RIEMANNIAN GEOMETRY
EXCERCISE 3

1. Let U be a normal neighborhood of $p \in M$ with coordinates $(x^1, \ldots, x^n)$. Consider the radial function on $(U \setminus \{p\}, x^1, \ldots, x^n)$:

$$r = \sqrt{\sum_i (x^i)^2}.$$

Show that $\operatorname{grad} r = \frac{\partial}{\partial r}$. (Hint: Use Riemannian polar coordinates.)

2. (Geodesics on $P^n(\mathbb{R})$)

(i) Prove that the antipodal mapping $A : S^n \to S^n$ given by $A(p) = -p$ is an isometry of S^n.

(ii) Introduce a Riemannian metric on the real projective space $P^n(\mathbb{R})$ such that the natural projection $\pi : S^n \to P^n(\mathbb{R})$ is a local isometry.

(iii) Show that the geodesics of $P^n(\mathbb{R})$ are periodic with period π.

3. (Lobatchevski plane II) Let M be the upper half-plane $\{(x, y) \in \mathbb{R}^2 : y > 0\}$, with the Riemannian metric

$$\frac{1}{y^2}(dx \otimes dx + dy \otimes dy).$$

(i) Prove that the transformation

$$z \to \frac{az + b}{cz + d}, \; z = x + iy, ad - bc = 1,$$

is an isometry of M.

(ii) Recall in Excercise 2, we show the curves $\gamma : (0, +\infty) \to M, \; t \mapsto (x_0, e^{ct})$, where $x_0 \in \mathbb{R}$, $c > 0$, are geodesics. Then, using (i), deduce the that the upper unit semicircle are geodesics.

3.(Surfaces of revolution) Consider the surface S of revolution obtained by rotating the curve $(x = 0, y = e^z, z) \in \mathbb{R}^3$, i.e. the graph of the exponential function, about the axis $0z$. This surface is the image of the map

$$\begin{aligned} \varphi : & U \subset \mathbb{R}^2 \to \mathbb{R}^3 \\ & (z, \theta) \mapsto (e^z \cos\theta, e^z \sin\theta, z), \end{aligned}$$

where

$$U := \{(z, \theta) \in \mathbb{R}^2 : -\infty < x < +\infty, 0 \le \theta < 2\pi\}.$$

The image of the curves $\theta = constant$ and $z = constant$ are called *meridians* and *parallels*, respectively, of S.

(i) Show that the induced metric in the coordinates (z, θ) is given by

$$g = \left(e^{2z} + 1\right) dz \otimes dz + e^{2z} d\theta \otimes d\theta.$$

(ii) Show that S with the induced metric is complete. (Hint: Using Hopf-Rinow Theorem. Which definition of completeness is more convenient for the purpose here?)