RIEMANNIAN GEOMETRY
EXCERCISE 5

1. Recall the Koszul formula states that: for any $X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle =& X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ & - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle \end{aligned}$$

Suppose we know the following fact: There exist three vector files $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on $\mathbb{S}^3 \subset \mathbb{R}^4$ which are linearly independent at any point of $\mathbb{S}^3$, such that

$$[\mathbf{i}, \mathbf{j}] = \mathbf{k},\ [\mathbf{j}, \mathbf{k}] = \mathbf{i},\ [\mathbf{k}, \mathbf{i}] = \mathbf{j}.$$

Assign to $\mathbb{S}^3$ a Riemmannian metric g such that $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are orthonormal at any point. Calculate the Levi-Civita connection ∇ of $(\mathbb{S}^3, g)$.

2. (Isometries preserve Levi-Civita connections) Let (M_1, g_1), (M_2, g_2) be two Riemannian manifolds. Let $\nabla^{(1)}$, $\nabla^{(2)}$ be their Levi-Civita connections, respectively. Suppose $\varphi : M_1 \to M_2$ be an isometry. Prove that for any $X, Y \in \Gamma(TM_1)$, we have

$$d\varphi\left(\nabla^{(1)}_X Y\right) = \nabla^{(2)}_{d\varphi(X)} d\varphi(Y).$$

3. (Induced connection) Let M, N be two smooth manifold and $\varphi : N \to M$ be a smooth map. A vector field along φ is an assignment

$$x \in N \mapsto T_{\varphi(x)}M.$$

Let $\{E_i\}_{i=1}^n$ be a frame field in a neighborhood U of $\varphi(x) \in M$. Then for any $y \in \varphi^{-1}(U)$, we have

$$V(x) = V^i(x)E_i(\varphi(x)).$$

Let $u \in T_xN$. We define

$$\widetilde{\nabla}_u V := u(V^i)(x)E_i(\varphi(x)) + V^i(x)\nabla_{d\varphi(u)}E_i, \tag{0.1}$$

where ∇ is an affine connection on M.

(i) Check that $\widetilde{\nabla}_u V$ is well defined, i.e., (0.1) is independent of the choices of $\{E_i\}$.

(ii) Let g be a Riemannian metric on M. Prove that if ∇ on M is compatible with g, then for vector fields V, W along φ, and $u \in T_xN$, we have

$$u\langle V, W\rangle = \langle \widetilde{\nabla}_u V, W\rangle + \langle V, \widetilde{\nabla}_u W\rangle.$$

(iii) Prove that if ∇ on M is torsion free, then for any $X, Y \in \Gamma(TN$, we have

$$\widetilde{\nabla}_X d\varphi(Y) - \widetilde{\nabla}_Y d\varphi(X) - d\varphi([X, y]) = 0.$$

4. (Variation of energy functional: A coordinate-free calculation) Let $\gamma : [a, b] \to M$ be a smooth curve, and

$$\alpha : (-\epsilon, \epsilon) \times [a, b] \to M,\ (s, t) \mapsto \alpha(s, t).$$

be a variation, where (M, g) is a Riemmanian manifold with a Levi-Civita connection.

Recall the energy functional of a curve γ is

$$E(\gamma) := \frac{1}{2}\int_a^b \left\langle d\gamma\left(\frac{\partial}{\partial t}\right), d\gamma\left(\frac{\partial}{\partial t}\right)\right\rangle dt.$$

For convenience, we denote by

$$W(t) := d\alpha\left(\frac{\partial}{\partial s}\right)(0,t), \text{ and } \dot\gamma(t) := d\gamma\left(\frac{\partial}{\partial t}\right).$$

Prove the following variation formula:

$$\frac{d}{ds}_{|_{s=0}} E(\bar\alpha(s)) = -\int_a^b \left\langle W(t), \frac{D\dot\gamma}{dt}(t)\right\rangle dt - \left\langle W(a), \frac{D\dot\gamma}{dt}(a)\right\rangle + \left\langle W(b), \frac{D\dot\gamma}{dt}(b)\right\rangle,$$

where $\bar\alpha(s) := \alpha(s,\cdot) : t \mapsto \alpha(s,t)$.

5. Let S^n be the sphere with the induced metric g from the Euclidean metric in $\mathbb{R}^{n+1}$. We denote by $\overline{\nabla}$ the canonical Levi-Civita connection on $\mathbb{R}^{n+1}$. For any $X, Y \in \Gamma(T\mathbb{S}^n)$, one can extend X, Y to smooth vector field $\overline{X}, \overline{Y}$ on $\mathbb{R}^{n+1}$, at least near $\mathbb{S}^n$.

By locality, the vector $\overline{\nabla}_{\overline{X}}\overline{Y}$ at any $p \in \mathbb{S}^n$ depends only on $\overline{X}(p) = X(p)$ and the vectors $\overline{Y}(q) = Y(q)$ for $q \in \mathbb{S}^n$. That is, $\overline{\nabla}_{\overline{X}}\overline{Y}$ is independent of the extension of X, Y we choose. So we will write $\overline{\nabla}_X Y$ instead of $\overline{\nabla}_{\overline{X}}\overline{Y}$ at points on $\mathbb{S}^n$.

We define $\nabla_X Y$ to be the orthogonal projection of $\overline{\nabla}_X Y$ onto the tangent space of $\mathbb{S}^n$, i.e.,

$$\nabla_X Y := \overline{\nabla}_X Y - \langle \overline{\nabla}_X Y, \mathbf{n}\rangle \mathbf{n},$$

where $\mathbf{n}$ is the unit out normal vector on $\mathbb{S}^n$.

(i) Prove that ∇ is an affine connection on $\mathbb{S}^n$.

(ii) Prove that ∇ is the Levi-Civita connection of $(\mathbb{S}^n, g)$.