# RIEMANNIAN GEOMETRY
# EXCERCISE 2

1. (Christoffel symbols) Let $(U, x = (x^1, \ldots, x^n))$ be a chart of a Riemannian manifold $M$. Let

$$(x^1, \ldots, x^n) \to (y^1, \ldots, y^n) \tag{0.1}$$

be a smooth coordinate change, and the Riemannian metric can be written as

$$g_{ij}(x)dx^i \otimes dx^j \text{ and } h_{\alpha\beta}(y)dy^\alpha \otimes dy^\beta$$

respectively.

(i) Show the transformation formula of $g^{ij}$ under the coordinate change (0.1) is

$$g^{ij}(x) = h^{\alpha\beta}(y(x))\frac{\partial x^i}{\partial y^\alpha}\frac{\partial x^j}{\partial y^\beta}.$$

(ii) Compute the transformation formulae of the Christoffel symbols $\Gamma^i_{jk}$ under the coordinate change (0.1). Do they define a tensor?

(iii) Let $\gamma : [a, b] \to U$ be a smooth curve. Denote $\dot{x}^i(t) := \frac{d}{dt}x^i(\gamma(t))$. Compute the transformation formula of

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t)$$

under the coordinate change (0.1).

2. (Lobatchevski plane) Let $M$ be the upper half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, with the Riemannian metric

$$\frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

(i) Compute the Christoffel symbols and write down the system of differential equations satisfied by the geodesics.

(ii) Prove that the curves $\gamma : (0, +\infty) \to M,\ t \mapsto (x_0, e^{ct})$, where $x_0 \in \mathbb{R}$, $c > 0$, are geodesics.

(iii) Let $0 < a < b < +\infty$. Let $\gamma$ be as in (ii) and $\sigma : [a, b] \to M$ be a piecewise smooth curve connecting $\gamma(a)$ and $\gamma(b)$. Prove that

$$\text{Length}(\sigma_{|[a,b]}) \geq \text{Length}(\gamma_{|[a,b]}).$$

Characterize the cases when the equality holds.

(iv) At $p = (0, 1) \in M$, compute $\exp_p((0, \eta))$, where $(0, \eta) \in T_pM$ (we identify $T_pM$ with $\mathbb{R}^2$ here); Compute $d\exp_p((0, \eta))$.

3. Consider $S^2 \subset \mathbb{R}^3$ and its spherical coordinate $\{(\varphi, \theta) : \varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}), \theta \in [-\pi, \pi)\}$. We know the induced Riemannian metric is

$$d\varphi \otimes d\varphi + \cos^2 \varphi d\theta \otimes d\theta.$$

(i) Compute the Christoffel symbols and write down the system of differential equations satisfied by the geodesics.

(ii) Prove that the curves $\gamma : (-\frac{\pi}{2}, \frac{\pi}{2}) \to S^2,\ t \mapsto (\theta_0, ct)$, where $\theta_0 \in (-\pi, \pi)$, $c > 0$, are geodesics.

(iii) Write down the exponential map $\exp_p$ and $d\exp_p$ at the north pole $p$ (i.e., at the point with $\varphi = -\frac{\pi}{2}$).