RIEMANNIAN GEOMETRY
EXCERCISE 11

1. (Jacobi fields in constant curved manifolds) Let M be a Riemannian manifold of constant sectional curvature k and $\gamma : [0,b] \to M$ be a normal geodesic. Let $Y_1(t), \ldots, Y_n(t)$ be a parallel orthonormal vector fields along γ such that $Y_1(t) = \dot{\gamma}(t)$.

(1) Show that

$$\langle R(Y_i, Y_1)Y_1, Y_j\rangle = k(\delta_{ij} - \delta_{j1}\delta_{i1}), \ \forall 1 \le i, j \le n.$$

(2) Let

$$U(t) = f^1(t)Y_1(t) + \sum_{i=1}^{n} f^i(t)Y_i(t)$$

be the Jacobi filed with initial values $f^i(0) = c^i$, $\dot{f}^i(0) = d^i$, $i = 1, 2, \ldots, n$. Compute the functions $f^i(t), i = 1, 2, \ldots, n$.

(Hint: Let

$$S_k(t) := \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}t), & \text{if } k > 0; \\ t, & \text{if } k = 0; \\ \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}t), & \text{if } k < 0. \end{cases}$$

and

$$C_k(t) := \begin{cases} \cos(\sqrt{k}t), & \text{if } k > 0; \\ 1, & \text{if } k = 0; \\ \cosh(\sqrt{-k}t), & \text{if } k < 0. \end{cases}$$

Once can check $f^i = c_i C_k(t) + d_i S_k(t)$ when $i = 2, \ldots, n$.)