RIEMANNIAN GEOMETRY
EXCERCISE 13

1.(Comparison estimates) Consider a simply-connected complete two dimensional Riemannian manifold (M, g) with Gauss curvature $\leq \beta$.

(1) Let $\gamma : [0, b] \to M$ be a normal geodesic in M, and there is no conjugate point of $\gamma(0)$ along γ. Let $U(t)$ be a normal Jacobi field along γ such that

$$U(0) = 0, |\dot{U}(0)| := \sqrt{g(\dot{U}(0), \dot{U}(0))} = 1.$$

Show that

$$|U(t)| \geq \begin{cases} \frac{1}{\sqrt{\beta}} \sin \sqrt{\beta} t, & \text{if } \beta > 0; \\ t, & \text{if } \beta = 0; \\ \frac{1}{\sqrt{-\beta}} \sinh \sqrt{-\beta} t, & \text{if } \beta < 0. \end{cases}$$

(2) Given $O \in M$. Denote by $c(r)$ the length of the curve

$$\{x \in M : d(x, O) = r\}.$$

Show that for any $r \geq 0$,

$$c(r) \geq \begin{cases} 2\pi r, & \text{if } \beta = 0; \\ \frac{2\pi}{\sqrt{-\beta}} \sinh \sqrt{-\beta} r, & \text{if } \beta < 0. \end{cases}$$

For the case of $\beta > 0$, prove that

$$\exp_O : T_O M \to M$$

is a diffeomorphism on

$$B\left(0, \frac{\pi}{\sqrt{\beta}}\right) := \left\{X \in T_O M : |X| < \frac{\pi}{\sqrt{\beta}}\right\}.$$

And for any $0 \leq r \leq \frac{\pi}{\sqrt{\beta}}$, it holds that

$$c(r) \geq \frac{2\pi}{\sqrt{\beta}} \sin \sqrt{\beta} r.$$