Rie. Geo. HW 3.

1. Let $(r, \theta_1, \dots, \theta_n)$ be a the Rie. polar coordinates.

then we have $\forall X \in T_pU$, $X = \cancel{X^i \frac{\partial}{\partial r} + X^i \frac{\partial}{\partial \theta^i}}$ $X^1 \frac{\partial}{\partial r} + X^i \frac{\partial}{\partial \theta^i}$

$\cancel{\langle \frac{\partial}{\partial r}, X \rangle = X^1}$

and use Gauss lemma, we have metric can be expressed as.

$$g = dr^2 + g_{ij}(r, \theta) d\theta^2 .$$

$$\Rightarrow \langle \frac{\partial}{\partial r}, X \rangle = X^1 = X^1 \frac{\partial}{\partial r} r + X^i \frac{\partial}{\partial \theta^i} r = \langle \nabla r, X \rangle .$$

$$\Rightarrow \frac{\partial}{\partial r} = \nabla r$$

2.

(i). $\forall p \in S^n$, set $\gamma$ be a great circle containing $p$.

and the rotate $S^n$ along $\gamma$ gives the antipodal map.

which is an isometry.

(ii). In order to make $\pi$ be an local isometry.

we can set $h$ be like. $\forall [x] \in \mathbb{R}P^n$, $u, v \in T_{[x]}\mathbb{R}P^n$.

$$h_{[x]}(u, v) := A\, g_x \left( (d\pi)_x^{-1} u,\ (d\pi)_x^{-1} v \right) .$$

check. $h$ is well-defined. i.e. $\forall x \sim x_1$. $\cancel{A g_{x_1}}$

$$g_{x_1}\left( (d\pi)_{x_1}^{-1} u,\ (d\pi)_{x_1}^{-1} v \right) = g_x \left( (d\pi)_x^{-1} u,\ (d\pi)_x^{-1} v \right) .$$

(iii). Since $\pi$ ~~→ open~~ $\pi$ is a diffeomorphism ~~and~~ and an local isometry on the open hemisphere. we conclude that

geodesics ~~in~~ on $S^n$ ~~are~~ geodesics on $\mathbb{R}P^n$ too.

but this shows that a semi great circle is a closed geodesic in $\mathbb{R}P^n$.

3. [illegible]

iii, Since a linear fractional linear transformation maps circle to circle and maps the vertical line to upper semi-circle we conclude that the upper unit semicircle are geodesics.

4. iii, Write S is a closed sq surface in $\mathbb{R}^3$.

here with the same topology as $\mathbb{R}^3$ a subspace of $\mathbb{R}^3$.

thus, bounded closed subset is also bounded closed in $\mathbb{R}^3$

which is cpt.