HW5.

We have.

~~$2\langle \nabla_i j, k\rangle = -\langle i, i\rangle$~~ $= -1$.

$\nabla_i j = a_1 i + a_2 j + a_3 k$. $\nabla_j k = b_1 i + b_2 j + b_3 k$

$\nabla_k i = c_1 i + c_2 j + c_3 k$. while $\nabla_i i = \nabla_j j = \nabla_k k = 0$.

by the Koszul formula. we have.

$\Rightarrow \nabla_i j = +\frac{1}{2}k$. $\nabla_j k = +\frac{1}{2}i$ $\nabla_k i = \frac{1}{2}j$.

2. By Koszul formula. we have.

$$2\langle \nabla_X Y, Z\rangle_{g_1} = X\langle Y, Z\rangle_{g_1} + Y\langle Z, X\rangle_{g_1} - Z\langle X, Y\rangle_{g_1}$$
$$- \langle X, [Y, Z]\rangle_{g_1} + \langle Y, [Z, X]\rangle_{g_1} + \langle Z, [X, Y]\rangle_{g_1}$$

and $2\langle \nabla^2_{d\varphi X}, d\varphi(Y), d\varphi Z\rangle_{g_2}$

$$= d\varphi X\langle d\varphi Y, d\varphi(Z)\rangle_{g_2} + \cdots$$

$$- \langle d\varphi(X), [d\varphi(Y), d\varphi(Z)]\rangle_{g_2} - \cdots$$

$$= X\langle Y, Z\rangle_{g_1} - \cdots$$

$$- \langle X, [Y, Z]\rangle - \cdots$$

$$= 2\langle \nabla_X Y, Z\rangle_{g_1}$$

3. suppose we have another frame field $\{e_i\}$

s.t $E_i\ e_i = a_i^j E_j$. $a_i^j \in C^\infty(U)$

$\forall y \in \varphi(U)$ set $V(y) = \tilde{v}^i(y) e_i(\varphi y) = v^i(y) E_i(\varphi y)$

$$\Rightarrow v^i(y) = \tilde{v}^j(y) a_j^i(y). \quad = \tilde{v}^i(y) a_i^j E_j(\varphi y)$$

hence

$$\tilde\nabla_u V := u(V^i)(x)\,E_i(\varphi(x)) + V^i(x)\,\nabla_{d\varphi(u)}E_i$$

$$= u(\tilde V^j a_j^i)(x)\,E_i(\varphi(x)) + \tilde V^j(x)\,a_j^i(x)\,\nabla_{d\varphi(u)}(b^k_i e_k)$$

$$= u(\tilde V^j)\,a_j^i E_i + \tilde V^j\,u(a_j^i)\,E_i + \tilde V^j(x)\,a_j^i(x)\,[d\varphi(u)\,b_i^k]\,e_k$$

$$+ \underline{\tilde V^j(x)\cdot a_j^i\cdot b_i^k\,\nabla_{d\varphi(u)}e_k}$$

$$= u(\tilde V^j)(x)\,e_j(\varphi(x)) + \tilde V^j(x)\,\nabla_{d\varphi(u)}e_j$$

here (b^k_i) is the inverse of (a^i_j)

and we use the fact that

$$a_j^i\,[d\varphi(u)\,b_i^k] = d\varphi(u)[a_j^i\,b_i^k] - [d\varphi(u)(a_j^i)]\,b_i^k$$

$$= 0 - u(a_j^i\circ\varphi(x))\cdot b_i^k$$

$$= -u(a_j^i(\varphi(x)))\cdot b_i^k(x)$$

(2). Locally. $V = V^i E_i$. $W = W^j E_j$.

~~we have $u\langle V, W\rangle = \langle \nabla_u V, W\rangle + \langle V, \nabla_u W\rangle$~~

we have $\langle \tilde\nabla_u V, W\rangle + \langle V, \tilde\nabla_u W\rangle$

$$= \langle u(V^i)E_i + V^i(x)\,\nabla_{d\varphi(u)}E_i,\ W^j E_j\rangle + \langle V^j E_j,\ u(W^i)E_i + W^i\,\nabla_{d\varphi(u)}E_i\rangle$$

$$= u(V^i)\,W^j g_{ij} + V^i W^j\,\langle \nabla_{d\varphi(u)}E_i, E_j\rangle$$

$$+ V^j u(W^i)\,g_{ij} + V^j W^i\,\langle E_j, \nabla_{d\varphi(u)}E_i\rangle$$

$$= u(V^i)\,W^j g_{ij} + V^i u(W^j)\,g_{ij} + V^i W^j\,(d\varphi(u)(g_{ij}))$$

$$= u(V^i W^j)\,g_{ij} + V^i W^j\,u(g_{ij}\circ\varphi)$$

$$= u\langle V, W\rangle$$

3. ~~$X=$~~: let $d\varphi(Y) = Y^j(\varphi(x))\bar{E}_j(\varphi(x))$. $d\varphi(X) = X^i\circ\varphi\,\bar{E}_i$.

we have, $\forall$ ~~$x\in M$~~ $p\in M$

$$LHS = \tilde{\nabla}_{X_p} d\varphi(Y) - \tilde{\nabla}_{Y_p} d\varphi(X) - (d\varphi)_p([X,Y]).$$

$$= X_p(Y^j)(\varphi(x))\bar{E}_j + Y^j(\varphi(x)) \nabla_{d\varphi(X_p)} \bar{E}_j$$

$$- Y_p(X^j)(\varphi(x))\bar{E}_j - X^j(\varphi(x)) \nabla_{d\varphi(Y_p)}\bar{E}_j - (d\varphi)_p([X,Y])$$

$$= \nabla_{d\varphi(X_p)}(Y^j\bar{E}_j) + \nabla_{d\varphi(Y_p)}(X^j\bar{E}_j) \cancel{(d\varphi)_p([X,Y])} [d\varphi_p(X), d\varphi_p(Y)]$$

$$= 0.$$

4. we have

$$\frac{d}{ds}\Big|_{s=0} E(\partial(s)) = \int_a^b \frac{d}{ds}\Big|_{s=0} \langle \dot\gamma, \dot\gamma\rangle dt$$

$$= 2\int_a^b \langle \nabla_{W(t)}\dot\gamma, \dot\gamma\rangle dt$$

$$= 2\int_a^b \langle \nabla_{\dot\gamma} W(t), \dot\gamma\rangle dt$$

$$= 2\int_a^b \dot\gamma\langle W(t), \dot\gamma\rangle - \langle W(t), \frac{D}{dt}\dot\gamma\rangle dt$$

$$= \langle W(b), \dot\gamma(b)\rangle - \langle W(a), \dot\gamma(a)\rangle - \int_a^b \langle W(t), \frac{D\dot\gamma}{dt}(t)\rangle dt$$

5. $\nabla_X Y = \bar{\nabla}_X Y - \langle \bar{\nabla}_X Y, \vec{n}\rangle_{g_0} \vec{n}$

here the metric is the standard Euclidean metric

then we have. $\nabla_X Y = \bar{\nabla}_X Y - (\cancel{X}\langle Y, \vec{n}\rangle - \langle Y, \bar{\nabla}_X \vec{n}\rangle)\vec{n}$

$$= \bar{\nabla}_X Y + \langle Y, X\rangle_{g_0} \vec{n}$$

Since $\vec{n}$ is the outer normal direction which is exactly the position vector.

we then we have.

For $X = x^i \frac{\partial}{\partial x^i}$. $\vec{n} = x^i \partial_i$.

$\bar{\nabla}_X \vec{n} = [X^i \frac{\partial}{\partial x^i}(x^j)] \frac{\partial}{\partial x^j} = x^j \frac{\partial}{\partial x^j} = X$

$\Rightarrow \quad \langle Y, \bar{\nabla}_X \vec{n} \rangle = \langle Y, X \rangle$

here $\quad \nabla_X Y = \bar{\nabla}_X Y + \langle X, Y \rangle \vec{n}$

we have. $\nabla_X (fY) = f \bar{\nabla}_X Y + Xf \, Y + f \langle X, Y \rangle \vec{n}$

$= f(\nabla_X Y) + Xf \, Y$.

$\nabla_{fX} Y = f \bar{\nabla}_X Y + f \langle X, Y \rangle \vec{n} = f \nabla_X Y$.

$\Rightarrow \nabla$ is a connection.

21. $\cancel{X \langle Y, Z \rangle_{g_{S^n}}} = \nabla_X Y - \nabla_Y X = \bar{\nabla}_X Y - \bar{\nabla}_Y X = [X, Y]$.

since $\bar{\nabla}$ is torsion free.

$2 \langle \nabla_X Y, Z \rangle_{g_{S^n}} = 2 \langle \bar{\nabla}_X Y, Z \rangle_{g_{S^n}} - 2 (\iota^* g_0) \langle \bar{\nabla}_X Y, Z \rangle$

$= 2 \langle \bar{\nabla}_{d\iota X}, d\iota Y, d\iota Z \rangle_{g_0} = 2 \, d\iota X \langle d\iota Y, d\iota Z \rangle + d\iota Y \langle d\iota Z, d\iota X \rangle$

$- d\iota Z \langle d\iota X, d\iota Y \rangle \rangle$

$= X \langle Y, Z \rangle_{g_{S^n}} + Y \langle Z, X \rangle_{g_{S^n}} - \langle Z \langle X, Y \rangle_{g_{S^n}}$

$\Rightarrow \quad \nabla$ is the Levi-Civita connection.