CHAPTER 3

THE CURVATURE OF SURFACES IN SPACE

A. HOW TO READ GAUSS

The single most important work in the history of differential geometry is Karl Friedrich Gauss' paper, in Latin, of 1827: *Disquisitiones generales circa superficies curvas.* The following translation of (part of) this paper is basically the one published* by The Princeton University Library, 1902, except that it adheres even more closely to the notation and typographic disposition of the original.

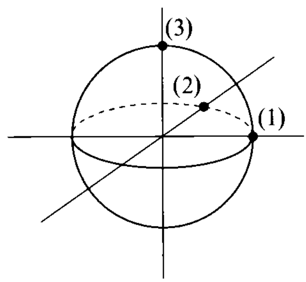
In addition, it has been supplemented with remarks designed to make this first confrontation with classical differential geometry much less painful. The translation of Gauss' paper appears to the right—on odd-numbered pages—while corresponding remarks appear to the left.

Although Part B of this chapter is an exposition of Gauss' results, in modern notation, a preliminary reading of Gauss' great work is heartily recommended; and since many of the difficulties will be clarified in Part B, as a general rule it is a good idea to read on, even if a particular section makes very little sense!

*A reprinting was produced in 1965 by Raven Press, but this is also out of print.

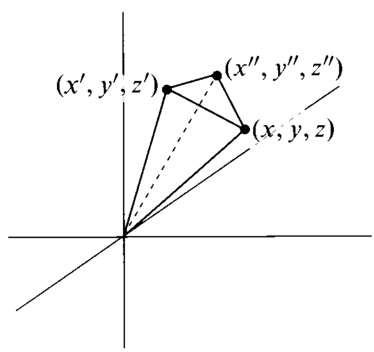
REMARKS ON GAUSS' PAPER

§1. Notice that (1), (2), (3) are used as the names of certain points [(1) = (1, 0, 0), etc.], a circumstance that is easy to forget later on.



§2. This section gives a complicated proof, using spherical trigonometry, that the volume of the pyramid shown below is

$$\frac{1}{6}\left|\det\begin{pmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{pmatrix}\right|,$$



and also includes remarks about the significance of the sign of the determinant. This result is equivalent to the well-known fact that $|\det A|$ is the volume of the parallelepiped spanned by the rows of A.

Almost all of this section can simply be skipped, except for noting that the

GENERAL INVESTIGATIONS

OF

CURVED SURFACES

1.

Investigations, in which the directions of various straight lines in space are to be considered, attain a high degree of clearness and simplicity if we employ, as an auxiliary, a sphere of radius $= 1$ described about an arbitrary center, and suppose the different points of the sphere to represent the directions of straight lines parallel to the radii ending at these points. As the position of every point in space is determined by three coordinates, that is to say, the distances of the point from three mutually perpendicular fixed planes, it is necessary to consider, first of all, the directions of the axes perpendicular to these planes. The points on the sphere, which represent these directions, we shall denote by (1), (2), (3). The distance of any one of these points from either of the other two will be a quadrant; and we shall suppose that the directions of the axes are those in which the corresponding coordinates increase.

2.

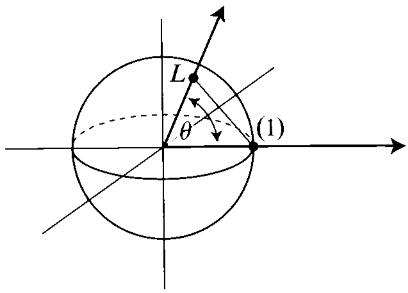
It will be advantageous to bring together here some propositions which are frequently used in questions of this kind.

I. The angle between two intersecting straight lines is measured by the arc between the points on the sphere which correspond to the directions of the lines.

II. The orientation of any plane whatever can be represented by the great circle on the sphere, the plane of which is parallel to the given plane.

III. The angle between two planes is equal to the spherical angle between the great circles representing them, and, consequently, is also measured by the arc intercepted between the poles of these great circles. And, in like manner, the angle of inclination of a straight line to a plane is measured by the arc drawn from the point which corresponds to the direction of the line, perpendicular to the great circle which represents the orientation of the plane.

IV. Letting x, y, z; x', y', z' denote the coordinates of two points, r the distance between them, and L the point on the sphere which represents the expression cos $(1)L$, which will also appear later on, means the cosine of the angle θ between the ray from $(0,0,0)$ through the point $L = (a,b,c)$ on the



sphere and the ray from $(0,0,0)$ through $(1) = (1,0,0)$. Thus, for the usual inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^3$ we have

$$a = \langle L, (1)\rangle = 1 \cdot 1 \cdot \cos\theta,$$

so cos $(1)L$ is the first component of L, and similarly for cos $(2)L$ and cos $(3)L$. [The original contains cos $(1)L^2$ instead of $\cos^2$ $(1)L$, etc., and multiplication is always indicated with a low dot . rather than a centered dot.]

direction of the line drawn from the first point to the second, we shall have

$$x' = x + r\cos(1)L, \quad y' = y + r\cos(2)L, \quad z' = z + r\cos(3)L$$

V. From this it follows at once that, generally,

$$\cos^2(1)L + \cos^2(2)L + \cos^2(3)L = 1$$

and also, if L' denote any other point on the sphere,

$$\cos(1)L \cdot \cos(1)L' + \cos(2)L \cdot \cos(2)L' + \cos(3)L \cdot \cos(3)L' = \cos LL'$$

VI. Theorem. *If L, L', L'', L''' denote four points on the sphere, and A the angle which the arcs $LL', L''L'''$ make at their point of intersection, then we shall have*

$$\cos LL'' \cdot \cos L'L''' - \cos LL''' \cdot \cos L'L'' = \sin LL' \cdot \sin L''L''' \cdot \cos A$$

Demonstration. Let A denote also the point of intersection itself, and set

$$AL = t, \quad AL' = t', \quad AL'' = t'', \quad AL''' = t'''$$

Then we shall have

$$\begin{aligned}
\cos LL'' &= \cos t \cos t'' + \sin t \sin t'' \cos A \\
\cos L'L''' &= \cos t' \cos t''' + \sin t' \sin t''' \cos A \\
\cos LL''' &= \cos t \cos t''' + \sin t \sin t''' \cos A \\
\cos L'L'' &= \cos t' \cos t'' + \sin t' \sin t'' \cos A
\end{aligned}$$

and consequently,

$$\begin{aligned}
&\cos LL'' \cdot \cos L'L''' - \cos LL''' \cdot \cos L'L'' \\
&\quad = \cos A(\cos t \cos t'' \sin t' \sin t''' + \cos t' \cos t''' \sin t \sin t''' \\
&\qquad\qquad - \cos t \cos t''' \sin t' \sin t'' - \cos t' \cos t'' \sin t \sin t''') \\
&\quad = \cos A(\cos t \sin t' - \sin t \cos t')(\cos t'' \sin t''' - \sin t'' \cos t''') \\
&\quad = \cos A \cdot \sin(t' - t) \cdot \sin(t''' - t'') \\
&\quad = \cos A \cdot \sin LL' \cdot \sin L''L'''
\end{aligned}$$

But as there are for each great circle two branches going out from the point A, these two branches form at this point two angles whose sum is 180^0. But our analysis shows that those branches are to be taken whose directions are in the

sense from the point L to L', and from the point L'' to L'''; and since great circles intersect in two points, it is clear that either of the two points can be chosen arbitrarily. Also, instead of the angle A, we can take the arc between the poles of the great circles of which the arcs LL', $L''L'''$ are parts. But it is evident that those poles are to be chosen which are similarly placed with respect to these arcs; that is to say, when we go from L to L' and from L'' to L''', both of the two poles are to be on the right, or both on the left.

VII. Let L, L', L'' be three points on the sphere and set, for brevity,

$$\begin{array}{lll} \cos(1)L = x, & \cos(2)L = y, & \cos(3)L = z \\ \cos(1)L' = x', & \cos(2)L' = y', & \cos(3)L' = z' \\ \cos(1)L'' = x'', & \cos(2)L'' = y'', & \cos(3)L'' = z'' \end{array}$$

and also

$$xy'z'' + x'y''z + x''yz' - xy''z' - x'yz'' - x''y'z = \Delta$$

Let λ denote the pole of the great circle of which LL' is a part, this pole being the one that is placed in the same position with respect to this arc as the point (1) is with respect to the arc (2)(3). Then we shall have, by the preceding theorem, $yz' - y'z = \cos(1)\lambda \cdot \sin(2)(3) \cdot \sin LL'$, or, because $(2)(3) = 90^0$,

$$\begin{aligned} yz' - y'z &= \cos(1)\lambda \cdot \sin LL', \quad \text{and similarly} \\ zx' - z'x &= \cos(2)\lambda \cdot \sin LL' \\ xy' - x'y &= \cos(3)\lambda \cdot \sin LL' \end{aligned}$$

Multiplying these equations by x'', y'', z'' respectively, and adding, we obtain, by means of the second of the theorems deduced in V,

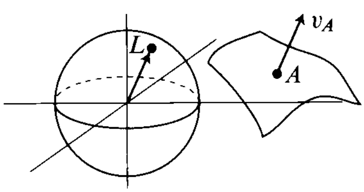
$$\Delta = \cos\lambda L'' \cdot \sin LL'$$

Now there are three cases to be distinguished. *First*, when L'' lies on the great circle of which the arc LL' is a part, we shall have $\lambda L'' = 90^0$, and consequently, $\Delta = 0$. If L'' does not lie on that great circle, the *second* case will be when L'' is on the same side as λ; the *third* case when they are on opposite sides. In the last two cases the points L, L', L'' will form a spherical triangle, and in the second case these points will lie in the same order as the points (1), (2), (3), and in the opposite order in the third case. Denoting the angles of this triangle simply by L, L', L'' and the perpendicular drawn on the sphere from the point L'' to the side LL' by p, we shall have

$$\sin p = \sin L \cdot \sin LL'' = \sin L' \cdot \sin L'L'', \quad \text{and} \quad \lambda L'' = 90^0 \mp p$$

§3. This section merely defines (or tries to define) a differentiable surface, and its tangent plane at a point.

§4. At a point $A = (x, y, z)$ in the surface we have a unit normal vector v_A, and $v \in S^2 \subset \mathbb{R}^3$ is what Gauss calls L. The expression $\cos(1)L$ means the cosine of



the angle between the rays from $(0,0,0)$ through L and through $(1) = (1,0,0)$ (c.f. page 58). So X, Y, Z are just the components of L. Thus X, Y, Z can be considered as functions on the surface [$X(A)$ = first component of v, for v_A a unit normal at A, etc.].

Gauss now nonchalantly introduces infinitely small quantities. The goal of his the upper sign being taken for the second case, the lower for the third. From this it follows that

$$\pm\Delta = \sin L \cdot \sin LL' \cdot \sin LL'' = \sin L' \cdot \sin LL' \cdot \sin L'L'' = \sin L'' \cdot \sin LL'' \cdot \sin L'L''$$

Moreover, it is evident that the first case can be regarded as contained in the second or third, and it is easily seen that the expression $\pm\Delta$ represents six time the volume of the pyramid formed by the points L, L', L'' and the center of the sphere. Whence, finally, it is clear that the expression $\pm\frac{1}{6}\Delta$ expresses generally the volume of any pyramid contained between the origin of coordinates and the three points whose coordinates are x, y, z; x', y', z'; x'', y'', z''.

3.

A curved surface is said to possess continuous curvature at one of its points A, if the directions of all the straight lines drawn from A to points of the surface at an infinitely small distance from A are deflected infinitely little from one and the same plane passing through A. This plane is said to *touch* the surface at the point A. If this condition is not satisfied for any point, the continuity of the curvature is here interrupted, as happens, for example, at the vertex of a cone. The following investigations will be restricted to such surfaces, or to such parts of surfaces, as have the continuity of their curvature nowhere interrupted. We shall only observe now that the methods used to determine the position of the tangent plane lose their meaning at singular points, in which the continuity of the curvature is interrupted, and must lead to indeterminate solutions.

4.

The orientation of the tangent plane is most conveniently studied by means of the direction of the straight line normal to the plane at the point A, which is also called the normal to the curved surface at the point A. We shall represent the direction of this normal by the point L on the auxiliary sphere, and we shall set

$$\cos(1)L = X, \quad \cos(2)L = Y, \quad \cos(3)L = Z$$

and denote the coordinates of the point A by x, y, z. Also let $x + \mathrm{d}x$, $y + \mathrm{d}y$, $z + \mathrm{d}z$ be the coordinates of another point A' on the curved surface; $\mathrm{d}s$ its distance from A, which is infinitely small; and finally, let λ be the point on the sphere representing the direction of the element AA'. Then we shall have

$$\mathrm{d}x = \mathrm{d}s \cdot \cos(1)\lambda, \quad \mathrm{d}y = \mathrm{d}s \cdot \cos(2)\lambda, \quad \mathrm{d}z = \mathrm{d}s \cdot \cos(3)\lambda$$

initial manipulations is the equation

$$X\,dx + Y\,dy + Z\,dz = 0.$$

If x, y, z are considered as functions on the surface (that is, as the restriction to the surface of the standard coordinate functions on $\mathbb{R}^3$), then this equation is literally true, interpreting dx, dy, dz as modern differentials. It should be easy to see this (remember how X, Y, Z are defined). Also try to follow Gauss' argument.

The rest of section 4 gives formulas for X, Y, Z in terms of different descriptions of the surface; in each case the formulas are paired with their negatives, since there are two different choices for the unit normal vector:

(1) If the surface is $\{p \in \mathbb{R}^3 : W(p) = 0\}$, for $W\colon \mathbb{R}^3 \to \mathbb{R}$, then

$$X = \frac{P}{\sqrt{P^2 + Q^2 + R^2}}, \quad \text{where } P = D_1 W,\ Q = D_2 W,\ R = D_3 W, \text{ etc.}$$

[The original has $XX + YY + ZZ = 1$, and $PP + QQ + RR$ for $P^2 + Q^2 + R^2$, and so forth, with a superscript ^{2} used only for the square of a term that is not a single letter.]

(2) If the surface is the image of $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ [Gauss writes dx for $d(x \circ f) = df^1$, etc.] and

$$\begin{aligned} a &= D_1 f^1, & a' &= D_2 f^1 \\ b &= D_1 f^2, & b' &= D_2 f^2 \\ c &= D_1 f^3, & c' &= D_2 f^3 \end{aligned}$$

and, since λL must be equal to 90^0,

$$X\cos(1)\lambda + Y\cos(2)\lambda + Z\cos(3)\lambda = 0$$

By combining these equations we obtain

$$X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z = 0$$

There are two general methods for defining the nature of a curved surface. The *first* uses the equation between the coordinates x, y, z, which we may suppose reduced to the form $W = 0$, where W will be a function of the indeterminants x, y, z. Let the complete differential of the function W be

$$\mathrm{d}W = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

and on the curved surface we shall have

$$P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = 0$$

and consequently,

$$P\cos(1)\lambda + Q\cos(2)\lambda + R\cos(3)\lambda = 0$$

Since this equation, as well as the one we have established above, must be true for the directions of all elements $\mathrm{d}s$ on the curved surface, we easily see that X, Y, Z must be proportional to P, Q, R respectively, and consequently, since

$$X^2 + Y^2 + Z^2 = 1$$

we shall have either

$$X = \frac{P}{\surd(P^2+Q^2+R^2)}, \quad Y = \frac{Q}{\surd(P^2+Q^2+R^2)}, \quad Z = \frac{R}{\surd(P^2+Q^2+R^2)}$$

or

$$X = \frac{-P}{\surd(P^2+Q^2+R^2)}, \quad Y = \frac{-Q}{\surd(P^2+Q^2+R^2)}, \quad Z = \frac{-R}{\surd(P^2+Q^2+R^2)}$$

The *second* method expresses the coordinates in the form of functions of two variables, p, q. Suppose that differentiation of these functions gives

$$\mathrm{d}x = a\,\mathrm{d}p + a'\,\mathrm{d}q$$
$$\mathrm{d}y = b\,\mathrm{d}p + b'\,\mathrm{d}q$$
$$\mathrm{d}z = c\,\mathrm{d}p + c'\,\mathrm{d}q$$

then

$$X = \frac{bc' - cb'}{\Delta}, \text{ etc.}$$

(3) If the surface is $\{(x, y, z) : z = f(x, y)\}$ for $f : \mathbb{R}^2 \to \mathbb{R}^3$, then

$$X = \frac{t}{\sqrt{1 + t^2 + u^2}}, \quad \text{where } t = D_1 f,\ u = D_2 f, \text{ etc.}$$

It should not be hard to work out these results, using our terminology. Again, it is instructive to follow Gauss' derivations as well.

§5. This section talks about orienting the surface, so that one can choose between the two unit normals.

Substituting these values in the formula given above, we obtain

$$(aX + bY + cZ)\,\mathrm{d}p + (a'X + b'Y + c'Z)\,\mathrm{d}q = 0$$

Since this equation must hold independently of the values of the differentials $\mathrm{d}p$, $\mathrm{d}q$, we evidently shall have

$$aX + bY + cZ = 0, \quad a'X + b'Y + c'Z = 0$$

From this we see that X, Y, Z will be proportional to the quantities

$$bc' - cb', \quad ca' - ac', \quad ab' - ba'$$

Hence, on setting, for brevity,

$$\sqrt{((bc' - cb')^2 + (ca' - ac')^2 + (ab' - ba')^2)} = \Delta$$

we shall have either

$$X = \frac{bc' - cb'}{\Delta}, \quad Y = \frac{ca' - ac'}{\Delta}, \quad Z = \frac{ab' - ba'}{\Delta}$$

or

$$X = \frac{cb' - bc'}{\Delta}, \quad Y = \frac{ac' - ca'}{\Delta}, \quad Z = \frac{ba' - ab'}{\Delta}$$

With these two general methods is associated a *third*, in which one of the coordinates, z, say, is expressed in the form of a function of the other two, x, y. This method is evidently only a particular case either of the first method, or of the second. If we set

$$\mathrm{d}z = t\,\mathrm{d}x + u\,\mathrm{d}y$$

we shall have either

$$X = \frac{-t}{\sqrt{(1+t^2+u^2)}}, \quad Y = \frac{-u}{\sqrt{(1+t^2+u^2)}}, \quad Z = \frac{1}{\sqrt{(1+t^2+u^2)}}$$

or

$$X = \frac{t}{\sqrt{(1+t^2+u^2)}}, \quad Y = \frac{u}{\sqrt{(1+t^2+u^2)}}, \quad Z = \frac{-1}{\sqrt{(1+t^2+u^2)}}$$

5.

The two solutions found in the preceding article evidently refer to opposite points of the sphere, or to opposite directions, as one would expect, since the

§6. In this section Gauss considers the map ν, from the surface to S^2, which takes A to the unit vector υ which is normal to the surface at that point. The map ν can be used to take any subset R of the surface to a subset $\nu(R)$ of S^2. The area of $\nu(R)$ is referred to by Gauss as the *total curvature* of R. Then the normal may be drawn toward either of the two sides of the curved surface. If we wish to distinguish between the two regions bordering upon the surface, and call one the exterior region and the other the interior region, we can then assign to each of the two normals its appropriate solution by aid of the theorem derived in Art. 2 (VII), and at the same time establish a criterion for distinguishing the one region from the other.

In the first method, such a criterion is to be drawn from the sign of the quantity W. Indeed, generally speaking, the curved surface divides those regions of space in which W keeps a positive value from those in which the value of W becomes negative. In fact, it is easily seen from this theorem that, if W takes a positive value toward the exterior region, and if the normal is supposed to be drawn outwardly, the first solution is to be taken. Moreover, it will be easy to decide in any case whether the same rule for the sign of W is to hold throughout the entire surface, or whether for different parts there will be different rules. As long as the coefficients P, Q, R have finite values and do not all vanish at the same time, the law of continuity will prevent any change.

If we follow the second method, we can imagine two systems of curved lines on the curved surface, one system for which p is variable, q constant; the other for which q is variable, p constant. The respective positions of these lines with reference to the exterior region will decide which of the two solutions must be taken. In fact, whenever the three lines, namely, the branch of the line of the former system going out from the point A as p increases, the branch of the line of the latter system going out from the point A as q increases, and the normal drawn toward the exterior region, are *similarly* placed as the x, y, z axes respectively from the origin of abscissas (e.g., if, both for the former three lines and for the latter three, we can conceive the first directed to the left, the second to the right, and the third upward), the first solution is to be taken. But whenever the relative position of the three lines is opposite to the relative position of the x, y, z axes, the second solution will hold.

In the third method, it is to be seen whether, when z receives a positive increment, x and y remaining constant, the point crosses toward the exterior or the interior region. In the former case, for the normal drawn outward, the first solution holds; in the latter case, the second.

6.

Just as each definite point on the curved surface is made to correspond to a definite point on the sphere, by the direction of the normal to the curved surface which is transferred to the surface of the sphere, so also any line whatever, or any figure whatever, on the latter will be represented by a corresponding line curvature at a point A in the surface is defined as

$$\frac{\text{total curvature of } R}{\text{area of } R}$$

where R is the "surface element" at A, which is supposed to have infinitely small area. As a first approximation to what Gauss is trying to say, we might define the curvature as

$$\lim \frac{\text{total curvature of } R}{\text{area of } R}$$

where the limit is taken as R approaches the point A. It is not *a priori* so clear whether this limit exists, or if it depends on the way in which R "approaches" A.

Gauss also gives considerable discussion to the sign of the curvature.

or figure on the former. In the comparison of two figures corresponding to one another in this way, one of which will be as the map of the other, two important points are to be considered, one when quantity alone is considered, the other when, disregarding quantitative relations, position alone is considered.

The first of these important points will be the basis of some ideas which it seems judicious to introduce into the theory of curved surfaces. Thus, to each part of a curved surface inclosed within definite limits we assign a *total* or *integral curvature*, which is represented by the area of the figure on the sphere corresponding to it. From this integral curvature must be distinguished the somewhat more specific curvature which we shall call the *measure of curvature*. The latter refers to a *point* of the surface, and shall denote the quotient obtained when the integral curvature of the surface element about a point is divided by the area of the element itself; and hence it denotes the ratio of the infinitely small areas which correspond to one another on the curved surface and on the sphere. The use of these innovations will be abundantly justified, as we hope, by what we shall explain below. As for the terminology, we have thought it especially desirable that all ambiguity be avoided. For this reason we have not thought it advantageous to follow strictly the analogy of the terminology commonly adopted (though not approved by all) in the theory of plane curves, according to which the measure of curvature should be called simply curvature, but the total curvature, the amplitude. But why not be free in the choice of words, provided they are not meaningless and not liable to a misleading interpretation?

The position of a figure on the sphere can be either similar to the position of the corresponding figure on the curved surface, or opposite (inverse). The former is the case when two lines going out on the curved surface from the same point in different, but not opposite directions, are represented on the sphere by lines similarly placed, that is, when the map of the line to the right is also to the right; the latter is the case when the contrary holds. We shall distinguish these two cases by the positive or negative *sign* of the measure of curvature. But evidently this distinction can hold only when on each surface we choose a definite face on which we suppose the figure to lie. On the auxiliary sphere we shall use always the exterior face, that is, that turned away from the center; on the curved surface also there may be taken for the exterior face the one already considered, or rather that face from which the normal is supposed to be drawn. For, evidently, there is no change in regard to the similitude of the figures, if on the curved surface both the figure and the normal be transferred to the opposite side, so long as the image itself is represented on the same side of the sphere.

The positive or negative sign, which we assign to the *measure* of curvature according to the position of the infinitely small figure, we extend also to the

§7. In this section Gauss finds a formula for the curvature k at A. His answer, at the top of page 77, is given for a surface which is the graph of $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. In this case, the functions X and Y can be thought of as functions on $\mathbb{R}^2$ (that is, we consider $X \circ f$ and $Y \circ f$), and Gauss' answer is

$$k = \frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial y}\frac{\partial Y}{\partial x} \quad [= D_1(X \circ f)D_2(Y \circ f) - D_2(X \circ f)D_1(Y \circ f)].$$

[The notation $\frac{dX}{dx}$, etc., in the original has been preserved. Similarly, a few integral curvature of a finite figure on the curved surface. However, if we wish to discuss the general case, some explanations will be necessary, which we can only touch here briefly. So long as the figure on the curved surface is such that to *distinct* points on itself there correspond distinct points on the sphere, the definition needs no further explanation. But whenever this condition is not satisfied, it will be necessary to take into account twice or several times certain parts of the figure on the sphere. Whence for a similar, or inverse position, may arise an accumulation of areas, or the areas may partially or wholly destroy each other. In such a case, the simplest way is to suppose the curved surface divided into parts, such that each part, considered separately, satisfies the above condition; to assign to each of the parts its integral curvature, determining this magnitude by the area of the corresponding figure on the sphere, and the sign by the position of this figure; and, finally, to assign to the total figure the integral curvature arising from the addition of the integral curvatures which correspond to the single parts. So, generally, the integral curvature of a figure is equal to $\int k\,\mathrm{d}\sigma$, $\mathrm{d}\sigma$ denoting the element of area of the figure, and k the measure of curvature at any point. The principal points concerning the geometric representation of this integral reduce to the following. To the perimeter of the figure on the curved surface (under the restriction of Art. 3) will correspond always a closed line on the sphere. If the latter nowhere intersect itself, it will divide the whole surface of the sphere into two parts, one of which will correspond to the figure on the curved surface; and its area (taken as positive or negative according as, with respect to its perimeter, its position is similar, or inverse, to the position of the figure on the curved surface) will represent the integral curvature of the figure on the curved surface. But whenever this line intersects itself once or several times, it will give a complicated figure, to which, however, it is possible to assign a definite area as legitimately as in the case of a figure without nodes; and this area, properly interpreted, will give always an exact value for the integral curvature. However, we must reserve for another occasion the more extended exposition of the theory of these figures viewed from this very general standpoint.

7.

We shall now find a formula which will express the measure of curvature for any point of a curved surface. Let $\mathrm{d}\sigma$ denote the area of an element of this surface; then $Z\,\mathrm{d}\sigma$ will be the area of the projection of this element on the plane of the coordinates x, y; and consequently, if $\mathrm{d}\Sigma$ is the area of the corresponding element on the sphere, $Z\,\mathrm{d}\Sigma$ will be the area of its projection on the same plane. The positive or negative sign of Z will, in fact, indicate that the position of the projection is similar or inverse to that of the projected element. Evidently these lines later the expressions $\frac{ddz}{dx^2}$ and $\frac{ddz}{dx \cdot dy}$ stand for $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial x \partial y}$, etc.]

Gauss obtains this answer by considering an infinitesimal triangle "$d\sigma$" with one vertex at $(x, y, f(x, y))$, one vertex at $(x+dx, y+dy, f(x+dx, y+dy))$, and one at $(x+\delta x, y+\delta y, f(x+\delta x, y+\delta y))$. It is a challenge both to follow Gauss' reasoning, and to put it in modern terms. Either way, one needs Gauss' preliminary observation that

$$\frac{\text{area } \nu(d\sigma)}{\text{area } d\sigma} = \frac{\text{area of projection on } (x, y)\text{-plane of } \nu(d\sigma)}{\text{area of projection on } (x, y)\text{-plane of } d\sigma}.$$

This mysterious equation really says that the tangent plane of M at A is parallel to the tangent plane of S^2 at $\nu(A)$. If this hint does not help, simply accept the formula for k, which will be derived later, using modern terminology.

The remainder of section 7 evaluates k in terms of partial derivatives of f (which Gauss denotes by z).

projections have the same ratio as to quantity and the same relation as to position as the elements themselves. Let us consider now a triangular element on the curved surface, and let us suppose that the coordinates of the three points which form its projection are

$$\begin{array}{ll} x, & y \\ x + \mathrm{d}x, & y + \mathrm{d}y \\ x + \delta x, & y + \delta y \end{array}$$

The double area of this triangle will be expressed by the formula

$$\mathrm{d}x \cdot \delta y - \mathrm{d}y \cdot \delta x$$

and this will be in a positive or negative form according as the position of the side from the first point to the third, with respect to the side from the first point to the second, is similar or opposite to the position of the y-axis of coordinates with respect to the x-axis of coordinates.

In like manner, if the coordinates of the three points which form the projection of the corresponding element on the sphere, from the center of the sphere as origin, are

$$\begin{array}{ll} X, & Y \\ X + \mathrm{d}X, & Y + \mathrm{d}Y \\ X + \delta X, & Y + \delta Y \end{array}$$

the double area of this projection will be expressed by

$$\mathrm{d}X \cdot \delta Y - \mathrm{d}Y \cdot \delta X$$

and the sign of this expression is determined in the same manner as above. Wherefore the measure of curvature at this point of the curved surface will be

$$k = \frac{\mathrm{d}X \cdot \delta Y - \mathrm{d}Y \cdot \delta X}{\mathrm{d}x \cdot \delta y - \mathrm{d}y \cdot \delta x}$$

If now we suppose the nature of the curved surface to be defined according to the third method considered in Art. 4, X and Y will be in the form of functions of the quantities x, y. We shall have, therefore,

$$\mathrm{d}X = (\tfrac{\mathrm{d}X}{\mathrm{d}x})\,\mathrm{d}x + (\tfrac{\mathrm{d}X}{\mathrm{d}y})\,\mathrm{d}y$$

$$\delta X = (\tfrac{\mathrm{d}X}{\mathrm{d}x})\,\delta x + (\tfrac{\mathrm{d}X}{\mathrm{d}y})\,\delta y$$

$$\mathrm{d}Y = (\tfrac{\mathrm{d}Y}{\mathrm{d}x})\,\mathrm{d}x + (\tfrac{\mathrm{d}Y}{\mathrm{d}y})\,\mathrm{d}y$$

$$\delta Y = (\tfrac{\mathrm{d}Y}{\mathrm{d}x})\,\delta x + (\tfrac{\mathrm{d}Y}{\mathrm{d}y})\,\delta y$$

When these values have been substituted, the above expression becomes

$$k = \left(\frac{\mathrm{d}X}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}Y}{\mathrm{d}y}\right) - \left(\frac{\mathrm{d}X}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}Y}{\mathrm{d}x}\right)$$

Setting, as above,

$$\frac{\mathrm{d}z}{\mathrm{d}x} = t, \quad \frac{\mathrm{d}z}{\mathrm{d}y} = u$$

and also

$$\frac{\mathrm{dd}z}{\mathrm{d}x^2} = T, \quad \frac{\mathrm{dd}z}{\mathrm{d}x \cdot \mathrm{d}y} = U, \quad \frac{\mathrm{dd}z}{\mathrm{d}y^2} = V$$

or

$$\mathrm{d}t = T\,\mathrm{d}x + U\,\mathrm{d}y, \quad \mathrm{d}u = U\,\mathrm{d}x + V\,\mathrm{d}y$$

we have from the formulæ given above

$$X = -tZ, \quad Y = -uZ, \quad (1 + t^2 + u^2)Z^2 = 1$$

and hence

$$\begin{aligned} \mathrm{d}X &= -Z\,\mathrm{d}t - t\,\mathrm{d}Z \\ \mathrm{d}Y &= -Z\,\mathrm{d}u - u\,\mathrm{d}Z \\ (1 + t^2 &+ u^2)\,\mathrm{d}Z + Z(t\,\mathrm{d}t + u\,\mathrm{d}u) = 0 \end{aligned}$$

or

$$\begin{aligned} \mathrm{d}Z &= -Z^3(t\,\mathrm{d}t + u\,\mathrm{d}u) \\ \mathrm{d}X &= -Z^3(1 + u^2)\,\mathrm{d}t + Z^3 t u\,\mathrm{d}u \\ \mathrm{d}Y &= +Z^3 t u\,\mathrm{d}t - Z^3(1 + t^2)\,\mathrm{d}u \end{aligned}$$

and so

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}x} &= Z^3(-(1 + u^2)T + tuU) \\ \frac{\mathrm{d}X}{\mathrm{d}y} &= Z^3(-(1 + u^2)U + tuV) \\ \frac{\mathrm{d}Y}{\mathrm{d}x} &= Z^3(tuT - (1 + t^2)U) \\ \frac{\mathrm{d}Y}{\mathrm{d}y} &= Z^3(tuU - (1 + t^2)V) \end{aligned}$$

Substituting these values in the above expression, it becomes

$$k = Z^6(TV - U^2)(1 + t^2 + u^2) = Z^4(TV - U^2) = \frac{TV - U^2}{(1 + t^2 + u^2)^2}$$

§8. This section, except for the last theorem, was already done in Chapter 2.

8.

By a suitable choice of origin and axes of coordinates, we can easily make the values of the quantities t, u, U vanish for a definite point A. Indeed, the first two conditions will be fulfilled at once if the tangent plane at this point be taken for the xy-plane. If, further, the origin is placed at the point A itself, the expression for the coordinate z evidently takes the form

$$z = \tfrac{1}{2}T^0x^2 + U^0xy + \tfrac{1}{2}V^0y^2 + \Omega$$

where Ω will be of higher degree than the second. Turning now the axes of x and y through an angle M such that

$$\tan 2M = \frac{2U^0}{T^0 - V^0}$$

it is easily seen that there must result an equation of the form

$$z = \tfrac{1}{2}Tx^2 + \tfrac{1}{2}Vy^2 + \Omega$$

In this way the third condition is also satisfied. When this has been done, it is evident that

I. If the curved surface be cut by a plane passing through the normal itself and through the x-axis, a plane curve will be obtained, the radius of curvature of which at the point A will be $= \frac{1}{T}$, the positive or negative sign indicating that the curve is concave or convex toward that region toward which the coordinates z are positive.

II. In like manner $\frac{1}{V}$ will be the radius of curvature at the point A of the plane curve which is the intersection of the surface and the plane through the y-axis and the z-axis.

III. Setting $x = r\cos\varphi$, $y = r\sin\varphi$, the equation becomes

$$z = \tfrac{1}{2}(T\cos^2\varphi + V\sin^2\varphi)r^2 + \Omega$$

from which we see that if the section is made by a plane through the normal at A and making an angle φ with the x-axis, we shall have a plane curve whose radius of curvature at the point A will

$$= \frac{1}{T\cos^2\varphi + V\sin^2\varphi}$$

IV. Therefore, whenever we have $T = V$, the radii of curvature in *all* the normal planes will be equal. But if T and V are not equal, it is evident that, since

§§9, 10, 11. These sections are essentially calculations, involving no new ideas. Every once in a while Gauss calculates a differential instead of some partial derivatives, but this should cause no difficulties.

The goal is the very last, four-line-long, equation at the end of section 11.

for any value whatever of the angle φ, $T\cos^2\varphi + V\sin^2\varphi$ falls between T and V, the radii of curvature in the principal sections considered in I and II refer to the extreme curvatures; that is to say, the one to the maximum curvature, the other to the minimum, if T and V have the same sign. On the other hand, one has the greatest convex curvature, the other the greatest concave curvature, if T and V have opposite signs. These conclusions contain almost all that the illustrious EULER was the first to prove on the curvature of curved surfaces.

V. The measure of curvature at the point A on the curved surface takes the very simple form $k = TV$, whence we have the

THEOREM. *The measure of curvature at any point whatever of the surface is equal to a fraction whose numerator is unity, and whose denominator is the product of the two extreme radii of curvature of the sections by normal planes.*

At the same time it is clear that the measure of curvature is positive for concavo-concave or convexo-convex surfaces (which distinction is not essential), but negative for concavo-convex surfaces. If the surface consists of parts of each kind, then on the lines separating the two kinds the measure of curvature ought to vanish. Later we shall make a detailed study of the nature of curved surfaces for which the measure of curvature everywhere vanishes.

9.

The general formula for the measure of curvature given at the end of Art. 7 is the most simple of all, since it involves only five elements. We shall arrive at a more complicated formula, indeed, one involving nine elements, if we wish to use the first method of representing a curved surface. Keeping the notation of Art. 4, let us set also

$$\frac{\mathrm{dd}W}{\mathrm{d}x^2} = P', \quad \frac{\mathrm{dd}W}{\mathrm{d}y^2} = Q', \quad \frac{\mathrm{dd}W}{\mathrm{d}z^2} = R'$$
$$\frac{\mathrm{dd}W}{\mathrm{d}y\cdot\mathrm{d}z} = P'', \quad \frac{\mathrm{dd}W}{\mathrm{d}x\cdot\mathrm{d}z} = Q'', \quad \frac{\mathrm{dd}W}{\mathrm{d}x\cdot\mathrm{d}y} = R''$$

so that

$$\begin{aligned}
\mathrm{d}P &= P'\,\mathrm{d}x + R''\,\mathrm{d}y + Q''\,\mathrm{d}z\\
\mathrm{d}Q &= R''\,\mathrm{d}x + Q'\,\mathrm{d}y + P''\,\mathrm{d}z\\
\mathrm{d}R &= Q''\,\mathrm{d}x + P''\,\mathrm{d}y + R'\,\mathrm{d}z
\end{aligned}$$

Now since $t = -\frac{P}{R}$, we find through differentiation

$$R^2\,\mathrm{d}t = -R\,\mathrm{d}P + P\,\mathrm{d}R = (PQ'' - RP')\,\mathrm{d}x + (PP'' - RR'')\,\mathrm{d}y + (PR' - RQ'')\,\mathrm{d}z$$

[As you can probably figure out for yourself, ϑ is an alternate form of β.]

or, eliminating $\mathrm{d}z$ by means of the equation $P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = 0$,

$$R^3\,\mathrm{d}t = (-R^2P' + 2PRQ'' - P^2R')\,\mathrm{d}x + (PRP'' + QRQ'' - PQR' - R^2R'')\,\mathrm{d}y$$

In like manner we obtain

$$R^3\,\mathrm{d}u = (PRP'' + QRQ'' - PQR' - R^2R'')\,\mathrm{d}x + (-R^2Q' + 2QRP'' - Q^2R')\,\mathrm{d}y$$

From this we conclude that

$$\begin{aligned} R^3T &= -R^2P' + 2PRQ'' - P^2R' \\ R^3U &= PRP'' + QRQ'' - PQR' - R^2R'' \\ R^3V &= -R^2Q' + 2QRP'' - Q^2R' \end{aligned}$$

Substituting these values in the formula of Art. 7, we obtain for the measure of curvature k the following symmetric expression:

$$\begin{aligned} &(P^2+Q^2+R^2)^2\,k \\ &\quad = P^2(Q'R' - P''^2) + Q^2(P'R' - Q''^2) + R^2(P'Q' - R''^2) \\ &\qquad + 2QR(Q''R'' - P'P'') + 2PR(P''R'' - Q'Q'') + 2PQ(P''Q'' - R'R'') \end{aligned}$$

10.

We obtain a still more complicated formula, indeed, one involving fifteen elements, if we follow the second general method of defining the nature of a curved surface. It is, however, very important that we develop this formula also. Retaining the notations of Art. 4, let us put also

$$\begin{aligned} &\frac{\mathrm{dd}x}{\mathrm{d}p^2} = \alpha, \quad \frac{\mathrm{dd}x}{\mathrm{d}p\cdot\mathrm{d}q} = \alpha', \quad \frac{\mathrm{dd}x}{\mathrm{d}q^2} = \alpha'' \\ &\frac{\mathrm{dd}y}{\mathrm{d}p^2} = \beta, \quad \frac{\mathrm{dd}y}{\mathrm{d}p\cdot\mathrm{d}q} = \beta', \quad \frac{\mathrm{dd}y}{\mathrm{d}q^2} = \beta'' \\ &\frac{\mathrm{dd}z}{\mathrm{d}p^2} = \gamma, \quad \frac{\mathrm{dd}z}{\mathrm{d}p\cdot\mathrm{d}q} = \gamma', \quad \frac{\mathrm{dd}z}{\mathrm{d}q^2} = \gamma'' \end{aligned}$$

and let us put, for brevity,

$$\begin{aligned} bc' - cb' &= A \\ ca' - ac' &= B \\ ab' - ba' &= C \end{aligned}$$

First we see that $A\,\mathrm{d}x + B\,\mathrm{d}y + C\,\mathrm{d}z = 0$, or $\mathrm{d}z = -\frac{A}{C}\,\mathrm{d}x - \frac{B}{C}\,\mathrm{d}y$; thus, inasmuch as z may be regarded as a function of x, y, we have

$$\frac{\mathrm{d}z}{\mathrm{d}x} = t = -\frac{A}{C}$$
$$\frac{\mathrm{d}z}{\mathrm{d}y} = u = -\frac{B}{C}$$

Then from the formulæ $\mathrm{d}x = a\,\mathrm{d}p + a'\,\mathrm{d}q$, $\mathrm{d}y = b\,\mathrm{d}p + b'\,\mathrm{d}q$, we have

$$C\,\mathrm{d}p = b'\,\mathrm{d}x - a'\,\mathrm{d}y$$
$$C\,\mathrm{d}q = -b\,\mathrm{d}x + a\,\mathrm{d}y$$

Thence we obtain for the total differentials of t, u

$$C^3\,\mathrm{d}t = \left(A\frac{\mathrm{d}C}{\mathrm{d}p} - C\frac{\mathrm{d}A}{\mathrm{d}p}\right)(b'\,\mathrm{d}x - a'\,\mathrm{d}y) + \left(C\frac{\mathrm{d}A}{\mathrm{d}q} - A\frac{\mathrm{d}C}{\mathrm{d}q}\right)(b\,\mathrm{d}x - a\,\mathrm{d}y)$$
$$C^3\,\mathrm{d}u = \left(B\frac{\mathrm{d}C}{\mathrm{d}p} - C\frac{\mathrm{d}B}{\mathrm{d}p}\right)(b'\,\mathrm{d}x - a'\,\mathrm{d}y) + \left(C\frac{\mathrm{d}B}{\mathrm{d}q} - B\frac{\mathrm{d}C}{\mathrm{d}q}\right)(b\,\mathrm{d}x - a\,\mathrm{d}y)$$

If now we substitute in these formulæ

$$\frac{\mathrm{d}A}{\mathrm{d}p} = c'\beta + b\gamma' - c\beta' - b'\gamma$$
$$\frac{\mathrm{d}A}{\mathrm{d}q} = c'\beta' + b\gamma'' - c\beta'' - b'\gamma'$$
$$\frac{\mathrm{d}B}{\mathrm{d}p} = a'\gamma + c\alpha' - a\gamma' - c'\alpha$$
$$\frac{\mathrm{d}B}{\mathrm{d}q} = a'\gamma' + c\alpha'' - a\gamma'' - c'\alpha'$$
$$\frac{\mathrm{d}C}{\mathrm{d}p} = b'\alpha + a\beta' - b\alpha' - a'\beta$$
$$\frac{\mathrm{d}C}{\mathrm{d}q} = b'\alpha' + a\beta'' - b\alpha'' - a'\beta'$$

and if we note that the values of the differentials $\mathrm{d}t$, $\mathrm{d}u$ thus obtained must be equal, independently of the differentials $\mathrm{d}x$, $\mathrm{d}y$, to the quantities $T\,\mathrm{d}x + U\,\mathrm{d}y$, $U\,\mathrm{d}x + V\,\mathrm{d}y$ respectively, we shall find, after some sufficiently obvious

transformations,

$$\begin{aligned} C^3T = {} & \alpha Ab'^2 + \beta Bb'^2 + \gamma Cb'^2 \\ & - 2\alpha' Abb' - 2\beta' Bbb' - 2\gamma' Cbb' \\ & + \alpha'' Ab^2 + \beta'' Bb^2 + \gamma'' Cb^2 \\ C^3U = {} & -\alpha Aa'b' - \beta Ba'b' - \gamma Ca'b' \\ & + \alpha' A(ab' + ba') + \beta' B(ab' + ba') + \gamma' C(ab' + ba') \\ & - \alpha'' Aab - \beta'' Bab - \gamma'' Cab \\ C^3V = {} & \alpha Aa'^2 + \beta Ba'^2 + \gamma Ca'^2 \\ & - 2\alpha' Aaa' - 2\beta' Baa' - 2\gamma' Caa' \\ & + \alpha'' Aa^2 + \beta'' Ba^2 + \gamma'' Ca^2 \end{aligned}$$

Hence, if we put, for the sake of brevity,

$$A\alpha + B\beta + C\gamma = D \qquad \ldots\ldots\ldots\ldots \quad (1)$$
$$A\alpha' + B\beta' + C\gamma' = D' \qquad \ldots\ldots\ldots\ldots \quad (2)$$
$$A\alpha'' + B\beta'' + C\gamma'' = D'' \qquad \ldots\ldots\ldots\ldots \quad (3)$$

we shall have

$$\begin{aligned} C^3T &= Db'^2 - 2D'bb' + D''b^2 \\ C^3U &= -Da'b' + D'(ab' + ba') - D''ab \\ C^3V &= Da'^2 - 2D'aa' + D''a^2 \end{aligned}$$

From this we find, after the reckoning has been carried out,

$$C^6(TV - U^2) = (DD'' - D'^2)(ab' - ba')^2 = (DD'' - D'^2)C^2$$

and therefore the formula for the measure of curvature

$$k = \frac{DD'' - D'^2}{(A^2 + B^2 + C^2)^2}$$

11.

By means of the formula just found we are going to establish another, which may be counted among the most productive theorems in the theory of curved

surfaces. Let us introduce the following notation:

$$\begin{aligned}
a^2 &+ b^2 &+ c^2 &= E \\
aa' &+ bb' &+ cc' &= F \\
a'^2 &+ b'^2 &+ c'^2 &= G \\
a\alpha &+ b\beta &+ c\gamma &= m \quad \ldots\ldots\ldots\ldots (4) \\
a\alpha' &+ b\beta' &+ c\gamma' &= m' \quad \ldots\ldots\ldots\ldots (5) \\
a\alpha'' &+ b\beta'' &+ c\gamma'' &= m'' \quad \ldots\ldots\ldots\ldots (6) \\
a'\alpha &+ b'\beta &+ c'\gamma &= n \quad \ldots\ldots\ldots\ldots (7) \\
a'\alpha' &+ b'\beta' &+ c'\gamma' &= n' \quad \ldots\ldots\ldots\ldots (8) \\
a'\alpha'' &+ b'\beta'' &+ c'\gamma'' &= n'' \quad \ldots\ldots\ldots\ldots (9) \\
A^2 &+ B^2 &+ C^2 &= EG - F^2 = \Delta
\end{aligned}$$

Let us eliminate from the equations 1, 4, 7 the quantities β, γ, which is done by multiplying them by $bc' - cb'$, $b'C - c'B$, $cB - bC$ respectively and adding: in this way we obtain

$$\begin{aligned}
&(A(bc' - cb') + a(b'C - c'B) + a'(cB - bC))\alpha \\
&= D(bc' - cb') + m(b'C - c'B) + n(cB - bC)
\end{aligned}$$

an equation which is easily transformed into

$$AD = \alpha\Delta + a(nF - mG) + a'(mF - nE)$$

Likewise the elimination of α, γ or α, β from the same equations gives

$$\begin{aligned}
BD &= \beta\Delta + b(nF - mG) + b'(mF - nE) \\
CD &= \gamma\Delta + c(nF - mG) + c'(mF - nE)
\end{aligned}$$

Multiplying these three equations by α'', β'', γ'' respectively and adding, we obtain

$$DD'' = (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'')\Delta + m''(nF - mG) + n''(mF - nE) \quad \ldots \quad (10)$$

If we treat the equations 2, 5, 8 in the same way, we obtain

$$\begin{aligned}
AD' &= \alpha'\Delta + a(n'F - m'G) + a'(m'F - n'E) \\
BD' &= \beta'\Delta + b(n'F - m'G) + b'(m'F - n'E) \\
CD' &= \gamma'\Delta + c(n'F - m'G) + c'(m'F - n'E)
\end{aligned}$$

§12. If $M, N \subset \mathbb{R}^3$ are surfaces, then a *development* of M on N is simply a map $f: M \to N$ which is an isometry (with respect to the induced Riemannian metrics).

and after these equations are multiplied by α', β', γ' respectively, addition gives

$$D'^2 = (\alpha'^2 + \beta'^2 + \gamma'^2)\Delta + m'(n'F - m'G) + n'(m'F - n'E)$$

A combination of this equation with equation (10) gives

$$\begin{aligned} DD'' - D'^2 = {} & (\alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2)\Delta \\ & + E(n'^2 - nn'') + F(nm'' - 2m'n' + mn'') + G(m'^2 - mm'') \end{aligned}$$

It is clear that we have

$$\frac{\mathrm{d}E}{\mathrm{d}p} = 2m, \quad \frac{\mathrm{d}E}{\mathrm{d}q} = 2m', \quad \frac{\mathrm{d}F}{\mathrm{d}p} = m' + n, \quad \frac{\mathrm{d}F}{\mathrm{d}q} = m'' + n', \quad \frac{\mathrm{d}G}{\mathrm{d}p} = 2n', \quad \frac{\mathrm{d}G}{\mathrm{d}q} = 2n''$$

or

$$\begin{aligned} m &= \tfrac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}p}, & m' &= \tfrac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}q}, & m'' &= \frac{\mathrm{d}F}{\mathrm{d}q} - \tfrac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}p} \\ n &= \frac{\mathrm{d}F}{\mathrm{d}p} - \tfrac{1}{2}\frac{\mathrm{d}E}{\mathrm{d}q}, & n' &= \tfrac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}p}, & n'' &= \tfrac{1}{2}\frac{\mathrm{d}G}{\mathrm{d}q} \end{aligned}$$

Moreover, it is easily shown that we shall have

$$\begin{aligned} \alpha\alpha'' + \beta\beta'' + \gamma\gamma'' - \alpha'^2 - \beta'^2 - \gamma'^2 &= \frac{\mathrm{d}n}{\mathrm{d}q} - \frac{\mathrm{d}n'}{\mathrm{d}p} = \frac{\mathrm{d}m''}{\mathrm{d}p} - \frac{\mathrm{d}m'}{\mathrm{d}q} \\ &= -\tfrac{1}{2}\cdot\frac{\mathrm{dd}E}{\mathrm{d}q^2} + \frac{\mathrm{dd}F}{\mathrm{d}p\cdot\mathrm{d}q} - \tfrac{1}{2}\cdot\frac{\mathrm{dd}G}{\mathrm{d}p^2} \end{aligned}$$

If we substitute these different expressions in the formula for the measure of curvature derived at the end of the preceding article, we obtain the following formula, which involves only the quantities E, F, G and their differential quotients of the first and second orders:

$$\begin{aligned} 4(EG - F^2)^2 k = {} & E\Big(\frac{\mathrm{d}E}{\mathrm{d}q}\cdot\frac{\mathrm{d}G}{\mathrm{d}q} - 2\frac{\mathrm{d}F}{\mathrm{d}p}\cdot\frac{\mathrm{d}G}{\mathrm{d}q} + \big(\frac{\mathrm{d}G}{\mathrm{d}p}\big)^2\Big) \\ & + F\Big(\frac{\mathrm{d}E}{\mathrm{d}p}\cdot\frac{\mathrm{d}G}{\mathrm{d}q} - \frac{\mathrm{d}E}{\mathrm{d}q}\cdot\frac{\mathrm{d}G}{\mathrm{d}p} - 2\frac{\mathrm{d}E}{\mathrm{d}q}\cdot\frac{\mathrm{d}F}{\mathrm{d}q} + 4\frac{\mathrm{d}F}{\mathrm{d}p}\cdot\frac{\mathrm{d}F}{\mathrm{d}q} - 2\frac{\mathrm{d}F}{\mathrm{d}p}\cdot\frac{\mathrm{d}G}{\mathrm{d}p}\Big) \\ & + G\Big(\frac{\mathrm{d}E}{\mathrm{d}p}\cdot\frac{\mathrm{d}G}{\mathrm{d}p} - 2\frac{\mathrm{d}E}{\mathrm{d}p}\cdot\frac{\mathrm{d}F}{\mathrm{d}q} + \big(\frac{\mathrm{d}E}{\mathrm{d}q}\big)^2\Big) \\ & - 2(EG - F^2)\Big(\frac{\mathrm{dd}E}{\mathrm{d}q^2} - 2\frac{\mathrm{dd}F}{\mathrm{d}p\cdot\mathrm{d}q} + \frac{\mathrm{dd}G}{\mathrm{d}p^2}\Big) \end{aligned}$$

12.

Since we always have

$$\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = E\,\mathrm{d}p^2 + 2F\,\mathrm{d}p\cdot\mathrm{d}q + G\,\mathrm{d}q^2$$

it is clear that $\sqrt{(E\,dp^2 + 2F\,dp \cdot dq + G\,dq^2)}$ is the general expression for the linear element on the curved surface. The analysis developed in the preceding article thus shows us that for finding the measure of curvature there is no need of finite formulæ, which express the coordinates x, y, z as functions of the indeterminants p, q; but that the general expression for the magnitude of any linear element is sufficient. Let us proceed to some applications of this very important theorem.

Suppose that our surface can be developed upon another surface, curved or plane, so that to each point of the former surface, determined by the coordinates x, y, z, will correspond a definite point of the latter surface, whose coordinates are x', y', z'. Evidently x', y', z' can also be regarded as functions of the indeterminants p, q, and therefore for the element $\sqrt{(dx'^2 + dy'^2 + dz'^2)}$ we shall have an expression of the form

$$\sqrt{(E'\,dp^2 + 2F'\,dp \cdot dq + G'\,dq^2)}$$

where E', F', G' also denote functions of p, q. But from the very notion of the *development* of one surface upon another it is clear that the elements corresponding to one another on the two surfaces are necessarily equal. Therefore we shall have identically

$$E = E', \quad F = F', \quad G = G'$$

Thus the formula of the preceding article leads of itself to the remarkable

THEOREM. *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

Also it is evident that *any finite part whatever of the curved surface will retain the same integral curvature after development upon another surface.*

Surfaces developable upon a plane constitute the particular case to which geometers have heretofore restricted their attention. Our theory shows at once that the measure of curvature at every point of such surfaces is equal to zero. Consequently, if the nature of these surfaces is defined according to the third method, we shall have at every point

$$\frac{ddz}{dx^2} \cdot \frac{ddz}{dy^2} - \left(\frac{ddz}{dx \cdot dy}\right)^2 = 0$$

a criterion which, though indeed known a short time ago, has not, at least to our knowledge, commonly been demonstrated with as much rigor as is desirable.

13.

What we have explained in the preceding article is connected with a particular method of studying surfaces, a very worthy method which may be thoroughly

§14. Throughout this section Gauss uses x, y, z to denote $x \circ c$, $y \circ c$, $z \circ c$, for the curve c under consideration. The integral in the second display, involving both d and δ, is what we would write as

$$\left.\frac{dL(\bar{\alpha}(u))}{du}\right|_{u=0} = \left.\frac{d}{du}\right|_{u=0} \int_a^b \sqrt{\left(\frac{\partial \alpha^1(u,t)}{\partial t}\right)^2 + \cdots}\; dt$$

$$= \int_a^b \frac{\dfrac{\partial \alpha^1(0,t)}{\partial u}\,\dfrac{\partial^2 \alpha^1(0,t)}{\partial u \partial t} + \cdots}{\sqrt{\left(\dfrac{\partial \alpha^1(0,t)}{\partial u}\right)^2 + \cdots}}\, dt = \int_a^b \frac{\dfrac{dc^1}{dt}\,\dfrac{\partial^2 \alpha^1(0,t)}{\partial u \partial t} + \cdots}{\sqrt{}}\, dt.$$

Thus, $\dfrac{dc^1}{dt}$ is $\dfrac{dx}{[dt]}$ and $\dfrac{\partial^2 \alpha^1(0,t)}{\partial u \partial t} = \dfrac{\partial^2 \alpha^1(0,t)}{\partial t \partial u}$ is $\dfrac{d\delta x}{[\partial t \partial u]}$. The next two lines show what this becomes after integration by parts. The integral is

$$-\int_a^b \frac{\partial \alpha^1}{\partial u}(0,t)\frac{d}{dt}\left(\frac{dc^1/dt}{\sqrt{}}\right) + \cdots\; dt;$$

here $\dfrac{\partial \alpha^1}{\partial u}(0,t)$ is $\dfrac{\delta x}{[\partial u]}$.

developed by geometers. When a surface is regarded, not as the boundary of a solid, but as a flexible, though not extensible solid, one dimension of which is supposed to vanish, then the properties of the surface depend in part upon the form to which we can suppose it reduced, and in part are absolute and remain invariable, whatever may be the form into which the surface is bent. To these latter properties, the study of which opens to geometry a new and fertile field, belong the measure of curvature and the integral curvature, in the sense which we have given to these expressions. To these belong also the theory of shortest lines, and a great part of what we reserve to be treated later. From this point of view, a plane surface and a surface developable on a plane, e.g., cylindrical surfaces, conical surfaces, etc., are to be regarded as essentially identical; and the generic method of defining in a general manner the nature of the surfaces thus considered is always based upon the formula $\sqrt{(E\,\mathrm{d}p^2 + 2F\,\mathrm{d}p \cdot \mathrm{d}q + G\,\mathrm{d}q^2)}$, which connects the linear element with the two indeterminants p, q. But before following this study further, we must introduce the principles of the theory of shortest lines on a given curved surface.

14.

The nature of a curved line in space is generally given in such a way that the coordinates x, y, z corresponding to the different points of it are given in the form of functions of a single variable, which we shall call w. The length of such a line from an arbitrary initial point to the point whose coordinates are x, y, z, is expressed by the integral

$$\int \mathrm{d}w \cdot \sqrt{\left(\left(\frac{\mathrm{d}x}{\mathrm{d}w}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}w}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^2\right)}$$

If we suppose that the position of the line undergoes an infinitely small variation, so that the coordinates of the different points receive the variations δx, δy, δz, the variation of the whole length becomes

$$= \int \frac{\mathrm{d}x \cdot d\,\delta x + \mathrm{d}y \cdot d\,\delta y + \mathrm{d}z \cdot d\,\delta z}{\sqrt{(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}}$$

which expression we can change into the form

$$\frac{\mathrm{d}x \cdot \delta x + \mathrm{d}y \cdot \delta y + \mathrm{d}z \cdot \delta z}{\sqrt{(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}}$$
$$-\int \left(\delta x \cdot \mathrm{d}\frac{\mathrm{d}x}{\sqrt{(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}} + \delta y \cdot \mathrm{d}\frac{\mathrm{d}y}{\sqrt{(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}} + \delta z \cdot \mathrm{d}\frac{\mathrm{d}z}{\sqrt{(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)}}\right)$$

We know that, in case the line is to be the shortest between its end points, all that stands under the integral sign must vanish. Since the line must lie on the given

Notice that Gauss has given $dL(\bar{\alpha}(u))/du\big|_{u=0}$ for an arbitrary variation in 3-space, not just a variation through curves in the surface. His $x = x \circ c$ is a coordinate function of c in 3-space, *not* a coordinate function with respect to some coordinate system on the surface. If the surface is $\{p : W(p) = 0\}$ for some $W \colon \mathbb{R}^3 \to \mathbb{R}$, so that on the surface we have

$$0 = dW = P\,dx + Q\,dy + R\,dz \qquad\qquad P = D_1 W,\ \ Q = D_2 W,\ \ R = D_3 W,$$

then for variations α through curves on the surface we will have

$$dW(\delta x, \delta y, \delta z) = dW\left(\frac{\partial \alpha^1}{\partial u}(0,t), \frac{\partial \alpha^2}{\partial u}(0,t), \frac{\partial \alpha^3}{\partial u}(0,t)\right) = 0,$$

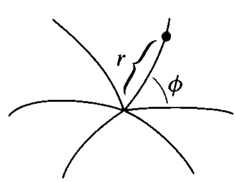
and any set of $\partial\alpha^i/\partial u(0,t)$ with this property comes from some variation on the surface. Using this, Gauss deduces a necessary and sufficient condition for a curve γ, parameterized by arclength, to be a geodesic on the surface. Unlike our equations for geodesics, this condition [the next-to-last displayed formula in this section] is expressed in terms of quantities which make sense only in $\mathbb{R}^3$:

$$\frac{\gamma^{1\prime\prime}(t)}{X(\gamma(t))} = \frac{\gamma^{2\prime\prime}(t)}{Y(\gamma(t))} = \frac{\gamma^{3\prime\prime}(t)}{Z(\gamma(t))},$$

i.e., $\gamma''(t)$ is a multiple of the normal vector at $\gamma(t)$. It takes a little detective work to see that Gauss is really considering a curved parameterized by arclength. Try to prove Gauss' result by modifying our proof of Euler's equations.

§15. The proof in this section is essentially our (first) proof of Gauss' Lemma (I.9-12). There are two main differences. First, Gauss uses the condition of section 14 rather than our equations. Second, for a surface it is unnecessary to choose a curve $v \colon \mathbb{R} \to M_q$ and manufacture the variation α that occurs in the proof of Lemma I.9-12. Instead, we just use

$\alpha(r,\phi) =$ point with "polar coordinates" (r,ϕ).



surface, whose nature is defined by the equation $P\,dx + Q\,dy + R\,dz = 0$, the variations $\delta x, \delta y, \delta z$ also must satisfy the equation $P\,\delta x + Q\,\delta y + R\,\delta z = 0$, and from this it follows at once, according to well-known rules, that the differentials

$$d\frac{dx}{\sqrt{(dx^2+dy^2+dz^2)}},\quad d\frac{dy}{\sqrt{(dx^2+dy^2+dz^2)}},\quad d\frac{dz}{\sqrt{(dx^2+dy^2+dz^2)}}$$

must be proportional to the quantities P, Q, R respectively. Let dr be the element of the curved line; λ the point on the sphere representing the direction of this element; L the point on the sphere representing the direction of the normal to the curved surface; finally, let ξ, η, ζ be the coordinates of the point λ, and X, Y, Z be those of the point L with reference to the center of the sphere. We shall then have

$$dx = \xi\,dr,\quad dy = \eta\,dr,\quad dz = \zeta\,dr$$

from which we see that the above differentials become $d\xi$, $d\eta$, $d\zeta$. And since the quantities P, Q, R are proportional to X, Y, Z, the character of shortest lines is expressed by the equations

$$\frac{d\xi}{X} = \frac{d\eta}{Y} = \frac{d\zeta}{Z}$$

Moreover, it is easily seen that $\sqrt{(d\xi^2 + d\eta^2 + d\zeta^2)}$ is equal to the small arc on the sphere which measures the angle between the directions of the tangents at the beginning and at the end of the element dr, and is thus $= \frac{dr}{\rho}$, if ρ denotes the radius of curvature of the shortest line at this point; thus we shall have

$$\rho\,d\xi = X\,dr,\quad \rho\,d\eta = Y\,dr,\quad \rho\,d\zeta = Z\,dr$$

15.

Suppose that an infinite number of shortest lines go out from a given point A on the curved surface, and suppose that we distinguish these lines from one another by the angle that the first element of each of them makes with the first element of one of them which we take for the first. Let φ be that angle, or, more generally, a function of that angle, and r the length of such a shortest line from the point A to the point whose coordinates are x, y, z. Since to definite values of the variables r, φ there correspond definite points of the surface, the coordinates x, y, z can be regarded as function of r, φ. We shall retain for the notation λ, L, ξ, η, ζ, X, Y, Z the same meaning as in the preceding article, this notation referring to any point whatever on any one of the shortest lines.

All the shortest lines that are of the same length r will end on another line whose length, measured from an arbitrary initial point, we shall denote by v. Thus v can be regarded as a function of the indeterminants r, φ, and if λ' denotes the point on the sphere corresponding to the direction of the element dv, and also ξ', η', ζ' denote the coordinates of this point with reference to the center of the sphere, we shall have

$$\frac{\mathrm{d}x}{\mathrm{d}\varphi} = \xi' \cdot \frac{\mathrm{d}v}{\mathrm{d}\varphi}, \qquad \frac{\mathrm{d}y}{\mathrm{d}\varphi} = \eta' \cdot \frac{\mathrm{d}v}{\mathrm{d}\varphi}, \qquad \frac{\mathrm{d}z}{\mathrm{d}\varphi} = \zeta' \cdot \frac{\mathrm{d}v}{\mathrm{d}\varphi}$$

From these equations and from the equations

$$\frac{\mathrm{d}x}{\mathrm{d}r} = \xi, \qquad \frac{\mathrm{d}y}{\mathrm{d}r} = \eta, \qquad \frac{\mathrm{d}z}{\mathrm{d}r} = \zeta$$

we have

$$\frac{\mathrm{d}x}{\mathrm{d}r} \cdot \frac{\mathrm{d}x}{\mathrm{d}\varphi} + \frac{\mathrm{d}y}{\mathrm{d}r} \cdot \frac{\mathrm{d}y}{\mathrm{d}\varphi} + \frac{\mathrm{d}z}{\mathrm{d}r} \cdot \frac{\mathrm{d}z}{\mathrm{d}\varphi} = (\xi\xi' + \eta\eta' + \zeta\zeta') \cdot \frac{\mathrm{d}v}{\mathrm{d}\varphi} = \cos\lambda\lambda' \cdot \frac{\mathrm{d}v}{\mathrm{d}\varphi}$$

Let S denote the first member of this equation, which will also be a function of r, φ. Differentiation of S with respect to r gives

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}r} &= \frac{\mathrm{dd}x}{\mathrm{d}r^2} \cdot \frac{\mathrm{d}x}{\mathrm{d}\varphi} + \frac{\mathrm{dd}y}{\mathrm{d}r^2} \cdot \frac{\mathrm{d}y}{\mathrm{d}\varphi} + \frac{\mathrm{dd}z}{\mathrm{d}r^2} \cdot \frac{\mathrm{d}z}{\mathrm{d}\varphi} + \frac{1}{2} \cdot \frac{\mathrm{d}\left(\left(\frac{\mathrm{d}x}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}r}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2\right)}{\mathrm{d}\varphi} \\ &= \frac{\mathrm{d}\xi}{\mathrm{d}r} \cdot \frac{\mathrm{d}x}{\mathrm{d}\varphi} + \frac{\mathrm{d}\eta}{\mathrm{d}r} \cdot \frac{\mathrm{d}y}{\mathrm{d}\varphi} + \frac{\mathrm{d}\zeta}{\mathrm{d}r} \cdot \frac{\mathrm{d}z}{\mathrm{d}\varphi} + \frac{1}{2} \cdot \frac{\mathrm{d}(\xi^2 + \eta^2 + \zeta^2)}{\mathrm{d}\varphi}\end{aligned}$$

But $\xi^2 + \eta^2 + \zeta^2 = 1$, and therefore its differential $= 0$; and by the preceding article we have, if ρ denotes the radius of curvature of the line r,

$$\frac{\mathrm{d}\xi}{\mathrm{d}r} = \frac{X}{\rho}, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}r} = \frac{Y}{\rho}, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}r} = \frac{Z}{\rho}$$

Thus we have

$$\frac{\mathrm{d}S}{\mathrm{d}r} = \frac{1}{\rho} \cdot (X\xi' + Y\eta' + Z\zeta') \cdot \frac{\mathrm{d}v}{\mathrm{d}\varphi} = \frac{1}{\rho} \cdot \cos L\lambda' \cdot \frac{\mathrm{d}v}{\mathrm{d}\varphi} = 0$$

since λ' evidently lies on the great circle whose pole is L. From this we see that S is independent of r, and is, therefore, a function of φ alone. But for $r = 0$ we evidently have $v = 0$, consequently $\frac{\mathrm{d}v}{\mathrm{d}\varphi} = 0$, and $S = 0$ independently of φ. Thus, in general, we have necessarily $S = 0$, and so $\cos\lambda\lambda' = 0$, i.e., $\lambda\lambda' = 90^0$. From this follows the

Gauss also gives a "geometric" proof of the lemma, using infinitesimal triangles. Perhaps the easiest way to make this rigorous would be to use our second proof of Gauss' Lemma.

§16. This section states a generalization of Gauss' Lemma, which has also been given in Problem I.9-28.

§17. In terms of a coordinate system (p, q) on a surface, the Riemannian metric that it acquires as a subset of $\mathbb{R}^3$ has the expression

$$\langle\ ,\ \rangle = E\,dp \otimes dp + F\,dp \otimes dq + F\,dq \otimes dp + G\,dq \otimes dq,$$

so that

$$\|\ \| = \sqrt{E\,dp \cdot dp + 2F\,dp \cdot dq + G\,dq \cdot dq}.$$

THEOREM. *If on a curved surface an infinite number of shortest lines of equal length be drawn from the same initial point, the lines joining their extremities will be normal to each of the lines.*

We have thought it worth while to deduce this theorem from the fundamental property of shortest lines; but the truth of the theorem can be made apparent without any calculation by means of the following reasoning. Let AB, AB' be two shortest lines of the same length including at A an infinitely small angle, and let us suppose that one of the angles made by the element BB' with the lines BA, $B'A$ differs from a right angle by a finite quantity. Then, by the law of continuity, one will be greater and the other less than a right angle. Suppose the angle at B is equal to $90^0 - \omega$, and take on the line AB a point C, such that $BC = BB' \cdot \operatorname{cosec} \omega$. Then, since the infinitely small triangle $BB'C$ may be regarded as plane, we shall have $CB' = BC \cdot \cos \omega$, and consequently

$$AC + CB' = AC + BC \cdot \cos \omega = AB - BC \cdot (1 - \cos \omega) = AB' - BC \cdot (1 - \cos \omega),$$

i.e., the path from A to B' through the point C is shorter than the shortest line, Q.E.D.

16.

With the theorem of the preceding article we associate another, which we state as follows: *If on a curved surface we imagine any line whatever, from the different points of which are drawn at right angles and toward the same side an infinite number of shortest lines of the same length, the curve which joins their other extremities will cut each of the lines at right angles.* For the demonstration of this theorem no change need be made in the preceding analysis, except that φ must denote the length of the *given* curve measured from an arbitrary point; or rather, a function of this length. Thus all of the reasoning will hold here also, with this modification, that $S = 0$ for $r = 0$ is now implied in the hypothesis itself. Moreover, this theorem is more general than the preceding one, for we can regard it as including the first one if we take for the given line the infinitely small circle described about the center A. Finally, we may say that here also geometric considerations may take the place of the analysis, which, however, we shall not take the time to consider here, since they are sufficiently obvious.

17.

We return to the formula $\sqrt{(E\,\mathrm{d}p^2 + 2F\,\mathrm{d}p \cdot \mathrm{d}q + G\,\mathrm{d}q^2)}$, which expresses generally the magnitude of a linear element on the curved surface, and investigate, first of all, the geometric meaning of the coefficients E, F, G. We have

Gauss uses ω to denote the angle between $\partial/\partial p$ and $\partial/\partial q$ (thus, ω is a function on the surface). Gauss' formula for $\cos\omega$ should be clear. Gauss also mentions that

$$dV = \sqrt{EG - F^2}\, dp \wedge dq,$$

a special case of the formula on pg. I.311.

To interpret the last two formulas in this section, we must divide ds, dp, and dq by dt in all places; it is to be understood that $dp/dt = (p \circ c)'(t)$, etc., where c is the curve we are considering. It is simplest to assume that c is parameterized by arclength, so that the terms ds/dt are 1. If

$$\theta(s) = \text{angle between } c'(s) \text{ and } \left.\frac{\partial}{\partial p}\right|_{c(s)},$$

then

$$\cos\theta = \frac{\left\langle c', \dfrac{\partial}{\partial p}\right\rangle}{\left\|\dfrac{\partial}{\partial p}\right\|} = \frac{E\dfrac{dp(c(s))}{ds} + F\dfrac{dq(c(s))}{ds}}{\sqrt{E}},$$

since

$$c' = \frac{dp(c(s))}{ds}\frac{\partial}{\partial p} + \frac{dq(c(s))}{ds}\frac{\partial}{\partial q}.$$

Moreover, the area of the parallelogram spanned by c' and $\partial/\partial p$ is

$$\sin\theta \cdot \left\|\frac{\partial}{\partial p}\right\|, \qquad \text{and also} \qquad dV\left(\frac{\partial}{\partial p}, c'\right),$$

from which we obtain

$$\sin\theta = \frac{\sqrt{EG - F^2}\,\dfrac{dq(c(s))}{ds}}{\sqrt{E}}.$$

§18. In this section Gauss deduces the conditions for a curve γ (having the component functions $\gamma^1 = p \circ \gamma$, $\gamma^2 = q \circ \gamma$) to be a critical point for the length function.

Unlike the condition in section 14, the result is expressed totally in terms of the Riemannian metric $\langle\ ,\ \rangle$ on the surface, and is essentially the condition for a geodesic that we obtained in Chapter I.9. However, the derivation is different, because the geodesic is assumed to satisfy $q(\gamma(t)) = \gamma^2(t) = t$ ["we regard p as a function of q"].

It is not necessary to actually follow the derivation. The really important point is simply the equation that constitutes the first line that appears in the already said in Art. 5 that two systems of lines may be supposed to lie on the curved surface, p being variable, q constant along each of the lines of the one system; and q variable, p constant along each of the lines of the other system. Any point whatever on the surface can be regarded as the intersection of a line of the first system with a line of the second; and then the element of the first line adjacent to this point and corresponding to a variation $\mathrm{d}p$ will be $= \sqrt{E} \cdot \mathrm{d}p$, and the element of the second line corresponding to the variation $\mathrm{d}q$ will be $= \sqrt{G} \cdot \mathrm{d}q$. Finally, denoting by ω the angle between these elements, it is easily seen that we shall have $\cos\omega = \frac{F}{\sqrt{EG}}$. Furthermore, the area of the surface element in the form of a parallelogram between the two lines of the first system, to which correspond $q, q + \mathrm{d}q$, and the two lines of the second system, to which correspond $p, p + \mathrm{d}p$, will be $\sqrt{(EG - F^2)}\, \mathrm{d}p \cdot \mathrm{d}q$.

Any line whatever on the curved surface belonging to neither of the two systems is determined when p and q are supposed to be functions of a new variable, or one of them is supposed to be a function of the other. Let s be the length of such a curve, measured from an arbitrary initial point, and in either direction chosen as positive. Let θ denote the angle which the element $\mathrm{d}s = \sqrt{(E\,\mathrm{d}p^2 + 2F\,\mathrm{d}p \cdot \mathrm{d}q + G\,\mathrm{d}q^2)}$ makes with the line of the first system drawn through the initial point of the element, and, in order that no ambiguity may arise, let us suppose that this angle is measured from that branch of the first line on which the values of p increase, and is taken as positive toward that side toward which the values of q increase. These conventions being made, it is easily seen that

$$\cos\theta \cdot \mathrm{d}s = \sqrt{E} \cdot \mathrm{d}p + \sqrt{G} \cdot \cos\omega \cdot \mathrm{d}q = \frac{E\,\mathrm{d}p + F\,\mathrm{d}q}{\sqrt{E}}$$

$$\sin\theta \cdot \mathrm{d}s = \sqrt{G} \cdot \sin\omega \cdot \mathrm{d}q = \frac{\sqrt{(EG - F^2)} \cdot \mathrm{d}q}{\sqrt{E}}$$

18.

We shall now investigate the condition that this line be a shortest line. Since its length s is expressed by the integral

$$s = \int \sqrt{(E\,\mathrm{d}p^2 + 2F\,\mathrm{d}p \cdot \mathrm{d}q + G\,\mathrm{d}q^2)}$$

the condition for a minimum requires that the variation of this integral arising from an infinitely small change in the position become $= 0$. The calculation, for our purpose, is more simply made in this case, if we regard p as a function of q.

second large display on page 105 (after the words "Thus we have"). For a curve parameterized by arclength, this equation says that

$$\frac{\partial E}{\partial p}(c(s))\left(\frac{dc^1}{ds}\right)^2 + 2\frac{\partial F}{\partial p}(c(s))\frac{dc^1}{ds}\frac{dc^2}{ds} + \frac{\partial G}{\partial p}(c(s))\left(\frac{dc^2}{ds}\right)^2$$
$$= 2\frac{d}{ds}\left[E(c(s))\frac{dc^1}{ds} + F(c(s))\frac{dc^2}{ds}\right].$$

It is a very useful exercise to write out the equations on pg. I.329 for the case of a 2-dimensional manifold, with $g_{11} = E$, $g_{12} = F$, $g_{22} = G$, and show that the first of these equations (the equation for $k = 1$) yields the above equation (it will be necessary to perform the differentiation on the right side).

Although Gauss performs various further manipulations, it is only necessary to follow the next step,

$$2\frac{d}{ds}\left[E\frac{dc^1}{ds} + F\frac{dc^2}{ds}\right] = 2\frac{d}{ds}\sqrt{E}\cos\theta,$$

where θ is defined in the previous section.

§19. In this section Gauss rewrites formulas from preceding sections for the case of a coordinate system (p, q) which is "orthogonal" ($\langle \partial/\partial p, \partial/\partial q\rangle = F = 0$). The important case for us is the last he considers, in which the coordinates are

When this is done, if the variation is denoted by the characteristic δ, we have

$$\delta s = \int \frac{(\frac{dE}{dp}\cdot dp^2 + \frac{2dF}{dp}\cdot dp\cdot dq + \frac{dG}{dp}\cdot dq^2)\,\delta p + (2E\,dP + 2F\,dq)\,d\,\delta p}{2\,ds}$$

$$= \frac{E\,dp + F\,dq}{ds}\cdot \delta p + \int \delta p \cdot \{\frac{\frac{dE}{dp}\cdot dp^2 + \frac{2dF}{dp}\cdot dp\cdot dq + \frac{dG}{dp}\cdot dq^2}{2\,ds} - d \cdot \frac{E\,dp + F\,dq}{ds}\}$$

and we know that what is included under the integral sign must vanish independently of δp. Thus we have

$$\frac{dE}{dp}\cdot dp^2 + \frac{2dF}{dp}\cdot dp\cdot dq + \frac{dG}{dp}\cdot dq^2 = 2\,ds \cdot d \cdot \frac{E\,dp + F\,dq}{ds}$$

$$= 2\,ds \cdot d \cdot \surd E \cdot \cos\theta = \frac{ds\cdot dE\cdot\cos\theta}{\surd E} - 2ds\cdot d\theta \cdot \surd E \cdot \sin\theta$$

$$= \frac{(E\,dp + F\,dq)\,dE}{E} - 2\surd(EG - F^2)\cdot dq \cdot d\theta$$

$$= (\frac{E\,dp + F\,dq}{E})\cdot(\frac{dE}{dp}\cdot dp + \frac{dE}{dq}\cdot dq) - 2\surd(EG - F^2)\cdot dq\cdot d\theta$$

This gives the following conditional equation for a shortest line:

$$\surd(EG - F^2)\cdot d\theta = \tfrac{1}{2}\cdot\frac{F}{E}\cdot\frac{dE}{dp}\cdot dp + \tfrac{1}{2}\cdot\frac{F}{E}\cdot\frac{dE}{dq}\cdot dq + \tfrac{1}{2}\cdot\frac{dE}{dq}\cdot dp - \frac{dF}{dp}\cdot dp - \tfrac{1}{2}\cdot\frac{dG}{dp}\cdot dq$$

which can also be written

$$\surd(EG - F^2)\cdot d\theta = \tfrac{1}{2}\cdot\frac{F}{E}\cdot dE + \tfrac{1}{2}\cdot\frac{dE}{dq}\cdot dp - \frac{dF}{dp}\cdot dp - \tfrac{1}{2}\cdot\frac{dG}{dp}\cdot dq$$

From this equation, by means of the equation

$$\cot\theta = \frac{E}{\surd(EG - F^2)}\cdot\frac{dp}{dq} + \frac{F}{\surd(EG - F^2)}$$

it is also possible to eliminate the angle θ, and to derive a differential equation of the second order between p and q, which, however, would become more complicated and less useful for applications than the preceding.

19.

The general formulæ, which we have derived in Arts. 11, 18 for the measure of curvature and the variation in the direction of a shortest line, become much simpler if the quantities p, q are so chosen that the lines of the first system cut the "polar coordinates" (r, ϕ) defined in terms of the geodesics emanating from a point A of the surface. Here Gauss obtains the formula

$$k = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial r^2},$$

and

$$\frac{d\theta}{ds} = -\frac{\partial \sqrt{G}}{\partial r} \frac{d\phi(c(s))}{ds},$$

where θ is the angle the geodesic c makes with the lines $\phi =$ constant. Notice that (r, ϕ) is not a coordinate system on a whole neighborhood of A; we must delete one geodesic ray, including the point A itself. Consequently, $\sqrt{G}$ and $\partial\sqrt{G}/\partial r$ are not even defined at A. Gauss' final assertions in this section should be interpreted as saying that

$$\lim_{B \to A} \sqrt{G}(B) = 0$$

$$\lim_{B \to A} \frac{\partial \sqrt{G}}{\partial r}(B) = 1.$$

everywhere orthogonally the lines of the second system; i.e., in such a way that we have generally $\omega = 90^0$, or $F = 0$. Then the formula for the measure of curvature becomes

$$4E^2G^2k = E \cdot \frac{\mathrm{d}E}{\mathrm{d}q} \cdot \frac{\mathrm{d}G}{\mathrm{d}q} + E(\frac{\mathrm{d}G}{\mathrm{d}p})^2 + G \cdot \frac{\mathrm{d}E}{\mathrm{d}p} \cdot \frac{\mathrm{d}G}{\mathrm{d}p} + G(\frac{\mathrm{d}E}{\mathrm{d}q})^2 - 2EG(\frac{\mathrm{dd}E}{\mathrm{d}q^2} + \frac{\mathrm{dd}G}{\mathrm{d}p^2})$$

and for the variation of the angle θ

$$\sqrt{EG} \cdot \mathrm{d}\theta = \tfrac{1}{2} \cdot \frac{\mathrm{d}E}{\mathrm{d}q} \cdot \mathrm{d}p - \tfrac{1}{2} \cdot \frac{\mathrm{d}G}{\mathrm{d}p} \cdot \mathrm{d}q$$

Among the various cases in which we have this condition of orthogonality, the most important is that in which all the lines of one of the two systems, e.g., the first, are shortest lines. Here for a constant value of q the angle θ becomes $= 0$, and therefore the equation for the variation of θ just given shows that we must have $\frac{\mathrm{d}E}{\mathrm{d}q} = 0$, or that the coefficient E must be independent of q; i.e., E must be either a constant or a function of p alone. It will be simplest to take for p the length of each line of the first system, which length, when all the lines of the first system meet in a point, is to be measured from this point, or, if there is no common intersection, from any line whatever of the second system. Having made these conventions, it is evident that p and q denote now the same quantities that were expressed in Arts. 15, 16 by r and φ, and that $E = 1$. Thus the two preceding formulæ become:

$$4G^2k = (\frac{\mathrm{d}G}{\mathrm{d}p})^2 - 2G\frac{\mathrm{dd}G}{\mathrm{d}p^2}$$

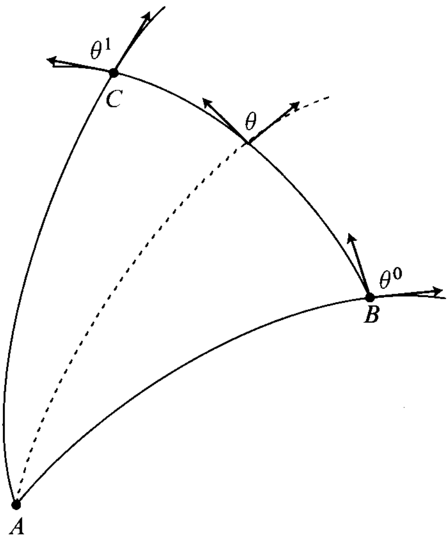
$$\sqrt{G} \cdot \mathrm{d}\theta = -\tfrac{1}{2} \cdot \frac{\mathrm{d}G}{\mathrm{d}p} \cdot \mathrm{d}q$$

or, setting $\sqrt{G} = m$,

$$k = -\frac{1}{m} \cdot \frac{\mathrm{dd}m}{\mathrm{d}p^2}, \qquad \mathrm{d}\theta = -\frac{\mathrm{d}m}{\mathrm{d}p} \cdot \mathrm{d}q$$

Generally speaking, m will be a function of p, q, and $m\,\mathrm{d}q$ the expression for the element of any line whatever of the second system. But in the particular case where all the lines p go out from the same point, evidently we must have $m = 0$ for $p = 0$. Furthermore, in the case under discussion we will take for q the angle itself which the first element of any line whatever of the first system makes with the element of any one of the lines chosen arbitrarily. Then, since for an infinitely small value of p the element of a line of the second system (which can be regarded as a circle described with radius p) is $= p\,\mathrm{d}q$, we shall have for

§20. If you have come this far, there should be no problem with this final section. Here is the picture.



an infinitely small value of p, $m = p$, and consequently, for $p = 0$, $m = 0$ at the same time, and $\frac{\mathrm{d}m}{\mathrm{d}p} = 1$.

20.

We pause to investigate the case in which we suppose that p denotes in a general manner the length of the shortest line drawn from a fixed point A to any other point whatever of the surface, and q the angle that the first element of this line makes with the first element of another given shortest line going out from A. Let B be a definite point in the latter line, for which $q = 0$, and C another definite point of the surface, at which we denote the value of q simply by A. Let us suppose the points B, C joined by a shortest line, the parts of which, measured from B, we denote in a general way, as in Art. 18, by s; and, as in the same article, let us denote by θ the angle which any element $\mathrm{d}s$ makes with the element $\mathrm{d}p$; finally, let us denote by θ^0, θ' the values of the angle θ at the points B, C. We have thus on the curved surface a triangle formed by shortest lines. The angles of this triangle at B and C we shall denote simply by the same letters, and B will be equal to $180^0 - \theta$, C to θ' itself. But, since it is easily seen from our analysis that all the angles are supposed to be expressed, not in degrees, but by numbers, in such a way that the angle $57^0 17' 45''$, to which corresponds an arc equal to the radius, is taken for the unit, we must set

$$\theta^0 = \pi - B, \quad \theta' = C$$

where 2π denotes the circumference of the sphere. Let us now examine the integral curvature of this triangle, which is $= \int k\,\mathrm{d}\sigma$, $\mathrm{d}\sigma$ denoting a surface element of the triangle. Wherefore, since this element is expressed by $m\,\mathrm{d}p \cdot \mathrm{d}q$, we must extend the integral $\iint m\,\mathrm{d}p \cdot \mathrm{d}q$ over the whole surface of the triangle. Let us begin by integration with respect to p, which, because $k = -\frac{1}{m} \cdot \frac{\mathrm{dd}m}{\mathrm{d}p^2}$, gives $\mathrm{d}q \cdot (\text{Const.} - \frac{\mathrm{d}m}{\mathrm{d}p})$, for the integral curvature of the area lying between the lines of the first system, to which correspond the values q, $q + \mathrm{d}q$ of the second indeterminate. Since this integral curvature must vanish for $p = 0$, the constant introduced by integration must be equal to the value of $\frac{\mathrm{d}m}{\mathrm{d}q}$ for $p = 0$, i.e., equal to unity. Thus we have $\mathrm{d}q(1 - \frac{\mathrm{d}m}{\mathrm{d}p})$, where for $\frac{\mathrm{d}m}{\mathrm{d}p}$ must be taken the value corresponding to the end of this area on the line CB. But on this line we have, by the preceding article, $\frac{\mathrm{d}m}{\mathrm{d}q} \cdot \mathrm{d}q = -\mathrm{d}\theta$, whence our expression is changed into $\mathrm{d}q + \mathrm{d}\theta$. Now by a second integration, taken from $q = 0$ to $q = A$, we find that the integral curvature $= A + \theta' - \theta^0 = A + B + C - \pi$.

The integral curvature is equal to the area of that part of the sphere which corresponds to the triangle, taken with the positive or negative sign according as the curved surface on which the triangle lies is concavo-concave or concavo-convex. For unit area will be taken the square whose side is equal to unity (the radius of the sphere), and then the whole surface of the sphere becomes $= 4\pi$. Thus the part of the surface of the sphere corresponding to the triangle is to the whole surface of the sphere as $\pm(A + B + C - \pi)$ is to 4π. This theorem, which, if we mistake not, ought to be counted among the most elegant in the theory of curved surfaces, may also be stated as follows:

The excess over 180^0 *of the sum of the angles of a triangle formed by shortest lines on a concavo-concave curved surface, or the deficit from* 180^0 *of the sum of the angles of a triangle formed by shortest lines on a concavo-convex curved surface, is measured by the area of the part of the sphere which corresponds, through the directions of the normals, to that triangle, if the whole surface of the sphere is set equal to* 720 *degrees.*

More generally, in any polygon whatever of n sides, each formed by a shortest line, the excess of the sum of the angles over $(2n - 4)$ right angles, or the deficit from $(2n - 4)$ right angles (according to the nature of the curved surface), is equal to the area of the corresponding polygon on the sphere, if the whole surface of the sphere is set equal to 720 degrees. This follows at once from the preceding theorem by dividing the polygon into triangles.