HW4.

1. ~~This question we will~~

Check the definition of "the connection"

and apply the "~~Partiti~~ Unit of Partition".

Pf: First we choose a coordinate patch ~~that~~ $\{U_\alpha, \varphi_\alpha\}$ covers $M$.

~~the~~ introduce a local coordinate, say $x$, on $U_\alpha$

then for any $X, Y \in T(TU_\alpha)$, define

a "local coordinate" $\nabla^{U_\alpha}$ on $U_\alpha$ by the directional derivative.

i.e. $(\nabla^{U_\alpha}_X Y)(p) := \lim_{t \to 0} \frac{Y(p+tX) - Y(p)}{t}$ $\forall p \in U$.

This is a connection defined only on $U_\alpha$

but we can always extend $\nabla^U$ on the whole $T(TM)$ by

trivially setting $(\nabla^{U_\alpha}_X Y)(q) = 0$ $\forall q \in M \setminus U_\alpha$.

Now, choose a "unit of partition" subordinate to $\{U_\alpha\}$.

i.e. $\sum \rho_\alpha = 1$ $\operatorname{supp} \rho_\alpha \subset U_\alpha$

and set $\nabla = \sum \rho_\alpha \nabla^{U_\alpha}$.

check: $\nabla$ is a connection which is untrivial.

i.e. $\forall f \in C^\infty(M)$, $X, Y \in T(TM)$.

$\nabla_{fX} Y = f \nabla_X Y$. $\nabla_X (fY) = f \nabla_X Y + X(f) Y$.

2. The This is what we called "locality of the connection".

Recall. two we prove $\nabla_X T|_{U} = 0$ for $X|_U = 0$.

choose $p \in U$. we only need to show $(\nabla_X T)(p) = 0$.

To show this. choose a small NB $V$ of $p$ s.t. $V \subset \bar{V} \subset U$.

then choose a cut-off $\rho$ s.t. $\operatorname{supp}\rho \subset U$ and $\rho \equiv 1$ on $V$.

hence $(1-\rho)X = X$. then

$$\nabla_X T|_V = (1-\rho)\nabla_X T|_V = 0.$$

In particular we have $(\nabla_X T)(p) = 0$.

So. we can similarly prove this.

i.e. choose ~~on~~ $\forall p \in U$. a NB $V$ s.t. $p \in V \subset \bar{V} \subset U$.

and then choose a cut-off function $\rho$ s.t. $\operatorname{supp}\rho \subset U$. $\rho \equiv 1$ on $V$.

hence. $(1-\rho)T = T$.

$$(\nabla_X (1-\rho)T)|_V = (\nabla_X(1-\rho))T|_V + (1-\rho)\nabla_X T|_V = 0$$

$$\Rightarrow (\nabla_X T)(p) = 0.$$

3. Apply the definition of parallel transport and understand what is a parallel transport

1) By definition.

$$\nabla_{X_p} K = \lim_{h\to 0}\frac{1}{h}\left[\tilde{P}^{-1}_{c,h}(K(c(h))) - K(p)\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left(\tilde{P}^{-1}_{c,h}(T(c(h))) \otimes \tilde{P}^{-1}_{c,h}(w(c(h))) \otimes \tilde{P}^{-1}_{c,h}(\eta(c(h))) - K(p)\right)$$

$$= \nabla_{X_p} T \otimes w \otimes \eta + T \otimes \nabla_{X_p} w \otimes \eta + T \otimes w \otimes \nabla_{X_p}\eta$$

(2) By definition.

$$\tilde{P}^{-1}_{c,h}(k(c(h))) - k(p)$$

$$= \omega(\gamma)(c(h))\,\tilde{P}^{-1}_{c,h}(\eta(c(h))) - \omega(\gamma)(p)\,\eta(p)$$

$$= \lim_{h\to 0} \frac{\omega(\gamma)(c(h)) - \omega(\gamma)(p)}{h}\,\eta(p) + \lim_{h\to 0} \omega(\gamma)(p)\cdot \frac{\tilde{P}^{-1}_{c,h}(\eta(c(h))) - \eta(p)}{h}$$

$$= \cancel{(\omega(\gamma)(p))}\ (X_p\,\omega(\gamma))(p)\,\eta(p) + \omega(\gamma)(p)\,\nabla_{X_p}\eta(p)$$

On the ~~han~~ other hand, we have

$$(\nabla_{X_p}k)(p) = \left(\omega(\nabla_{X_p}\gamma)\,\eta + (\nabla_{X_p}\omega)(\gamma)\,\eta + \omega(\gamma)\,\nabla_{X_p}\eta\right)\Big|_p$$

$$= X_p(\omega(\gamma))(p)\,\eta(p) + \omega(\gamma)(p)\,(\nabla_{X_p}\eta)(p)$$

$$+ \omega(\nabla_{X_p}\gamma)(p)\,\eta(p) - \omega(\nabla_{X_p}\gamma)(p)\,\eta(p)$$

$$= X_p(\omega(\gamma))(p)\,\eta(p) + \omega(\gamma)(p)\,(\nabla_{X_p}\eta)(p).$$