Gamma^1_{12} = \Gamma^1_{21}, \Gamma^1_{22}, \Gamma^2_{11}, \Gamma^2_{12} = \Gamma^2_{21}, \Gamma^2_{22}$.

（这是所有的 Christoffel 符号）.

根据定义 $\Gamma^1_{11} = \frac{1}{2} g^{1s}\left(\frac{\partial g_{s1}}{\partial u^1} + \frac{\partial g_{1s}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^s}\right)$

$$= \frac{1}{2} g^{11}\frac{\partial g_{11}}{\partial u^1} + \frac{1}{2} g^{12}\left(\frac{\partial g_{21}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\right)$$

$$\left.\begin{array}{l} g^{11} = \frac{G}{EG-F^2} \\ g^{12} = \frac{-F}{EG-F^2} \\ g^{22} = \frac{E}{EG-F^2} \end{array}\right\} \Rightarrow = \frac{1}{2}\frac{G}{EG-F^2}\cdot E_u + \frac{1}{2}\frac{-F}{EG-F^2}(2F_u - E_v)$$

$$= \frac{1}{EG-F^2}\left(\frac{G}{2}E_u + \frac{F}{2}E_v - FF_u\right).$$

类似可得: $\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{EG-F^2}\left(\frac{G}{2}E_v - \frac{F}{2}G_u\right)$

$$\Gamma^1_{22} = \frac{1}{EG-F^2}\left(GF_v - \frac{G}{2}G_u - \frac{F}{2}G_v\right)$$

$$\Gamma^2_{11} = \frac{1}{EG-F^2}\left(-\frac{F}{2}E_u - \frac{E}{2}E_v + EF_u\right) \quad (☆☆)$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{EG-F^2}\left(-\frac{F}{2}E_v + \frac{E}{2}G_u\right)$$

$$\Gamma^2_{22} = \frac{1}{EG-F^2}\left(-FF_v + \frac{F}{2}G_u + \frac{E}{2}G_v\right)$$

作业: 证明 (☆☆)

我们考虑一个特别的参数表示: 当 (u,v) 是曲面的曲率线网时, (即参数曲线 $u=$常数和 $v=$常数是曲率线, 由上一章习题, 知这等价于 $F=M=0$), 计算会比较简单。这时我们有 (C1) $\Leftrightarrow$

$$\cancel{L_v = L\cdot\frac{1}{2}\frac{E_u}{E} + M\frac{1}{2}\frac{G_u}{G} - M\frac{1}{2}\frac{E_u}{E} + \frac{1}{2}\frac{E_v}{G}N}$$

$$\cancel{= \frac{1}{2}\frac{LGE_u + NEE_v}{EG}}$$

$$L_v = L\Gamma^1_{12} - N\Gamma^2_{11} = L\cdot\frac{1}{EG}\cdot\frac{G}{2}E_v - N\frac{1}{EG}\left(-\frac{E}{2}E_v\right)$$

$$= \frac{1}{2}\left(\frac{L}{E}E_v + \frac{N}{G}E_v\right) = \frac{1}{2}\frac{LG+NE}{EG}\cdot E_v.$$

(回42) 平均曲率 $H = \frac{1}{2} tr \left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \right)$

$$= \frac{1}{2} \frac{1}{EG-F^2} tr \left(\begin{pmatrix} L & M \\ M & N \end{pmatrix} \cdot \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \right)$$

$$= \frac{1}{2} \frac{1}{EG-F^2} (LG - 2MF + NE)$$

特别地，$\overset{M}{\overset{\shortparallel}{F}} = 0 \Rightarrow H = \frac{1}{2} \frac{LG+NE}{EG}$.

综上 (C1) $\Leftrightarrow L_v = HE_v \Leftrightarrow H = \frac{L_v}{E_v}$.

同样地 (C2) $\Leftrightarrow -N_u = L \cdot \Gamma^1_{22} - \Gamma^2_{21} N$

$$= L \cdot \frac{1}{EG}\left(-\frac{G}{2}G_u\right) - \frac{N}{EG}\left(\frac{E}{2}G_u\right)$$

$$= -\frac{L}{2E}G_u - \frac{N}{2G}G_u$$

$$\Leftrightarrow N_u = \frac{1}{2}\frac{LG+NE}{EG}G_u = HG_u.$$

总结起来，当 (u,v) 是曲率线网时，Codazzi 方程等价于

$$\boxed{\begin{cases} L_v = HE_v \\ N_u = HG_u \end{cases}}$$

作业：证明 [PC], P.78. (2,28), P.82. (3,16) 和 (3,17). □

§3. 曲面论基本定理

有了以上曲面结构方程的讨论，我们可以开始曲面论基本定理了，也就是给定两个二次微分形式，相应曲面的存在性和唯一性。这个问题归结为一阶偏微分方程组解的存在和唯一性。