RIEMANNIAN GEOMETRY
EXCERCISE 8

1. (The Second Variation Formula for length) Let $\gamma : [a,b] \to M$ be a smooth curve and

$$F : [a,b] \times (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$$

be a 2-parameter variation of γ. Denote by

$$V(t) := \frac{\partial F}{\partial v}(t,0,0), \quad W(t) = \frac{\partial F}{\partial w}(t,0,0)$$

the two corresponding variational fields. Let $L(v,w) := L(\gamma_{v,w})$ be the length of the curve $\gamma_{v,w}(t) := F(t,v,w), t \in [a,b]$.

(1) Show that

$$\frac{\partial}{\partial w \partial v} L(v,w) = \int_a^b \frac{1}{\left\| \frac{\partial F}{\partial t} \right\|} \Big\{ \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w} \right\rangle - \left\langle R\left(\frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right) \frac{\partial F}{\partial t}, \frac{\partial F}{\partial v} \right\rangle$$
$$+ \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \widetilde{\nabla}_{\frac{\partial}{\partial w}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle$$
$$- \frac{1}{\left\| \frac{\partial F}{\partial t} \right\|^2} \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial v}, \frac{\partial F}{\partial t} \right\rangle \left\langle \widetilde{\nabla}_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial w}, \frac{\partial F}{\partial t} \right\rangle \Big\} dt,$$

where $\left\| \frac{\partial F}{\partial t} \right\| := \left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial t} \right\rangle^{\frac{1}{2}}$.

(2) Let γ be a normal geodesic. Show that

$$\frac{\partial}{\partial w \partial v}_{|v=w=0} L(v,w) = \int_a^b \left(\langle \nabla_T V, \nabla_T W \rangle - \langle R(W,T)T, V \rangle - T\langle V,T \rangle T\langle W,T \rangle \right) dt$$
$$+ \langle \nabla_W V, T \rangle |_a^b,$$

where $T(t) := \dot{\gamma}(t)$ is the velocity field along γ.

(3) Consider the orthogonal component $V^\perp, W^\perp$ of V, W with respect to T, that is

$$V^\perp := V - \langle V,T \rangle T,$$
$$W^\perp := W - \langle W,T \rangle T.$$

Show that

$$\frac{\partial}{\partial w \partial v}_{|v=w=0} L(v,w) = \int_a^b \left(\langle \nabla_T V^\perp, \nabla_T W^\perp \rangle - \langle R(W^\perp, T)T, V^\perp \rangle \right) dt$$
$$+ \langle \nabla_W V, T \rangle |_a^b,$$