RIEMANNIAN GEOMETRY
EXCERCISE 12

1. (Conjugate points)
   (1) Prove that the antipodal points p, q in the n-dimensional sphere $\mathbb{S}^n(\frac{1}{\sqrt{k}})$ of radius $\frac{1}{\sqrt{k}}$ are conjugate points.
   (2) Compute the multiplicity of the antipodal points $p, q \in \mathbb{S}^n(\frac{1}{\sqrt{k}})$ as conjugate points.

2. Prove the following version of *Gauss lemma*: Given $p \in M$ and $V \in T_pM$ in the domain of definition of $\exp_p$, let $\gamma(t) := \exp_p(tV), t \in [0,1]$ be the geodesic with $\gamma(0) = p$, $\dot{\gamma}(0) = V$. Then for any $W \in T - pM$, we have

$$\langle V, W\rangle = \langle (d\exp_p)_{(V)}(V), (d\exp_p)_{(V)}(W)\rangle,$$

where $d\exp_p)_{(V)}$, the derivative of $\exp_p$ at the point $V \in T_pM$, is applied to the vectors V and W considered as vectors in $T_V(T_pM)$, i.e., as vectors tangent to T_pM at the point V.

(Hint: Using the fact that $U(t) := (d\exp_p)_{(tV)}(tW)$ is the variation field of the geodesic variation, $F(t,x) := \exp_p t(V + sW)$ and, therefore, is a Jacobi field with $\dot{U}(0) = W$. Hence $\langle V, W\rangle = \langle \dot{\gamma}(0), \dot{U}(0)\rangle$).