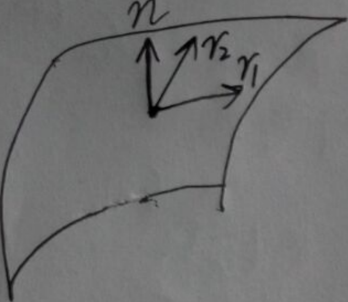
设 $r=r(u^1,u^2)$ 为曲面Σ的参数表示，记 $r_1=\frac{\partial r}{\partial u^1}$，$r_2=\frac{\partial r}{\partial u^2}$，

$n=\frac{r_1\wedge r_2}{|r_1\wedge r_2|}$ 为曲面Σ的单位法向量，则曲面上任意一点，我们有自然标架 $\{r;r_1,r_2,n\}$，$\{r_1,r_2,n\}$ 构成 $\mathbb{R}^3$ 的一组基.



下面我们导出 r_1,r_2,n 的一阶导数所满足的方程. 首先 $\frac{\partial r_1}{\partial u^\beta},\frac{\partial r_2}{\partial u^\beta},\frac{\partial n}{\partial u^\beta}$ 都是 $\mathbb{R}^3$ 中的向量，故它在基 $\{r_1,r_2,n\}$ 下有一个坐标表示

记
$$\begin{cases}\dfrac{\partial r_\alpha}{\partial u^\beta}=C_{\alpha\beta}^{\gamma}\dfrac{\partial r}{\partial u^\gamma}+B_{\alpha\beta}n\\[2ex]\dfrac{\partial n}{\partial u^\alpha}=D_\alpha^{\beta}\dfrac{\partial r}{\partial u^\beta}+E_\alpha n\end{cases}$$

我们计算出各系数 $C_{\alpha\beta}^{\gamma}$，$B_{\alpha\beta}$，D_α^{γ}，E_α.

首先，$\langle n,n\rangle=1$，故 $\langle n,\frac{\partial n}{\partial u^\alpha}\rangle=0$，$E_\alpha=0$.

$B_{\alpha\beta}=\langle r_{\alpha\beta},n\rangle$ 为第二基本形式矩阵，记为 $b_{\alpha\beta}$

$$\begin{pmatrix}n_1\\n_2\end{pmatrix}=-W\begin{pmatrix}r_1\\r_2\end{pmatrix}=-\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix}\begin{pmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{pmatrix}^{-1}$$

故 $D_\alpha^{\beta}=-b_{\alpha\gamma}g^{\gamma\beta}$. 我们记 $b_\alpha^{\beta}=b_{\alpha\gamma}g^{\gamma\beta}$，记 $\Gamma_{\alpha\beta}^{\gamma}=$

下面我们计算 $\Gamma_{\alpha\beta}^{\gamma}$，因为 $\frac{\partial r_\alpha}{\partial u^\beta}=\frac{\partial r_\beta}{\partial u^\alpha}$，故 $C_{\alpha\beta}^{\gamma}=C_{\beta\gamma}^{\alpha}$

$g_{\beta\gamma}=\langle r_\beta,r_\gamma\rangle$

$$\frac{\partial g_{\beta\gamma}}{\partial u^\alpha}=\langle\frac{\partial r_\beta}{\partial u^\alpha},r_\gamma\rangle+\langle r_\beta,\frac{\partial r_\gamma}{\partial u^\alpha}\rangle$$

$$=\langle\Gamma_{\beta\alpha}^{\theta}r_\theta,r_\gamma\rangle+\langle r_\beta,\Gamma_{\gamma\alpha}^{\theta}r_\theta\rangle$$

$$=\Gamma_{\beta\alpha}^{\theta}g_{\theta\gamma}+\Gamma_{\gamma\alpha}^{\theta}g_{\theta\beta}$$

i.e. $$\frac{\partial g_{\beta\gamma}}{\partial u^\alpha}=\Gamma_{\beta\alpha}^{\theta}g_{\theta\gamma}+\Gamma_{\gamma\alpha}^{\theta}g_{\theta\beta}\qquad ①$$

把指标α,β,γ进行轮换，我们得.

$$\frac{\partial g_{\gamma\alpha}}{\partial u^{\beta}} = \Gamma_{\gamma\beta}^{\theta} g_{\theta\alpha} + \Gamma_{\alpha\beta}^{\theta} g_{\theta\gamma} \quad ②$$

$$\frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}} = \Gamma_{\alpha\gamma}^{\theta} g_{\theta\beta} + \Gamma_{\beta\gamma}^{\theta} g_{\theta\alpha} \quad ③$$

① − ② + ③ 得 $\frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} - \frac{\partial g_{\gamma\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}} = 2 g_{\theta\beta} \Gamma_{\alpha\gamma}^{\theta}$

$$g^{\beta\xi}\left(\frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} - \frac{\partial g_{\gamma\alpha}}{\partial u^{\beta}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}}\right) = 2 g_{\theta\beta} g^{\beta\xi} \Gamma_{\alpha\gamma}^{\theta}$$

$$= 2\delta_{\theta}^{\xi} \Gamma_{\alpha\gamma}^{\theta} = 2\Gamma_{\alpha\xi}^{\theta}$$

i.e. $\Gamma_{\alpha\xi}^{\theta} = \frac{1}{2} g^{\beta\xi}\left(\frac{\partial g_{\beta\gamma}}{\partial u^{\alpha}} + \frac{\partial g_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial g_{\gamma\alpha}}{\partial u^{\beta}}\right)$.

换指标得 $\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2} g^{\gamma\delta}\left(\frac{\partial g_{\alpha\delta}}{\partial u^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\delta}}\right)$

$C_{\alpha\beta}^{\gamma}$.

习题 III

1. 证明：

(1) $g^{\alpha\beta}g_{\alpha\beta}=2$

(2) $\dfrac{\partial \ln\sqrt{g}}{\partial u^\alpha}=\Gamma^1_{1\alpha}+\Gamma^2_{2\alpha}$

证明：

(1) 注意到 $tr(AB)=a^{ij}b_{ij}$，

故 $g^{\alpha\beta}g_{\alpha\beta}=tr\begin{pmatrix}1&0\\0&1\end{pmatrix}=2$.

(2)

$$\frac{\partial}{\partial u^\alpha}g_{\beta\gamma}=\frac{\partial}{\partial u^\alpha}\langle r_\beta,r_\gamma\rangle=\langle\frac{\partial r_\beta}{\partial u^\alpha},r_\gamma\rangle+\langle r_\beta,\frac{\partial r_\gamma}{\partial u^\alpha}\rangle$$

$$=\Gamma^\theta_{\beta\alpha}g_{\theta\gamma}+\Gamma^\theta_{\gamma\alpha}g_{\theta\beta}$$

$$\frac{\partial \ln\sqrt{g}}{\partial u^\alpha}=\frac{1}{2g}\frac{\partial g}{\partial u^\alpha}=\frac{1}{2g}\frac{\partial}{\partial u^\alpha}(g_{11}g_{22}-g_{12}^2)$$

$$=\frac{1}{2g}\left(\frac{\partial g_{11}}{\partial u^\alpha}g_{22}+g_{11}\frac{\partial g_{22}}{\partial u^\alpha}-2g_{12}\frac{\partial g_{12}}{\partial u^\alpha}\right)$$

$$=\frac{1}{2g}\Big(2\Gamma^\theta_{1\alpha}g_{\theta1}g_{22}+2\Gamma^\theta_{2\alpha}g_{\theta2}g_{11}$$
$$-2g_{12}\big(\Gamma^\theta_{1\alpha}g_{\theta2}+\Gamma^\theta_{2\alpha}g_{\theta1}\big)\Big)$$

$$=\frac{1}{2g}\Big(2\Gamma^1_{1\alpha}g_{11}g_{22}+\cancel{2\Gamma^2_{1\alpha}g_{21}g_{22}}+\cancel{2\Gamma^1_{2\alpha}g_{12}g_{11}}+2\Gamma^2_{2\alpha}g_{22}g_{11}$$
$$-2\Gamma^1_{1\alpha}g_{12}^2-\cancel{2\Gamma^2_{1\alpha}g_{12}g_{22}}-\cancel{2\Gamma^1_{2\alpha}g_{11}g_{12}}-2\Gamma^2_{2\alpha}g_{12}^2\Big)$$

$$=\frac{1}{g}\Big(\Gamma^1_{1\alpha}(g_{11}g_{22}-g_{12}^2)+\Gamma^2_{2\alpha}(g_{11}g_{22}-g_{12}^2)\Big)$$

$$=\Gamma^1_{1\alpha}+\Gamma^2_{2\alpha}$$

2.

(1)

- $\tilde{g}_{ij} = \langle \frac{\partial r}{\partial \tilde{u}^i}, \frac{\partial r}{\partial \tilde{u}^j} \rangle = \langle \frac{\partial r}{\partial u^\alpha} \cdot \frac{\partial u^\alpha}{\partial \tilde{u}^i}, \frac{\partial r}{\partial u^\beta} \cdot \frac{\partial u^\beta}{\partial \tilde{u}^j} \rangle$

$$= g_{\alpha\beta} a_i^\alpha a_j^\beta$$

- $\tilde{b}_{ij} = \langle \frac{\partial^2 r}{\partial \tilde{u}^i \partial \tilde{u}^j}, n \rangle$

注意到 $\frac{\partial r}{\partial \tilde{u}^i} = \frac{\partial r}{\partial u^\alpha} \cdot \frac{\partial u^\alpha}{\partial \tilde{u}^i} = a_i^\alpha \frac{\partial r}{\partial u^\alpha}$

$$\frac{\partial^2 r}{\partial \tilde{u}^i \partial \tilde{u}^j} = \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \frac{\partial r}{\partial u^\alpha} + a_i^\alpha \frac{\partial}{\partial \tilde{u}^j}\left(\frac{\partial r}{\partial u^\alpha}\right)$$

$$= \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \frac{\partial r}{\partial u^\alpha} + a_i^\alpha a_j^\beta \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta}$$

故 $\tilde{b}_{ij} = \langle a_i^\alpha a_j^\beta \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta}, n \rangle = b_{\alpha\beta} a_i^\alpha a_j^\beta$

- 令 $G = (g_{\alpha\beta})_{\alpha\beta}$, $\tilde{G} = (\tilde{g}_{ij})_{ij}$, $A = (a_i^\alpha)_{i\alpha}$

由第一个等式 $\tilde{G} = AGA^T$，故 $\tilde{G}^{-1} = (A^T)^{-1} G^{-1} A^{-1}$

故 $G^{-1} = A^T \tilde{G}^{-1} A$, i.e. $g^{\alpha\beta} = \tilde{g}^{ij} a_i^\alpha a_j^\beta$

(2)

一方面，$\left(\frac{\partial^2 r}{\partial \tilde{u}^i \partial \tilde{u}^j}\right)^T = \tilde{\Gamma}_{ij}^k \frac{\partial r}{\partial \tilde{u}^k}$ (*) 这里T表示在切平面的投影

另一方面，$\left(\frac{\partial^2 r}{\partial \tilde{u}^i \partial \tilde{u}^j}\right)^T = \left(\frac{\partial}{\partial \tilde{u}^j}\left(\frac{\partial r}{\partial u^\alpha} \frac{\partial u^\alpha}{\partial \tilde{u}^i}\right)\right)^T$

$$= \left(\frac{\partial u^\beta}{\partial \tilde{u}^j} \frac{\partial}{\partial u^\beta}\left(\frac{\partial u^\alpha}{\partial \tilde{u}^i} \cdot \frac{\partial r}{\partial u^\alpha}\right)\right)^T$$

$$= \left(a_j^\beta a_i^\alpha \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} + a_j^\beta \frac{\partial a_i^\alpha}{\partial u^\beta} \frac{\partial r}{\partial u^\alpha}\right)^T$$

$$= a_j^\beta a_i^\alpha \Gamma_{\alpha\beta}^\gamma \frac{\partial r}{\partial u^\gamma} + a_j^\beta \frac{\partial a_i^\alpha}{\partial u^\beta} \frac{\partial r}{\partial u^\alpha}$$

$$= a_j^\beta a_i^\alpha \Gamma_{\alpha\beta}^\gamma \tilde{a}_\gamma^k \frac{\partial r}{\partial \tilde{u}^k} + a_j^\beta \frac{\partial a_i^\alpha}{\partial u^\beta} \tilde{a}_\alpha^k \frac{\partial r}{\partial \tilde{u}^k}$$

$$= \left(\Gamma_{\alpha\beta}^\gamma a_i^\alpha a_j^\beta \tilde{a}_\gamma^k + \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \tilde{a}_\alpha^k\right) \frac{\partial r}{\partial \tilde{u}^k} \qquad (**)$$

比较(*)(**)即得 $\tilde{\Gamma}_{ij}^k = \Gamma_{\alpha\beta}^\gamma a_i^\alpha a_j^\beta \tilde{a}_\gamma^k + \frac{\partial a_i^\alpha}{\partial \tilde{u}^j} \cdot a_\alpha^k$.

4. 令 $u^1 = r, u^2 = s$

$g_{11} = 1,\ g_{12} = 0,\ g_{22} = r^2,\ g^{11} = 1,\ g^{12} = 0,\ g^{22} = \frac{1}{r^2}$

$$\Gamma_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\delta}\left(\frac{\partial g_{\alpha\delta}}{\partial u^\beta} + \frac{\partial g_{\beta\delta}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\delta}\right)$$

$$\Gamma_{11}^1 = \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial u^1}\right) = 0,\ \Gamma_{22}^1 = \frac{1}{2} g^{11}\left(2\frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{22}}{\partial u^1}\right) = -r$$

$$\Gamma_{11}^2 = \frac{1}{2} g^{22}\left(2\frac{\partial g_{12}}{\partial u^2} - \frac{\partial g_{11}}{\partial u^2}\right) = 0$$

$$\Gamma_{12}^1 = \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1}\right) = 0$$

$$\Gamma_{12}^2 = \frac{1}{2} g^{22}\left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^2}\right) = \frac{1}{r}$$

$$\Gamma_{22}^2 = \frac{1}{2} g^{22} \cdot \left(\frac{\partial g_{22}}{\partial u^2}\right) = 0$$

故 $\Gamma_{22}^1 = -r,\ \Gamma_{21}^2 = \Gamma_{12}^2 = \frac{1}{r},\quad \Gamma_{11}^1 = \Gamma_{22}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = \Gamma_{11}^2 = 0$

1.(2) 的另一种解法.

先证一个引理

设 $A(t)=(a_{ij}(t))_{n\times n}$ 为对称方阵, $\det A(t)>0$, 其中 $a_{ij}(t)$ 为关于t的可微函数, 则 $\frac{d}{dt}\log\det A = tr(A^{-1}\frac{dA}{dt})$.

证明: 令 A_{ij} 为 a_{ij} 的代数余子式, 则 $\det A=\sum_{k=1}^{n} a_{ik}A_{ik}$, $\forall i$.

故 $\frac{\partial \det A}{\partial a_{ij}} = A_{ij}$, $\forall i,j$. 设 A^* 为A的伴随方阵, 因为A对称,

故 $A_{ij}=A_{ji}$

$$\frac{d}{dt}\log\det A = \frac{1}{\det A}\frac{d}{dt}\det A = \frac{1}{\det A}\frac{\partial\det A}{\partial a_{ij}}\frac{da_{ij}}{dt} = \frac{1}{\det A}A_{ij}\frac{da_{ij}}{dt}$$

$$= \frac{1}{\det A}A_{ji}\frac{da_{ij}}{dt} = \frac{1}{\det A}(A^*)_{ij}\frac{da_{ij}}{dt} = (A^{-1})_{ij}\frac{da_{ij}}{dt}$$

$$= tr(A^{-1}\frac{dA}{dt})$$

回到原题, $\frac{\partial \ln\sqrt{g}}{\partial u^\alpha} = \frac{1}{2}\frac{\partial \ln g}{\partial u^\alpha} = \frac{1}{2}tr(G^{-1}\frac{\partial G}{\partial u^\alpha}) = \frac{1}{2}g^{ij}\frac{\partial g_{ij}}{\partial u^\alpha}$

$$= \frac{1}{2}g^{ij}(\Gamma^\theta_{i\alpha}g_{\theta j}+\Gamma^\theta_{j\alpha}g_{\theta i})$$

$$= \frac{1}{2}(\Gamma^\theta_{i\alpha}\delta^i_\theta + \Gamma^\theta_{j\alpha}\delta^j_\theta)$$

$$= \frac{1}{2}(\Gamma^i_{i\alpha}+\Gamma^j_{j\alpha})$$

$$= \Gamma^i_{i\alpha} = \Gamma^1_{1\alpha}+\Gamma^2_{2\alpha}$$