RIEMANNIAN GEOMETRY
EXCERCISE 9

Given two orientable manifold M_1, M_2, we say a C^∞ map $f : M_1 \to M_2$ *preserves the orientation* if $w_1(e_1, \ldots, e_n) > 0$ implies $w_2(df(e_1), \ldots, df(e_n)) > 0$ where w_i is the C^∞ nowhere vanishing n-form on M_i determining the orientation, $i = 1, 2$.

(1) Let (M, g) be a compact, orientable, even-dimensional Riemmanian manifold with positive sectional curvatures. Prove that any isometry $f : M \to M$ which preserves the orientation has a fixed point. (Hint: Mimic the proof for the odd-dimensional case we discussed in the lecture.)

(2) Derive the following theorem of Synge from (1) and Bonnet-Myers theorem:

Theorem 0.1 (Synge)**.** *Any compact, orientable, even-dimensional Riemannian manifold with positive sectional curvatures is simply connected.*