RIEMANNIAN GEOMETRY
EXCERCISE 6

1. Let N_1, N_2 be two submanifolds of a complete Riemannian manifold (M, g) without boundary, and let $\gamma : [0, a] \to M$ be a geodesic such that $\gamma(0) \in N_1$, $\gamma(a) \in N_2$ and γ is the shortest curve from N_1 to N_2. Prove that $\dot\gamma(0)$ is perpendicular to $T_{\gamma(0)}N_1$, and $\dot\gamma(a)$ is perpendicular to $T_{\gamma(t)}N_2$. (Hint: Use the First Variation Formula.)

2. Let $c : [0, a] \to M$ be a piecewise smooth curve. That is, there exists a subdivision

$$0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = a$$

such that c is smooth on each interval $[t_i, t_{i+1}]$.

(i) At the break points t_i, there are two possible values for the velocity vector filed along c: a right derivative and a left derivative:

$$\dot c(t_i^+) = \frac{dc}{dt}_{|[t_i, t_{i+1}]}(t_i), \ \dot c(t_i^-) = \frac{dc}{dt}_{|[t_{i-1}, t_i]}(t_i).$$

Let $F : [0, a] \times (-\epsilon, \epsilon) \to M$ be a piecewise smooth variation of c. Derive the First Variation Formula of the energy functional. (Hint: Make use of the formula for smooth curves we discussed during the course.)

(ii) Let $V(t)$ be a piecewise smooth vector filed along the curve c. Show that there exists a variation $F : [0, a] \times (-\epsilon, \epsilon) \to M$ such that $V(t)$ is the variational field of F; in addition, If $V(0) = V(a) = 0$, it is possible to choose F as a proper variation. (Hint: Use exponential maps.)

(iii) (Characterization of geodesics) Prove that a piecewise smooth curve $c : [0, a] \to M$ is a geodesic if and only if, for every proper variation F of c, we have

$$E'(0) = 0.$$