RIEMANNIAN GEOMETRY
EXCERCISE 7

Recall for $X, Y, Z \in \Gamma(TM)$, the curvature tensor is defined as

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

1. Let $s : \mathbb{R}^2 \to M$ be a parametrized surface, and V be a smooth vector field along s. Prove that

$$\widetilde{\nabla}_{\frac{\partial}{\partial x}} \widetilde{\nabla}_{\frac{\partial}{\partial y}} V - \widetilde{\nabla}_{\frac{\partial}{\partial y}} \widetilde{\nabla}_{\frac{\partial}{\partial x}} V = R\left(\frac{\partial s}{\partial x}, \frac{\partial s}{\partial y}\right) V,$$

where $\frac{\partial s}{\partial x} := ds\left(\frac{\partial}{\partial x}\right)$. (Hint: Compute in local coordinates.)

2. (Ricci Identity)
  (i) Let $X, Y \in \Gamma(TM)$, $\phi \in \Gamma(\otimes^{r,s} TM)$. Prove that

$$\begin{aligned}&\nabla^2 \phi(w_1, \dots, w_r, X_1, \dots, X_s, X, Y) - \nabla^2 \phi(w_1, \dots, w_r, X_1, \dots, X_s, Y, X) \\ =& - R(X,Y)\phi(w_1, \dots, w_r, X_1, \dots, X_s),\end{aligned}$$

  for any $w_1, \dots, w_r \in \Gamma(T^*M)$, $X_1, \dots, X_s \in \Gamma(TM)$.
  (ii) Let $Z \in \Gamma(TM)$. Derive from Ricci Identity the following identity in local coordinate $(U, x^1, \dots, x^n)$:

$$Z^\ell{}_{,jk} - Z^\ell{}_{,kj} = -Z^i R^\ell{}_{ijk},$$

  where $R^\ell{}_{ijk} \frac{\partial}{\partial x^\ell} := R\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) \frac{\partial}{\partial x^i}$, and $\nabla^2 Z = Z^\ell{}_{,jk} \frac{\partial}{\partial x^\ell} \otimes dx^j \otimes dx^k$.

3.(Curvature tensor expressed in local coordinates)
  (i) Show that

$$R^k{}_{\ell ij} = \frac{\partial \Gamma^k_{j\ell}}{\partial x^i} - \frac{\partial \Gamma^k_{i\ell}}{\partial x^j} + \Gamma^\gamma_{j\ell}\Gamma^k_{i\gamma} - \Gamma^\gamma_{i\ell}\Gamma^k_{j\gamma}.$$

  (ii) Using (i), show that

$$\begin{aligned} R_{k\ell ij} :=& \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k} \right\rangle \\ =& \frac{1}{2}\left(\frac{\partial^2 g_{jk}}{\partial x^i \partial x^\ell} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^\ell} - \frac{\partial^2 g_{j\ell}}{\partial x^i \partial x^k} + \frac{\partial^2 g_{i\ell}}{\partial x^j \partial x^k}\right) \\ &+ g_{mp}\left(\Gamma^m_{i\ell}\Gamma^p_{jk} - \Gamma^m_{j\ell}\Gamma^p_{ik}\right). \end{aligned}$$

4. Show the following identities:
  (i) $\langle R(X,Y)Z, W\rangle = -\langle Z, R(X,Y)W\rangle$. (Hint: using compatibility of ∇ with the metric. Of course, one can also use the local expression in 3(ii).)
  (ii) $\nabla R(W, Z, X, Y, V) = \langle \nabla_V R(X,Y)Z, W\rangle$.

5. (Curvature tensor R is determined by the values $\langle R(X,Y)Y, X\rangle$)
  (i) Show that if $\langle R(X,Y)Y, X\rangle = 0$ for any $X, Y \in \Gamma(TM)$, then $\langle R(X,Y)W, X\rangle = 0$ for any $X, Y, W \in \Gamma(TM)$.

(ii) Show that if $\langle R(X,Y)W, X\rangle = 0$ for any $X, Y, W \in \Gamma(TM)$, then

$$2\langle R(Y,Z)X, W\rangle = \langle R(X,Z)Y, W\rangle,$$

for any $X, Y, Z, W \in \Gamma(TM)$. (Hint: using the first Bianchi identity.)

(iii) Show that if $2\langle R(Y,Z)X, W\rangle = \langle R(X,Z)Y, W\rangle$, $\forall X, Y, Z, W \in \Gamma(TM)$, then

$$\langle R(X,Y)Z, W\rangle = 0,$$

for any $X, Y, Z, W \in \Gamma(TM)$.

6.(An alternative proof of 5)
Prove that for any $X, Y, Z, W \in \Gamma(TM)$, we have

$$-6\langle R(X,Y)Z, W\rangle = \frac{\partial^2}{\partial s \partial t}_{|s=t=0} [\langle R(X+sZ, Y+tW)(Y+tW), X+sZ\rangle - \langle R(X+sW, Y+tZ)(Y+tZ), X+sW\rangle]$$

(Hint: using the first Bianchi identity.)